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On 2-Resolving Sets in the Join and Corona of Graphs†

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Abstract. Let G be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set in G if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. If G has a 2-resolving set, we denote the least size of a 2-resolving set by $\dim_2(G)$, the 2-metric dimension of G. A 2-resolving set of size $\dim_2(G)$ is called a 2-metric basis for G. This study deals with the concept of 2-resolving set of a graph. It characterizes the 2-resolving set in the join and corona of graphs and determine the exact values of the 2-metric dimension of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: 2-resolving set, 2-metric dimension, 2-metric basis, corona

1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [10], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [4] where metric generators were called resolving sets. In [6], Monsanto, Acal and Rara discussed the strong resolving dominating sets in the join and corona of graphs while in [5], Monsanto and Rara discussed the resolving restrained domination in graphs.

Bailey and Yero in [1] demonstrated a construction of error-correcting codes from graphs by means of k-resolving sets, and present a decoding algorithm which makes use of covering designs.

The distance between two vertices u and v of a graph is the length of a shortest path

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between u and v, and we denote this by $d_G(u,v)$. In recent years, much attention has been paid to the *metric dimension* of graphs: this is the smallest size of a subset of vertices (called a *resolving set*) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by $\dim(G)$.

According to the paper of Saenpholphat et al. [9], for an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, the k-vector (ordered k-tuple)

$$r(v/W) = (d_G(v,w_1), d_G(v,w_2), ..., d_G(v,w_k))$$

is referred to as the *(metric) representation of* v *with respect to* W. The set W is called a *resolving set* for G if distinct vertices have distinct representation with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G.

In the paper of Bailey et al.[1], an ordered set of vertices $W = \{w_1, ..., w_l\}$ is a k-*resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r(u/W)$ and $r(v/W)$ of u and v, respectively differ in at least k positions. If $k = 1$, then the k-resolving set is called a *resolving set* for G. If G has a k-resolving set, the minimum cardinality $\dim_k(G)$ is called the k-*metric dimension* of G.

In this paper, the concept of 2-resolving set in the join and corona of graphs is discussed.

2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [3].

Remark 1. Let G be any connected graph of order $n \geq 2$. Then the vertex set of G is a 2-resolving set in G. Hence, $2 \leq \dim_2(G) \leq n$.

Proposition 1.[7] $\dim_2(G) = 2$ if and only if $G \cong P_n, n \geq 2$.

Proposition 2. For any complete graph K_n of order $n \geq 2$, $\dim_2(K_n) = n$.

Theorem 1. Every 2-resolving set in a connected graph G is a resolving set in G. Hence, $\dim(G) \leq \dim_2(G)$.

Remark 2. A superset of a 2-resolving set is a 2-resolving set.

Remark 3. Let $S \subseteq V(G)$. For any pair of vertices $x, y \in S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions. Hence, to prove that S is a 2-resolving set in G, we only need to show that for every pair of vertices $x, y \in V(G)$ where $x \in S$ and $y \in V(G)\backslash S$ or both $x, y \in V(G)\backslash S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions.

3. 2-Resolving Sets in the Join of Graphs

Definition 1.[2] The *join* $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Note that the star $K_{1,n}$ can be expressed as the join of the trivial graph K_1 and the empty graph $\overline{K}_n$ of order n, that is, $K_{1,n} = K_1 + \overline{K}_n$. The graphs $F_n = K_1 + P_n$ and $W_n = K_1 + C_n$ of orders $n + 1$ are called *fan* and *wheel*, respectively.

Definition 2. Let $G = ((V(G), E(G))$ be a connected graph. The open *neighborhood* $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element u of $N_G(v)$ is called a *neighbor* of v.

The notation $x \in V(G)\backslash S$ means that $x \in V(G)$ but not in S.

Definition 3. Let G be any nontrivial connected graph and $S \subseteq V(G)$. Then S is a 2-*locating set* of G if $\forall x, y \in V(G)$, $x \neq y$, the following are satisfied:
(i) If $x, y \in V(G)\backslash S$, then $\exists w, z \in S$, $w \neq z$ such that either:

(a) $w, z \in (N_G(x))\backslash N_G(y)$, or

(b) $w, z \in (N_G(y))\backslash N_G(x)$, or

(c) $w \in (N_G(x))\backslash N_G(y)$ and $z \in (N_G(y))\backslash N_G(x)$.

