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Asymptotic Approximations of Apostol-Genocchi Numbers and Polynomials

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Abstract. Asymptotic approximations of the Apostol-Genocchi numbers and polynomials are derived using Fourier series and ordering of poles of the generating function. Asymptotic formulas for the Apostol-Euler numbers and polynomials are obtained as consequence. Asymptotic formulas for special cases which include the Genocchi numbers and polynomials are also explicitly stated.

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1. Introduction

The Apostol-Genocchi polynomials $G_n(x;\lambda)$ are defined by the generating function

$$\frac{2te^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}G_n(x;\lambda)\frac{t^n}{n!}, \tag{1.1}$$

where $|t|<\pi$ when $\lambda=1$ and $|t+\log\lambda|<\pi$ when $\lambda\neq 1$. When $\lambda=1$, the above equation gives the generating function of the Genocchi polynomials [3].

When $x=0$, (1.1) reduces to the generating function of the Apostol-Genocchi numbers $G_n(0;\lambda)$ given by

$$\frac{2t}{\lambda e^t+1}=\sum_{n=0}^{\infty}G_n(0;\lambda)\frac{t^n}{n!}. \tag{1.2}$$

For λ not zero, the set of poles of the generating function (1.1) is

$$T_\lambda:=\{(2k+1)\pi i-\log\lambda : k\ \in\ \mathbb{Z}\}, \tag{1.3}$$

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which is also the set of poles of (1.2), where the logarithm is taken to be the principal branch.

Bayad [2] and Luo [13] derived Fourier series of Apostol-Genocchi polynomials expressed in terms of these poles. The Fourier series they obtained is given in the next section. Fourier expansion of higher-order Apostol-Genocchi polynomials was derived in [4] and was shown to be reducible to those obtained in [2] and [13] when the order is 1.

New identities involving the Apostol-Genocchi polynomials were established in [9]. Some generalizations and properties of these polynomials were presented in [14]. Multiplication and explicit recursive formulas of higher-order Apostol-Genocchi polynomials were obtained in [12]. A new generalization of Apostol type Hermite-Genocchi polynomials is studied in [1] while products of the Apostol-Genocchi polynomials were studied in [10]. Moreover, the higher-order convolutions of these polynomials using generating-function methods and summation-transform techniques were established in [11].

Inspired by the work of Kim and Kim [7], a new class of the Frobenius-Genocchi polynomials was considered in [6] by means of the polyexponential function and new relations and properties were obtained. New relations on q-Genocchi polynomials where the relations were stated by symmetric group of degree n were done in [5].

Navas, Ruiz and Varona [15] obtained asymptotic estimates of the Apostol-Bernoulli and Apostol-Euler numbers and polynomials and further analyzed the asymptotic behavior of the Apostol-Bernoulli polynomials in detail. The starting point of their analysis is the Fourier series of the polynomials on the closed interval $[0, 1]$ followed by ordering the poles of the generating function.

In this paper, asymptotic approximations of the Apostol-Genocchi numbers and polynomials for $\lambda \in \mathbb{C}\backslash\{0\}$ are obtained. The method used in [15] is applied to the Apostol-Genocchi numbers and polynomials to obtain asymptotic formulas of these numbers and polynomials. A more detailed proof of the results is provided so as to reach a bigger group of readers. Asymptotic formulas of Genocchi numbers and Euler numbers are obtained as special cases. Asymptotic formulas of the Apostol-Euler numbers and Apostol-Euler polynomials are also derived. The results in this paper will complete the results of [15] as the latter considered only the Apostol-Bernoulli and Apostol-Euler polynomials. Moreover, the results can be used as check formulas of those in [15].

2. Asymptotic Approximations

Fourier series of the Apostol-Genocchi polynomials in terms of the poles in T_λ is given in the following theorem.

Theorem 2.1. *([2], [13]) Let* $\lambda \in \mathbb{C}\backslash\{0\}$*. For* $n \geq 1$, $0 \leq x \leq 1$,

$$\frac{G_n(x;\lambda)}{n!} = \frac{2}{\lambda^x} \sum_{k\in\mathbb{Z}} \frac{e^{(2k+1)\pi i x}}{[(2k+1)\pi i - \log \lambda]^n}, \tag{2.1}$$

where the logarithm is taken to be the principal branch.

Taking $x = 0$ in (2.1) gives the Fourier series of the Apostol-Genocchi numbers given by

$$\frac{G_n(0;\lambda)}{n!} = 2\sum_{k\in\mathbb{Z}} \frac{1}{[(2k+1)\pi i - \log\lambda]^n}, \tag{2.2}$$

where the logarithm is taken to be the principal branch.

Proceeding as in [15], ordering of the poles of the generating function (1.1) is done in the following lemma.

Lemma 2.2. *Let* $u_k = (2k+1)\pi i - \log\lambda$ *with* $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}\backslash\{0\}$ *and* $\gamma = (\log\lambda)/2\pi i$, *where the logarithm is taken to be the principal branch.*

a) If $\mathfrak{Im}\ \lambda > 0$ *then* $0 < \mathfrak{Re}\ \gamma < \frac{1}{2}$ *and for* $k \geq 1$,

$$|u_0| < |u_{-1}| < |u_1| < |u_{-2}| < |u_2| < \cdots < |u_{-k}| < |u_k| < \cdots \tag{2.3}$$

b) If $\mathfrak{Im}\ \lambda < 0$ *then* $-\frac{1}{2} < \mathfrak{Re}\ \gamma < 0$ *and for* $k \geq 1$,

$$\begin{aligned} &|u_{-1}| < |u_0| < |u_{-2}| < |u_1| < |u_{-3}| < \cdots \\ &\qquad < |u_{-k}| < |u_{k-1}| < |u_{-(k+1)}| < |u_k| < \cdots . \end{aligned} \tag{2.4}$$

c) If $\lambda > 0$ *(positive real number), then* $\mathfrak{Re}\ \gamma = 0$, *and for* $k \geq 1$,

$$\begin{aligned} &|u_0| = |u_{-1}| < |u_1| = |u_{-2}| < |u_2| < \cdots \\ &\qquad < |u_{-k}| < |u_k| = |u_{-(k+1)}| < |u_{k+1}| < \cdots . \end{aligned} \tag{2.5}$$

d) If $\lambda < 0$ *(negative real number), then* $\mathfrak{Re}\ \gamma = \frac{1}{2}$, *and for* $k \geq 1$,

$$|u_0| < |u_1| = |u_{-1}| < |u_2| = |u_{-2}| < \cdots < |u_k| = |u_{-k}| < |u_{k+1}| < \cdots . \tag{2.6}$$

Moreover, $|u_k| \geq 2\pi(|k| - 1)$ *if* $|k| \geq 1$.

Proof. With the logarithm taken to be the principal branch, γ (as a function of λ) maps $\lambda \in \mathbb{C}\backslash\{0\}$ to the strip $-\frac{1}{2} < \mathfrak{Re}\ \gamma \leq \frac{1}{2}$ (see [15]). To see this write

$$\gamma = \frac{\theta}{2\pi} - i\frac{\ln|\lambda|}{2\pi},$$

from which we have

$$\mathfrak{Re}\ \gamma = \frac{\theta}{2\pi} \text{ and } \mathfrak{Im}\ \gamma = -\frac{\ln|\lambda|}{2\pi}.$$

With $-\pi < \theta \leq \pi$,

$$\frac{-\pi}{2\pi} \leq \mathfrak{Re}\ \gamma = \frac{\theta}{2\pi} \leq \frac{\pi}{2\pi} \Rightarrow \frac{-1}{2} < \mathfrak{Re}\ \gamma \leq \frac{1}{2},$$

where $\mathfrak{Re}\ \gamma = 0$ when $\lambda > 0$ and $\mathfrak{Re}\ \gamma = \frac{1}{2}$ when $\lambda < 0$.

If $\mathfrak{Im}\ \lambda > 0$, then $0 < \theta < \pi$, hence $0 < \mathfrak{Re}\ \gamma < \frac{1}{2}$. If $\mathfrak{Im}\ \lambda < 0$, then $-\pi < \theta < 0$, hence $-\frac{1}{2} < \mathfrak{Re}\ \gamma < 0$.

To verify the chains in (2.3), (2.4), (2.5), (2.6), let $x = \mathfrak{Re}\ \gamma$ and $y = \mathfrak{Im}\ \gamma$. Then for $k \in \mathbb{Z}$,

$$u_k = 2\pi\sqrt{\left(k + \frac{1}{2} - x\right)^2 + y^2}.$$

a) If $\mathfrak{Im}\ \lambda > 0$, then $0 < x < \frac{1}{2}$ and

$$|u_0| = 2\pi\sqrt{\left(\frac{1}{2} - x\right)^2 + y^2}$$

$$|u_1| = 2\pi\sqrt{\left(\frac{3}{2} - x\right)^2 + y^2}$$

$$|u_2| = 2\pi\sqrt{\left(\frac{5}{2} - x\right)^2 + y^2}$$

$$|u_{-1}| = 2\pi\sqrt{\left(-\frac{1}{2} - x\right)^2 + y^2} = 2\pi\sqrt{\left(\frac{1}{2} + x\right)^2 + y^2}$$

$$|u_{-2}| = 2\pi\sqrt{\left(-\frac{3}{2} - x\right)^2 + y^2} = 2\pi\sqrt{\left(\frac{3}{2} + x\right)^2 + y^2}$$

$$|u_{-3}| = 2\pi\sqrt{\left(-\frac{5}{2} - x\right)^2 + y^2} = 2\pi\sqrt{\left(\frac{5}{2} + x\right)^2 + y^2}$$

$$|u_3| = 2\pi\sqrt{\left(\frac{7}{2} - x\right)^2 + y^2}$$

$$\cdots$$

From which one can see that the order of magnitude of u_k, $k \in \mathbb{Z}$ given in (2.3) holds.

b) The second case can be derived similarly.

The last two cases are belonging to the case $\mathfrak{Im}\ \lambda = 0$. This means that λ is a real number which is either positive or negative but not zero. Hence the cases c and d.

c) If $\lambda > 0$, then $\mathfrak{Re}\ \gamma = 0$. For $k \geq 0$,

$$|u_k| = 2\pi\sqrt{\left(k + \frac{1}{2}\right)^2 + y^2}.$$

In particular,

$$|u_0| = 2\pi\sqrt{\left(\frac{1}{2}\right)^2 + y^2}$$

$$|u_1| = 2\pi\sqrt{\left(1+\frac{1}{2}\right)^2 + y^2}$$

$$|u_{-1}| = 2\pi\sqrt{\left(-1+\frac{1}{2}\right)^2 + y^2}$$

$$|u_2| = 2\pi\sqrt{\left(2+\frac{1}{2}\right)^2 + y^2}$$

$$|u_{-2}| = 2\pi\sqrt{\left(-2+\frac{1}{2}\right)^2 + y^2}$$

$$|u_3| = 2\pi\sqrt{\left(3+\frac{1}{2}\right)^2 + y^2}$$

From which we have the chain

$$|u_0| = |u_{-1}| < |u_1| = |u_{-2}| < |u_2| < \cdots \\ < |u_k| = |u_{-(k+1)}| < |u_{k+1}| < \cdots ,$$

which is exactly (2.5).