(ii) If $x \in S$, $y \in V(G)\backslash S$, then $\exists p \in (N_G(x) \cap S)\backslash N_G(y)$ or $p \in (N_G(y) \cap S)\backslash N_G(x)$.
The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as ln_2-set of G.

Example 1. The sets $S_1 = \{c, d, e, f\}$ and $S_2 = \{a, b, c, f\}$ are 2-locating sets in G in Figure1. Moreover, S_1 and S_2 are ln_2-set in G. Thus, $ln_2(G) = |S_1| = |S_2| = 4$.

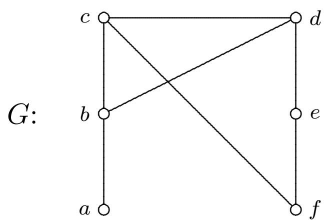


Figure 1: A graph G with $ln_2 = 4$

Remark 4. Every 2-locating set in G is a 2-resolving set in G. However, a 2-resolving set in G need not be a 2-locating set in G. Thus, $\dim_2(G) \leq ln_2(G)$.

Example 2. Let $P_6 = [v_1, v_2, ..., v_6]$ be a path of order 6 and $S_1 = \{v_1, v_3, v_5, v_6\}$. Then S_1 is both 2-locating and 2-resolving set in P_6. On the other hand, $S_2 = \{v_2, v_4, v_6\}$ is a 2-resolving set but not 2-locating.

Example 3. For all $n \geq 2$, $ln_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Example 4. For all $n \geq 5$, $ln_2(C_n) = \left\lceil \frac{n}{2} \right\rceil$ and $ln_2(C_3) = 3$, $ln_2(C_4) = 4$.

Definition 4. Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a *strictly 2-locating* (*strictly 1-locating*) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$), $\forall y \in V(G)$. The *strictly 2-locating* (*strictly 1-locating*) *number* of G, denoted by $sln_2(G)$ ($sln_1(G)$), is the smallest cardinality of a strictly 2-locating (strictly 1-locating) set in G. A strictly 2-locating (strictly 1-locating) set in G of cardinality $sln_2(G)$ ($sln_1(G)$) is referred to as sln_2-*set* (sln_1-*set*) in G.

Example 5. The set $S_2 = \{a, b, c, f\}$ is a strictly 1-locating set in G in Figure 1. Moreover, S_2 is a sln_1-set in G. Thus, $sln_1(G) = 4$.

Example 6. The set $S = \{u_1, u_3, u_5, u_7\}$ is a strictly 2-locating set in P_7 in Figure 2. Moreover, S is a sln_2-set in P_7. Thus, $sln_2(P_7) = 4$.

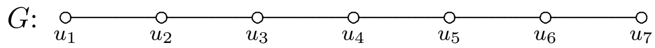


Figure 2: A graph P_7 with $sln_2 = 4$

Example 7. For all $n \geq 4$, $sln_1(P_n) = \begin{cases} \frac{n}{2} + 1, n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, n \text{ is odd} \end{cases}$

Example 8. For all $n \geq 5$, $sln_1(C_n) = \begin{cases} \frac{n}{2}, n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, n \text{ is odd} \end{cases}$

Example 9. For all $n \geq 6$, $sln_2(P_n) = \begin{cases} \frac{n}{2} + 1, n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, n \text{ is odd} \end{cases}$

Example 10. For all $n \geq 7$, $sln_2(C_n) = \begin{cases} \frac{n}{2}, n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, n \text{ is odd} \end{cases}$

Remark 5. Every strictly 2-locating set in G is strictly 1-locating. However, strictly 1-locating set in G need not be a strictly 2-locating set in G.

Theorem 2. A proper subset S of $V(K_1 + \overline{K}_n)$ is a 2-resolving set in $K_1 + \overline{K}_n$ if and only if $S = V(\overline{K}_n)$, $\forall n \geq 2$.