d) If $\lambda < 0$, $\theta = \pi$, hence $x = \frac{1}{2}$. For $k \geq 0$,

$$|u_k| = 2\pi\sqrt{k^2 + y^2} = |u_{-k}|,$$

from which it can be observed easily that

$$|u_0| < |u_1| = |u_{-1}| < |u_2| = |u_{-2}| < |u_3| = |u_{-3}| \\ < \cdots < |u_k| = |u_{-k}| < \cdots ,$$

which is exactly the chain in (2.6).

Moreover,

$$\begin{aligned} |u_k| &= 2\pi\left|k + \frac{1}{2} - \gamma\right| \\ &= 2\pi\sqrt{\left(k+\frac{1}{2} - x\right)^2 + y^2} \\ &\geq 2\pi\sqrt{\left(k+\frac{1}{2} - x\right)^2} \end{aligned}$$

$$
\begin{aligned}
&= 2\pi \left|k + \frac{1}{2} - x\right|, \quad \text{with } -\frac{1}{2} \le x \le \frac{1}{2} \\
&= 2\pi \left|k - \left(x - \frac{1}{2}\right)\right| \\
&\ge 2\pi \left(|k| - \left|x - \frac{1}{2}\right|\right) \\
&\ge 2\pi \left(|k| - \left|\frac{1}{2} - x\right|\right) \\
&\ge 2\pi \left(|k| - 1\right).
\end{aligned}
$$

An asymptotic expansion of the Apostol-Genocchi numbers $G_n(0;\lambda)$ is given in the next theorem.

Theorem 2.3. *Given $\lambda \in \mathbb{C}\backslash\{0\}$, let H be a finite subset of T_λ satisfying*

$$\max\{|u| \,:\, u \in H\} < \min\{|u| \,:\, u \in T_\lambda\backslash H\} := \nu.$$

For all integers $n \ge 2$,

$$\frac{G_n(0;\lambda)}{n!} = 2\sum_{u\in H} \frac{1}{u^n} + O(\nu^{-n}).$$

Proof. Write the series in (2.2) as $\sum_k \frac{1}{(u_k)^n}$. By Lemma 2.2 we can relabel the set of poles in increasing order of magnitude as

$$|\mu_0| \le |\mu_1| \le \cdots \le |\mu_M| \le \cdots.$$

Since $|\mu_k| \ge 2\pi(|k|-1)$, for $k \ge 2$, the series $\sum_k \frac{1}{(\mu_k)^n}$ is absolutely convergent for $n \ge 2$. For any $M > 2$, the tail of the series is

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} = \frac{1}{|\mu_{M+1}|^n} \sum_{k=M+1}^{\infty} \left|\frac{\mu_{M+1}}{\mu_k}\right|^n.$$

Since for $k > M+1$, $\left|\frac{\mu_{M+1}}{\mu_k}\right| \le 1$, we have $\left|\frac{\mu_{M+1}}{\mu_k}\right|^n \le \left|\frac{\mu_{M+1}}{\mu_k}\right|^2$ for $n \ge 2$.
Hence,

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \le \frac{1}{|\mu_{M+1}|^n} \sum_{k=M+1}^{\infty} \left|\frac{\mu_{M+1}}{\mu_k}\right|^2.$$

Let

$$C_{M,\lambda} = \sum_{k=M+1}^{\infty} \left|\frac{\mu_{M+1}}{\mu_k}\right|^2.$$

Then

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \le \frac{C_{M,\lambda}}{|\mu_{M+1}|^n}.$$

Consider $C_{M,\lambda}$:

$$\begin{aligned}
C_{M,\lambda} &= \sum_{k=M+1}^{\infty} \frac{|\mu_{M+1}|^2}{|\mu_k|^2} \\
&= |\mu_{M+1}|^2 \sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^2} \\
&= (2\pi)^2 \left| M+1+\frac{1}{2}-\gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{(2\pi)^2 \left| k+\frac{1}{2}-\gamma \right|^2} \\
&\leq \left| M+\frac{3}{2}-\gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{(|k|-1)^2} \\
&\leq 2\left| M+\frac{3}{2}-\gamma \right|^2 \sum_{l=0}^{\infty} \frac{1}{(M+l)^2} \\
&\leq 2\left| M+\frac{3}{2}-\gamma \right|^2 \left(\frac{1}{M^2} + \sum_{l=1}^{\infty} \frac{1}{(M+l)^2} \right).
\end{aligned}$$

With

$$\sum_{l=1}^{\infty} \frac{1}{(M+l)^2} \leq \int_1^{\infty} \frac{1}{(M+x)^2}\, dx = \frac{1}{M+1},$$

$$\begin{aligned}
C_{M,\lambda} &\leq 2\left| M+\frac{3}{2}-\gamma \right|^2 \left(\frac{1}{M^2} + \frac{1}{M+1} \right) \\
&= \frac{2\left| M+\frac{3}{2}-\gamma \right|^2}{M^2} + \frac{2\left| M+\frac{3}{2}-\gamma \right|^2}{M+1}.
\end{aligned}$$

Let

$$\epsilon_1 = \frac{\left| M+\frac{3}{2}-\gamma \right|^2}{M^2} \leq \left| \frac{5}{2}-\gamma \right|^2,$$

and

$$\epsilon_2 = \frac{\left| M+\frac{3}{2}-\gamma \right|}{M+1} \leq 1 + \frac{|1/2-\gamma|}{|M+1|} \leq 1 + \left| \frac{1}{2}-\gamma \right|.$$

Consequently,

$$\begin{aligned}
\frac{C_{m,\lambda}}{|\mu_{M+1}|^n} &\leq 2\frac{\epsilon_1}{|\mu_{M+1}|^n} + 2\frac{\epsilon_2}{|\mu_{M+1}|^n} \cdot \left| M+\frac{3}{2}-\gamma \right| \\
&\leq \frac{2\epsilon_1}{|\mu_{M+1}|^n} + \frac{2\epsilon_2 \cdot |M+3/2-\gamma|}{|\mu_{M+1}|^n},
\end{aligned}$$

where

$$|\mu_{M+1}| = \left|M + \frac{3}{2} - \gamma\right| = \sqrt{\left(M + \frac{3}{2} - \mathfrak{Re}\,\gamma\right)^2 + (\mathfrak{Im}\,\gamma)^2} \geq |M| - 2.$$

$$\begin{aligned} C_{M,\lambda} &\leq \frac{\epsilon_1}{2^{n-1}\pi^n\ |M + 3/2 - \gamma|^n} + \frac{\epsilon_2}{2^{n-1}\pi^n\ |M + 3/2 - \gamma|^{n-1}} \\ &\leq \frac{\epsilon_1}{2^{n-1}\pi^n\ (|M| - 2)^n} + \frac{\epsilon_2}{2^{n-1}\pi^n\ (|M| - 2)^n} \\ &\leq \frac{|5/2 - \gamma|^2}{2^{n-1}\pi^n\ (|M| - 2)^n} + \frac{1 + |1/2 - \gamma|}{2^{n-1}\pi^n\ (|M| - 2)^n} \\ &\leq \frac{|5/2 - \gamma|^2}{2^{n-1}\pi^n} + \frac{1 + |1/2 - \gamma|}{2^{n-1}\pi^n}. \end{aligned}$$

We can see that $C_{M,\lambda} \to 0$ as $n \to \infty$ for $|M| > 2$. Thus, the tail of the series,

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^n} \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, for fixed $M > 2$ and $n \gg 0$, $C_{M,\lambda}$ is bounded and independent of M. Hence, we can replace $C_{M,\lambda}$ by C_λ. This completes the proof of the theorem.

When $\lambda = 1, \log\lambda = 0$ and $u_k = (2k+1)\pi i,\ k \in \mathbb{Z}$. Take $H = \{\pi i, -\pi i\}$. Then $\nu = 3\pi$ and the ordinary Genocchi numbers $G_n = G_n(0; 1)$ satisfy

$$\frac{G_n}{2(n!)} = \frac{G_n(0;1)}{2(n!)} = \frac{1}{(\pi i)^n} + \frac{1}{(-\pi i)^n} + O((3\pi)^{-n}). \tag{2.7}$$

An approximation of $G_n(0; 1)$ is given by

$$\frac{G_n}{2(n!)} \approx \frac{1}{(\pi i)^n} + \frac{1}{(-\pi i)^n}. \tag{2.8}$$

For odd $n, n \geq 3$, it is known that $G_n = 0$ which is also true when we use (2.8). For even indices,

$$G_{2n} \approx \frac{(-1)^n 4((2n)!)}{\pi^{2n}}, \quad n \geq 2 \tag{2.9}$$

Taking $n = 4$,

$$G_8 \approx \frac{4(8!)}{\pi^8} \approx 16.99.$$

This value is very close to the exact value of G_8 which is 17.

It is proved in the next theorem that an asymptotic approximation of the Apostol-Genocchi polynomials can be obtained from its Fourier series (2.1) by choosing an appropriate subset of T_λ.

Theorem 2.4. *Given* $\lambda \in \mathbb{C}\backslash\{0\}$, *let* H *be a finite subset of* T_λ *satisfying*

$$\max\{|u| : u \in H\} < min\{|u| : u \in T_\lambda \setminus H\} := \nu.$$

For all integers $n \geq 2$, *we have, uniformly for* x *in a compact subset* K *of* $\mathbb{C}$,

$$\frac{G_n(x;\lambda)}{n!} = 2\sum_{u \in H} \frac{e^{ux}}{u^n} + O\left(\frac{e^{\nu|x|}}{\nu^n}\right),$$

where the constant implicit in the order term depends on λ, H *and* K. *Moreover, for* $n \gg 0$, *this constant can be made independent of* K, *equal to the constant for the Apostol-Genocchi numbers, corresponding to the case* $x = 0$.