Proof. Let S be a proper subset of $V(K_1 + \overline{K}_n)$. Suppose S is a 2-resolving set in $K_1 + \overline{K}_n$ and suppose $\exists x \in V(\overline{K_n})\backslash S$. Then $r(x/S)$ and $r(y/S)$ differ in at most one position for each $y \in V(\overline{K_n})$. Thus, $S = V(\overline{K_n})$.

Conversely, let $S = V(\overline{K_n})$ and $x \in V(K_1)$. Then, $r(x/S) = (1, ..., 1)$ and $r(y/S) = (..., 2, 2, 2, 0, 2, ...)$ for each $y \in V(\overline{K_n})$. Thus, $r(x/S)$ and $r(y/S)$ differ in at least two positions. Therefore S is a 2-resolving set of $K_1 + \overline{K}_n$.

□

Corollary 1. $\dim_2(K_1 + \overline{K}_n) = |V(\overline{K}_n)|$.

Theorem 3. Let G be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and S is strictly 2-locating set of G or $S = \{v\} \cup T$, where T is a strictly 1-locating set in G.

Proof. Let $S \subseteq V(K_1 + G)$ be a 2-resolving set of $K_1 + G$. If $v \notin S$, then $S \subseteq V(G)$ is 2-locating set in G. Suppose there exists $y \in V(G)$ such that $|N_G(y) \cap S| > |S| - 2$. Then $r(v/S)$ and $r(y/S)$ differ in at most one position, contrary to our assumption that S is a 2-resolving set in $K_1 + G$. Hence, S is a strictly 2-locating set of $K_1 + G$. Next, suppose that $S = T \cup \{v\}$, where $T = V(G) \cap S$. Then $\emptyset \neq T \subseteq V(G)$. Thus, T is a 2-locating set in G. Since S is a 2-resolving set and $v \in S$, T is strictly 1-locating set in G.

For the converse, let $x, y \in V(K_1 + G)$. First, assume that $v \notin S$ and S is a strictly 2-locating set in G. Consider the following cases.

Case 1. $x, y \in S$

By Remark 3, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions, the x^{th} and y^{th} positions.

Case 2. $x, y \in V(G)\backslash S$

By Definition 3(i), $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the z^{th} and w^{th} positions, for some distinct vertices $z, w \in S$.

Case 3. $x \in S$, $y \in V(G)\backslash S$

By Definition 3(ii), there exists $z \in (N_G(x) \cap S)\backslash N_G(y)$ or $z \in (N_G(y) \cap S)\backslash N_G(x)$. Hence, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the x^{th} and z^{th} positions.

Case 4. $x = v$, $y \in V(G)$.

By Definition 4, $\exists u, w \in S\backslash N_G(y)$, $u \neq w$. Thus, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the u^{th} and w^{th} positions.

Next, suppose $S = \{v\} \cup T$ where T is strictly 1-locating set in G. Consider the following cases.

Case 1. $x, y \in S$

By Remark 3, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions, the x^{th} and y^{th} positions.

Case 2. $x, y \in V(K_1 + G)\backslash S$.

Then $x, y \in V(G)\backslash T$. By Definition 3(i), $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions.

Case 3. $x = v$, $y \in V(G)$.

By Definition 4, $\exists z \in T\backslash N_G(y)$. Thus, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the x^{th}

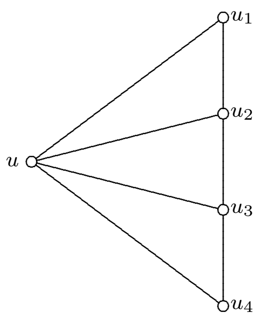
and z^{th} positions.

Case 4. $x \in T$, $y \in V(G) \backslash T$.

Since T is 2-locating set in G, $r_G(x/T)$ and $r_G(y/T)$ differ in at least 2 positions. Hence, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ also in at least 2 positions.

Therefore, S is a 2-resolving set in $K_1 + G$. □

The sets $\{u, u_1, u_3, u_4\}$ and $\{v, v_1, v_3, v_5\}$ are 2-resolving sets in the join $\langle u \rangle + P_5$ and $\langle v \rangle + C_6$, respectively, in Figure 3.