Proof. From the generating function (1.1) we have

$$\frac{2ze^{(x+y)z}}{\lambda e^z + 1} = \sum_{n=0}^{\infty} G_n(x+y;\lambda)\frac{z^n}{n!}.$$

The LHS can be written

$$\begin{aligned}\frac{2ze^{xz}}{\lambda e^z + 1} \cdot e^{yz} &= \left(\sum_{n=0}^{\infty} G_n(x;\lambda)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(yz)^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} G_{n-k}(x;\lambda)\frac{z^{n-k}}{(n-k)!}\frac{(yz)^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}(x;\lambda)y^k\right)\frac{z^n}{n!},\end{aligned}$$

from which

$$G_n(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}G_{n-k}(x;\lambda)y^k.$$

For $z \in \mathbb{C}$, writing $z = 0 + z$ (here $y = z, x = 0$),

$$\begin{aligned}G_n(z;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}G_{n-k}(0,\lambda)z^k,\\ \frac{G_n(z;\lambda)}{n!} &= \sum_{k=0}^{n}\frac{G_{n-k}(0;\lambda)}{(n-k)!}\frac{z^k}{k!}\\ &= 2\sum_{k=0}^{n}\left(\sum_{u \in H}\frac{1}{u^{n-k}} + O(\nu^{-(n-k)})\right)\frac{z^k}{k!} \quad \text{(by Theorem 2.3)}\\ &= 2\sum_{k=0}^{n}\left(\sum_{u\in H}\frac{1}{u^{n-k}}\frac{z^k}{k!}\right) + \sum_{k=0}^{n} O(\nu^{-(n-k)})\frac{z^k}{k!},\end{aligned}$$

where the implicit constant c in the order term is that corresponding to $z = 0$ and only depends on H and λ. Note also that

$$\begin{aligned}\left|\sum_{k=0}^{n} O(\nu^{-n+k})\frac{z^k}{k!}\right| &\leq \sum_{k=0}^{n} c\nu^{-n+k}\frac{|z^k|}{k!}\\ &= c\nu^{-n}\sum_{k=0}^{n}\nu^k\frac{|z^k|}{k!}\\ &\leq cv^{-n}e_n(\nu|z|),\end{aligned}$$

where $e_n = \sum_{k=0}^{n} \frac{w^k}{k!}$.

To prove the theorem, it remains to show that

$$\frac{e_n^*(uz)}{u^n} = \frac{e^{uz} - e_n(uz)}{u^n}$$

is bounded.

Using MVT for Banach spaces (see also [15])

$$\begin{aligned}e_n^*(w) &= \frac{w^{n+1}}{(n+1)!} + \frac{w^{n+2}}{(n+2)!} + \cdots\\ &= \frac{w^{n+1}}{(n+1)!}\left\{1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots\right\},\end{aligned}$$

from which

$$\begin{aligned}|e_n^*(w)| &\leq \left|\frac{w^{n+1}}{(n+1)!}\right|\left|1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots\right|\\ &\leq \frac{|w|^{n+1}}{(n+1)!}e^{\mathfrak{Re}^+(w)},\end{aligned}$$

where $\mathfrak{Re}^+(w) = \max\{\mathfrak{Re}(w), 0\}$.

Since $|u| \leq \nu$, for all $u \in H$, we have

$$\begin{aligned}\frac{|e_n^*(uz)|}{|u^n|} &\leq \frac{e^{|uz|}|uz|^{n+1}}{|u^n|(n+1)!}\\ &= |u|e^{|uz|}\frac{|z^{n+1}|}{(n+1)!}\\ &< \nu e^{\nu|z|}\frac{|z|^{n+1}}{(n+1)!},\end{aligned}$$

so that

$$\left|\sum_{u\in H}\frac{e_n^*(uz)}{u^n}\right| \leq \sum_{u\,\in\,H}\frac{|e_n^*(uz)|}{|u^n|}$$

$$< \#H\nu e^{\nu|z|}\frac{|z|^{n+1}}{(n+1)!},$$

where $\#H =$ no. of elements in H.
We give the argument that

$$\#H\nu e^{\nu|z|}\frac{|z|^{n+1}}{(n+1)!} < ce^{\nu|z|}\nu^{-n}$$

if

$$\#H\frac{(\nu|z|)^{n+1}}{(n+1)!} < c,$$

which certainly holds for $n \gg 0$, uniformly for z in a compact subset $K \subset \mathbb{C}$.

Corollary 2.5. *Let K be an arbitrary compact subset of $\mathbb{C}$. The Genocchi polynomials satisfy uniformly on K the estimates*

$$\frac{G_{2n}(x)}{(2n)!} = \frac{(-1)^n\ 4\cos\pi x}{\pi^{2n}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right), \qquad n \geq 2,$$

$$\frac{G_{2n+1}(x)}{(2n+1)!} = \frac{(-1)^n\ 4\sin\pi x}{\pi^{2n+1}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right), \qquad n \geq 3,$$

where the implicit constant in the order term depends on the set K. Moreover, for $n \gg 0$, this constant can be made independent of K, equal to the constant for the Genocchi numbers, corresponding to the case $x = 0$.

Proof. The Genocchi polynomials correspond to the case $\lambda = 1$ so that $u_k = (2k+1)\pi i$, for $k \in \mathbb{Z}$. Thus, $T_1 = \{(2k+1)\pi i\ :\ k \in \mathbb{Z}\}$. Taking $H = \{(2k+1)\pi i \mid k = -1, 0\} = \{-\pi i, \pi i\}$, then $\nu = |3\pi i| = 3\pi$. From Theorem 2.4,

$$\frac{G_n(x;1)}{n!} = 2\sum_{u\in H}\frac{e^{ux}}{u^n} + O\left(\frac{e^{\nu|x|}}{\nu^n}\right)$$
$$= 2\left(\frac{e^{-\pi ix}}{(-\pi i)^n} + \frac{e^{\pi ix}}{(\pi i)^n}\right) + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right).$$

For even indices,

$$\frac{G_{2n}(x)}{(2n)!} = \frac{G_{2n}(x;1)}{(2n)!}$$
$$= 2\left(\frac{e^{-\pi ix}}{(\pi i)^{2n}} + \frac{e^{\pi ix}}{(\pi i)^{2n}}\right) + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{2n}}\right)$$
$$= \frac{4\cos\pi x}{(\pi i)^{2n}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{2n}}\right)$$

$$= \frac{(-1)^n \ 4\cos\pi x}{\pi^{2n}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right).$$

For odd indices,

$$\begin{aligned}
\frac{G_{2n+1}(x)}{(2n+1)!} &= \frac{G_{2n+1}(x;1)}{(2n+1)!} \\
&= 2\left(\frac{e^{-\pi i x}}{(-\pi i)^{2n+1}} + \frac{e^{\pi i x}}{(\pi i)^{2n+1}}\right) + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{2n+1}}\right) \\
&= 2\left(\frac{(-1)^n \ 2\sin\pi x}{(\pi)^{2n+1}}\right) + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{2n+1}}\right) \\
&= \frac{(-1)^n(4\sin\pi x)}{\pi^{2n+1}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right).
\end{aligned}$$

Notice the resemblance of the results in Corollary 2.5 and of (33) in [3]. Since, for $k = 2n$,

$$\cos(\pi x - \frac{k\pi}{2}) = \pm\cos\pi x = (-1)^n\cos\pi x,$$

(33) in [3] can be written as

$$\begin{aligned}
G_{2n}(x) &= \frac{4((2n)!)}{\pi^{2n}}\left[(-1)^n\cos\pi x + O(3^{-n})\right] \\
\frac{G_{2n}(x)}{(2n)!} &= \frac{(-1)^n 4\cos\pi x}{\pi^{2n}} + O\left(\frac{3^{-n}}{\pi^{2n}}\right) \\
&= \frac{(-1)^n 4\cos\pi x}{\pi^{2n}} + O\left(\frac{1}{(3\pi)^n}\right) \\
&= \frac{(-1)^n 4\cos\pi x}{\pi^{2n}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right), \quad \text{for } x \in K.
\end{aligned}$$

For odd k ($k = 2n+1$),

$$\cos\pi x - \frac{k\pi}{2} = (-1)^n\sin\pi x.$$

Then (33) in [3] can be written as

$$\begin{aligned}
G_{2n+1}(x) &= \frac{4((2n+1)!)}{\pi^{2n+1}}\left[(-1)^n\sin\pi x + O\left(3^{-(2n+1)}\right)\right] \\
\frac{G_{2n+1}(x)}{(2n+1)!} &= \frac{(-1)^n \ 4\sin\pi x}{\pi^{2n+1}} + O\left(\frac{3^{-(2n+1)}}{\pi^{2n+1}}\right) \\
&= \frac{(-1)^n \ 4\sin\pi x}{\pi^{2n+1}} + O\left(\frac{1}{(3\pi)^{2n+1}}\right)
\end{aligned}$$

$$= \frac{(-1)^n \; 4 \sin \pi x}{\pi^{2n+1}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{2n+1}}\right)$$

$$= \frac{(-1)^n \; 4 \sin \pi x}{\pi^{2n+1}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^{n}}\right).$$

Thus, the asymptotic formulas in Corollary 2.5 are equivalent to (33) in [3].

3. λ is a negative real number

When λ is a negative real number, writing $\lambda = -|\lambda|$, the generating function is given by

$$\frac{2te^{xt}}{-|\lambda|e^t + 1} = \sum_{n=0}^{\infty} G_n(x;\lambda)\frac{t^n}{n!}. \tag{3.1}$$

The poles of the generating function (3.1) is

$$T_{-|\lambda|} = \{2k\pi i - \log|\lambda| : k \in \mathbb{Z}\}.$$

The next theorem follows from Theorem 2.4.

Theorem 3.1. *Given that λ is a negative real number, let F be a finite subset of $T_{-|\lambda|}$ satisfying*

$$\max\{|a| \;:\; a \in F\} < \min\{|a| \;:\; a \in T_{-|\lambda|}\backslash F\} := \mu.$$

For all integers $n \geq 2$, we have, uniformly for x in a compact subset K of $\mathbb{C}$,

$$\frac{G_n(x;\lambda)}{n!} = 2\sum_{a\in F}\frac{e^{ax}}{a^n} + O\left(\frac{e^{\mu|x|}}{\mu^n}\right), \tag{3.2}$$

where the constant implicit in the order term depends on λ, F and K.

The Apostol-Genocchi numbers $G_n(0;-1)$ corresponding to the case $\lambda = -1$ has generating function

$$\frac{2t}{-e^t + 1} = \sum_{n=0}^{\infty} G_n(0;-1)\frac{t^n}{n!}, \tag{3.3}$$

The set of poles is $T_{-1} = \{2k\pi i : k \in \mathbb{Z}\backslash\{0\}\}$. An asymptotic formula for $G_n(0;-1)$ is given in the following theorem.