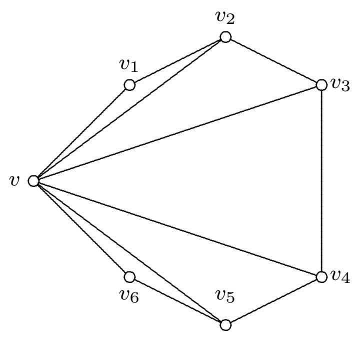


Figure 3: The join $\{u\} + P_5$ with $\dim_2(\{u\} + P_5) = 4$ and the join $\{v\} + C_6$ with $\dim_2(\{v\} + C_6) = 4$

The next result follows immediately from Theorem 3.

Corollary 2. $\dim_2(K_1 + G) = \min\{sln_2(G), sln_1(G) + 1\}$.

Example 11.[8] For any integer $n \geq 6$, $\dim_2(F_{1,n}) = \left\lceil \dfrac{(n+1)}{2} \right\rceil = sln_2(P_n)$.

Example 12.[8] For any $n \geq 7$, $\dim_2(W_{1,n}) = \left\lceil \dfrac{n}{2} \right\rceil = sln_2(C_n)$.

Theorem 4. Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G + H$ if and only if $S_G = V(H) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H respectively where S_G or S_H is strictly 2-locating set or S_G and S_H are strictly 1-locating sets.

Proof. Suppose S is a proper subset of $V(G + H)$. Let S be a 2-resolving set in $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Then $S = S_G \cup S_H$. Suppose $S_G = \emptyset$. Then $S = S_H$. Let $x, y \in V(G)$, $x \neq y$. Then $r_{G+H}(x/S) = r_{G+H}(y/S) = (1, ..., 1)$. A contradiction to the assumption of S. Thus, $S_G \neq \emptyset$. Similarly, $S_H \neq \emptyset$.

Next, suppose S_G or S_H, say S_G is not 2-locating set in G. Then there exist $x, y \in V(G)$, $x \neq y$ such that $r_G(x/S_G)$ and $r_G(y/S_G)$ differ in at most 1 position. Hence, $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ also in at most one position. Thus, S is not 2-resolving set in $G + H$, contrary to our assumption. Therefore S_G and S_H are 2-locating sets in G

and H, respectively. Now, suppose that both S_G and S_H are not strictly 2-locating sets. Then $|N_G(x) \cap S_G| > |S_G| - 2$, $\forall x \in V(G)$ and $|N_H(y) \cap S_H| > |S_H| - 2$, $\forall y \in V(H)$. Hence either $N_G(x) \cap S_G = S_G$ or $\exists p \in S_G \backslash N_G(x)$ and either $N_H(y) \cap S_H = S_H$ or $\exists q \in S_H \backslash N_H(y)$. Since S is a 2-locating set, $\exists p \in S_G \backslash N_G(x)$ and $\exists q \in S_H \backslash N_H(y)$. Thus, S_G and S_H are both strictly 1-locating sets.

For the converse, suppose that S_G and S_H are 2-locating sets in G and H, respectively where S_G or S_H is strictly 2-locating set or S_G and S_H are both strictly 1-locating sets. Let $x, y \in V(G+H)$ with $x \neq y$. If $x, y \in V(G)$, then $r_G(x/S_G)$ and $r_G(y/S_G)$ differ in at least 2 positions since S_G is a 2-locating set in G. Hence, $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ also differ in at least 2 positions. Similarly, if $x, y \in V(H)$, then $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in at least 2 positions. Suppose that $x \in V(G)$ and $y \in V(H)$ and S_G is strictly 2-locating set. Then, $\exists w, z \in S_G \backslash N_G(x)$. Then $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in the z^{th} and w^{th} positions. On the other hand, if S_G and S_H are strictly 1-locating sets, then $\exists p \in S_G \backslash N_G(x)$ and $q \in S_H \backslash N_H(y)$. Hence $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in p^{th} and q^{th} positions. Therefore, S is a 2-resolving set in $G + H$. □

Corollary 3. Let G and H be connected nontrivial graphs. Then,

$$\dim_2(G+H) = \min\left\{sln_2(G) + ln_2(H), ln_2(G) + sln_2(H), sln_1(G) + sln_1(H)\right\}.$$