Theorem 3.2. *For $n \geq 3$, the Apostol-Genocchi numbers $G_n(0;-1)$ satisfy*

$$\frac{G_n(0;-1)}{n!} = 2\left(\frac{1}{(-2\pi i)^n} + \frac{1}{(2\pi i)^n}\right) + O\left((4\pi)^{-n}\right). \tag{3.4}$$

In particular,

$$\frac{G_{2n}(0;-1)}{(2n)!} = \frac{(-1)^n 4}{(2\pi)^{2n}} + O\left((4\pi)^{-2n}\right), \qquad n \geq 2. \tag{3.5}$$

Proof. Taking $x = 0$, $F = \{-2\pi i, 2\pi i\}$ in Theorem 3.1, then $\mu = 4\pi$. Hence,

$$\frac{-\frac{1}{2}G_n(0;-1)}{n!} = -\left(\frac{1}{(-2\pi i)^n} + \frac{1}{(2\pi i)^n}\right) + O\left((4\pi)^{-n}\right), \tag{3.6}$$

from which (3.4) follows.

For $(n \geq 3)$, (3.6) gives $G_{2n+1}(0;-1) \approx 0$. Indeed $G_{2n+1}(0;-1) = 0$, $\forall n \geq 1$.

For $n \geq 2$,

$$\frac{G_{2n}(0;-1)}{(2n)!} = 4\left(\frac{(-1)^n}{(2\pi)^{2n}}\right) + O\left((4\pi)^{-2n}\right). \tag{3.7}$$

From (3.7) we have the approximation

$$G_{2n}(0;-1) \approx \frac{(-1)^n\ 4(2n)!}{(2\pi)^{2n}}. \tag{3.8}$$

Taking $n = 4$,

$$G_8(0;-1) = \frac{4(8!)}{(2\pi)^8} \approx .06638.$$

The actual value of $G_8(0;-1) = -2B_8 = \frac{1}{15} \approx .06667$.

The Apostol-Genocchi polynomials, $G_n(x;-1)$ correspond to the case $\lambda = -1$. These polynomials have generating function

$$\frac{2te^{xt}}{-e^t + 1} = \sum_{n=0}^{\infty} G_n(x;-1)\frac{t^n}{n!}. \tag{3.9}$$

We will prove the following theorem.

Theorem 3.3. *Let K be a compact subset of $\mathbb{C}$. The Apostol-Genocchi polynomials $G_n(x;-1)$ satisfy uniformly on K the estimates*

$$\frac{G_{2n}(x;-1)}{(2n)!} = \frac{(-1)^n 4\cos 2\pi x}{(2\pi)^{2n}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right), \tag{3.10}$$

$$\frac{G_{2n+1}(x;-1)}{(2n+1)!} = \frac{(-1)^n 4\sin 2\pi x}{(2\pi)^{2n+1}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right), \tag{3.11}$$

where the implicit constant in the order term depends on the set K. Moreover, for $n \gg 0$, this constant can be made independent of K, equal to the constant for the Apostol-Genocchi numbers $G_n(0;-1)$ corresponding to the case $x = 0$.

Proof. Taking $F = \{-2\pi i, 2\pi i\}$, then $\mu = 4\pi$. Hence, it follows from Theorem 3.1 that

$$\frac{\frac{-1}{2}G_n(x;-1)}{n!} = -\frac{e^{2\pi ix}}{(2\pi i)^n} - \frac{e^{-2\pi ix}}{(-2\pi i)^n} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right). \tag{3.12}$$

For odd indices,

$$\frac{\frac{-1}{2}G_{2n+1}(x;-1)}{(2n+1)!} = -\left(\frac{e^{2\pi ix}}{(2\pi i)^{2n+1}} + \frac{e^{-2\pi ix}}{(-2\pi i)^{2n+1}}\right) + O\left(\frac{e^{4\pi|x|}}{(4\pi)^{2n+1}}\right) \tag{3.13}$$

$$\frac{G_{2n+1}(x;-1)}{(2n+1)!} = \frac{(-1)^n 4\sin 2\pi x}{(2\pi)^{2n+1}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right). \tag{3.14}$$

For even indices,

$$\frac{G_{2n}(x;-1)}{(2n)!} = 2\left(\frac{e^{2\pi ix}}{(2\pi i)^{2n}} + \frac{e^{-2\pi ix}}{(-2\pi i)^{2n}}\right) + O\left(\frac{e^{4\pi|x|}}{(4\pi)^{2n}}\right) \tag{3.15}$$

$$= \frac{(-1)^n 4\cos 2\pi x}{(2\pi)^{2n}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right). \tag{3.16}$$

4. Apostol-Euler Numbers and Polynomials

The Apostol-Euler numbers are defined by the generating function

$$\frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(0;\lambda)\frac{t^n}{n!}. \tag{4.1}$$

Multiplying both sides of (4.1) by t gives

$$\sum_{n=0}^{\infty} G_n(0;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(n+1)E_n(0;\lambda)\frac{t^{n+1}}{(n+1)!},$$

from which we have, for $n \geq 1$

$$E_{n-1}(0;\lambda) = \frac{G_n(0;\lambda)}{n} = (n-1)!\frac{G_n(0;\lambda)}{n!}. \tag{4.2}$$

Thus, from Theorem 2.3,

$$E_{n-1}(0;\lambda) = 2(n-1)!\left[\sum_{n=0}^{\infty}\frac{1}{u^n} + O\left(\nu^{-n}\right)\right], \tag{4.3}$$

where $F \subseteq T_\lambda = \{(2k+1)\pi i - \log\lambda \mid k \in \mathbb{Z}\}$ and F satisfies

$$\max\{|u| \ : \ u \in F\} < \min\{|u| \ : \ u \in T_\lambda \backslash F\} = \nu.$$

For odd n, say $n = 2k+1$, from (4.2), we have

$$E_{2k}(0;\lambda) = \frac{G_{2k+1}(0;\lambda)}{2k+1}, \tag{4.4}$$

while for even n, say $n = 2k$,

$$E_{2k-1}(0;\lambda) = \frac{G_{2k}(0;\lambda)}{2k}. \tag{4.5}$$

The case $\lambda = 1$, corresponds to the Euler numbers E_n. From (4.2),

$$E_{n-1} = \frac{G_n}{n}. \tag{4.6}$$

Since $G_n = 0$ for all odd $n \geq 3$, $E_{2k} = 0$ for $k \geq 1$.