Proof. Let S be a minimum 2-resolving set of $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. By Theorem 4, S_G and S_H are 2-locating sets in G and H, respectively where S_G or S_H is strictly 2-locating set or S_G and S_H are strictly 1-locating sets. If S_G is strictly 2-locating set in G, then $sln_2(G) + ln_2(H) \leq |S_G| + |S_H| = |S| = \dim_2(G+H)$. If S_H is strictly 2-locating set in H, then $sln_2(H) + ln_2(G) \leq |S_H| + |S_G| = |S| = \dim_2(G+H)$. If S_G and S_H are both strictly 1-locating sets, then $sln_1(G) + sln_1(H) \leq |S_G| + |S_H| = |S| = \dim_2(G+H)$. Thus, $\dim_2(G+H) \geq \min\left\{sln_2(G) + ln_2(H), ln_2(G) + sln_2(H), sln_1(G) + sln_1(H)\right\}$.

Next suppose that $sln_1(G) + sln_1(H) \leq sln_2(G) + ln_2(H)$ and $sln_1(G) + sln_1(H) \leq ln_2(G) + sln_2(H)$. Let S_G be a minimum strictly 1-locating set in G and S_H be a minimum strictly 1-locating set in H. Then $S = S_G \cup S_H$ is a 2-resolving set in $G + H$, by Theorem 4. Hence $\dim_2(G+H) \leq |S| = |S_G| + |S_H| = sln_1(G) + sln_1(H)$. Therefore, $\dim_2(G+H) \leq sln_1(G) + sln_1(H)$. Similarly, if $sln_2(G) + ln_2(H) \leq sln_1(G) + sln_1(H)$ and $sln_2(G) + ln_2(H) \leq ln_2(G) + sln_2(H)$, then $\dim_2(G+H) \leq sln_2(G) + ln_2(H)$. Also, if $ln_2(G) + sln_2(H) \leq sln_2(G) + ln_2(H)$ and $ln_2(G) + sln_2(H) \leq sln_1(G) + sln_1(H)$, then $\dim_2(G+H) \leq ln_2(G) + sln_2(H)$. Therefore,
$\dim_2(G+H) = \min\left\{sln_2(G) + ln_2(H), ln_2(G) + sln_2(H), sln_1(G) + sln_1(H)\right\}$. □

Example 13. For any $n, m \geq 4$,

$$\dim_2(P_n + P_m) = \begin{cases} \left(\frac{n}{2} + 1\right) + \left(\frac{m}{2} + 1\right), & \text{if } n, m \text{ even} \\ \left(\frac{n}{2} + 1\right) + \left\lceil \frac{m}{2} \right\rceil, & \text{if } n \text{ is even}, m \text{ is odd} \\ \left\lceil \frac{n}{2} \right\rceil + \left(\frac{m}{2} + 1\right), & \text{if } n \text{ is odd}, m \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil, & \text{if } n, m \text{ odd} \end{cases}$$

In particular, for $n = 2, 3$ and $m = 2, 3$,
$\dim_2(P_n + P_m) = n + m$

4. 2-Resolving Sets in the Corona of Graphs

Definition 5. [2] The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.

The sets $\{u_1, u_2, v_1, v_2, w_1, w_2\}$ and $\{a_1, a_3, b_1, b_3, c_1, c_3, d_1, d_3\}$ are 2-resolving sets in the coronas $P_3 \circ P_2$ and $C_4 \circ P_3$, respectively, in Figure 4.