For odd indices, using (2.9) we have

$$E_{2n-1} = (2n-1)!\frac{G_{2n}}{(2n)!} = (2n-1)!\left(\frac{(-1)^n(4)}{\pi^{2n}} + O\left((3\pi)^{-n}\right)\right), \quad n \geq 2. \tag{4.7}$$

Taking $n = 2$,

$$E_3 \approx 3!\left(\frac{4}{\pi^4}\right) = \frac{24}{\pi^4} = 0.24638.$$

The Actual value of $E_3 = 0.25$.

The Apostol-Euler Polynomials $E_n(x;\lambda)$ are defined by the generating function

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(x;\lambda)\frac{t^n}{n!}, \tag{4.8}$$

which can be written

$$\sum_{n=0}^{\infty} \frac{G_n(x;\lambda)t^n}{n!} = \sum_{n=0}^{\infty}(n+1)E_n(x;\lambda)\frac{t^{n+1}}{(n+1)!}. \tag{4.9}$$

Thus,

$$E_{n-1}(x;\lambda) = \frac{G_n(x;\lambda)}{n}. \tag{4.10}$$

From Theorem 2.4,

$$\begin{aligned} E_{n-1}(x;\lambda) &= \frac{G_n(x;\lambda)}{n} \cdot \frac{(n-1)!}{(n-1)!} \\ &= (n-1)!\frac{G_n(x;\lambda)}{n!} \\ &= (n-1)!\left(2\sum_{u\in F}\frac{e^{uz}}{u^n} + O\left(\frac{e^{\nu|x|}}{\nu^n}\right)\right). \end{aligned}$$

Hence, we have the following corollary.

Corollary 4.1. *Given $\lambda \in \mathbb{C}\backslash\{0\}$, let F be a finite subset of T_λ satisfying*

$$\max\{|u| \,:\, u \in F\} < \min\{|u| \,:\, u \in T_\lambda \backslash F\} = \nu.$$

Let K be an arbitrary compact subset of $\mathbb{C}$. The Apostol-Euler polynomials satisfy uniformly on K the estimates,

$$\frac{E_{n-1}(x;\lambda)}{(n-1)!} = 2\sum_{u\in F} \frac{e^{ux}}{u^n} + O\left(\frac{e^{\nu|x|}}{\nu^n}\right),$$

where the constant implicit in the order term depends on λ, F and K. Moreover, for $n \gg 0$, this constant can be made independent of K, equal to the constant for the Apostol-Euler numbers, corresponding to the case $x = 0$.

It follows from Corollary 2.5 that the Euler polynomials which correspond to $\lambda = 1$, satisfy, uniformly on a compact subset K of $\mathbb{C}$ the estimates

$$\frac{E_{2n-1}(x)}{(2n-1)!} = \frac{G_{2n}(x)}{(2n)!} = \frac{(-1)^n 4\cos\pi x}{\pi^{2n}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right), \tag{4.11}$$

$$\frac{E_{2n}(x)}{(2n)!} = \frac{G_{2n+1}(x)}{(2n+1)!} = \frac{(-1)^n 4\sin\pi x}{\pi^{2n+1}} + O\left(\frac{e^{3\pi|x|}}{(3\pi)^n}\right), \tag{4.12}$$

as $n \to \infty$, for $n \geq 1$.

The Apostol-Euler polynomials $E_{n-1}(x;-1)$ correspond to the special case $\lambda = -1$. From (4.10),

$$E_{n-1}(x;-1) = \frac{G_n(x;-1)}{n}. \tag{4.13}$$

It follows from (3.10) and (3.11), respectively that

$$\frac{E_{2n}(x;-1)}{(2n)!} = \frac{(-1)^n 4\sin 2\pi x}{(2\pi)^{2n+1}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right), \tag{4.14}$$

$$\frac{E_{2n-1}(x;-1)}{(2n-1)!} = \frac{(-1)^n 4\cos 2\pi x}{(2\pi)^{2n}} + O\left(\frac{e^{4\pi|x|}}{(4\pi)^n}\right), \tag{4.15}$$

on a compact subset K of $\mathbb{C}$.

5. Conclusion

Asymptotic approximations of the Apostol-Genocchi numbers and polynomials were obtained for values of the parameter λ in $\mathbb{C}\backslash\{0\}$. Unlike in [15] we have considered explicitly the case when λ is negative and obtained corresponding asymptotic formulas.

Moreover, the asymptotic formulas for $\lambda = 1$ are explicitly obtained for each of the Apostol-Genocchi and Apostol-Euler numbers and polynomials. The tangent polynomials [8] have generating function very similar to that of the Apostol-Genocchi polynomials. The author recommends finding Fourier expansion and asymptotic approximations of these polynomials.

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