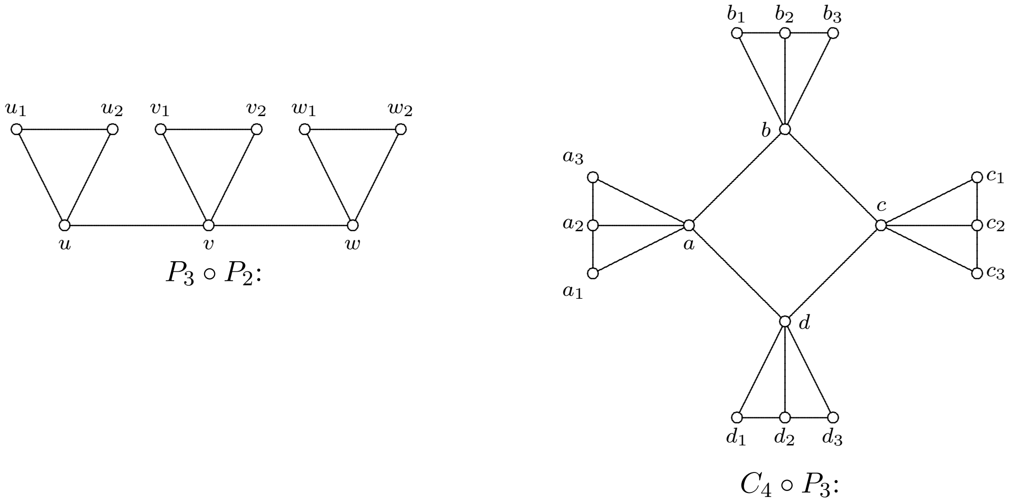


Figure 4: The corona $P_3 \circ P_2$ with $\dim_2(P_3 + P_2) = 6$ and the corona $C_4 \circ P_3$ with $\dim_2(C_4 \circ P_3) = 8$

Remark 6. Let $v \in V(G)$. For every $x, y \in V(H^v)$, $d_{G \circ H}(x, w) = d_{G \circ H}(y, w)$ and $d_{G \circ H}(v, w) + 1 = d_{G \circ H}(x, w)$ for every $w \in V(G \circ H) \backslash V(H^v)$.

Remark 7. Let G and H be non-trivial connectd graphs, $C \subseteq V(G \circ H)$ and $S_v = V(H^v) \cap C$ where $v \in V(G)$. For each $x \in V(H^v) \setminus S_v$ and $z \in S_v$,

$$d_{G \circ H}(x, z) = \begin{cases} 1 & \text{if } z \in N_{H^v}(x) \\ 2 & \text{otherwise} \end{cases}$$

Theorem 5. Let G and H be nontrivial connected graphs. A proper subset S of $V(G \circ H)$ is a 2-resolving set of $G \circ H$ if and only if $S = A \cup B$, where $A \subseteq V(G)$ and

$$B = \bigcup \{S_v : S_v \text{ is a 2-resolving set of } H^v, \forall v \in V(G)\}.$$

Proof. Suppose S is a 2-resolving set in $G \circ H$. Let $A = V(G) \cap C$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$. Suppose $S_v = \emptyset$ for some $v \in V(G)$. Let $x, y \in V(H^v)$. Then $r_{G\circ H}(x/S) = r_{G\circ H}(y/S)$ which is a contradiction to the assumption of S. Thus $S_v \neq \emptyset$. Now, we claim that S_v is a 2-resolving set in H^v for each $v \in V(G)$. Let $p, q \in V(H^v)$ where $p \neq q$. Since S is a 2-resolving set in $G \circ H$, $r_{G\circ H}(p/S)$ and $r_{G\circ H}(q/S)$ differ in at least 2 positions. By Remark 6, $r_{H^v}(p/S_v)$ and $r_{H^v}(q/S_v)$ must differ in at least 2 positions. Thus S_v is a 2-resolving set in H^v.

Conversely, let $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ satisfying the given conditions. Let $x, y \in V(G \circ H)$ with $x \neq y$ and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$.

Case 1. $u = v$

Subcase 1.1 $x, y \in V(H^v)$

Since S_v is a 2-resolving set, $r_{H^v}(x/S_v)$ and $r_{H^v}(y/S_v)$ differ in at least 2 positions. By Remark 6, $r_{G\circ H}(x/S)$ and $r_{G\circ H}(y/S)$ differ in at least 2 positions.

Subcase 1.2 $x = v$ and $y \in V(H^v)$

Since G is nontrivial and connected, $\exists w \in N_G(v)$ and $|S_w| \geq 2$. By Remark 6, $r_{G\circ H}(x/S)$ and $r_{G\circ H}(y/S)$ differ in at least 2 positions.

Case 2. $u \neq v$

Subcase 2.1 $x \in V(H^u)$, $y \in V(H^v)$

Note that $r_{G\circ H}(x/S_v)$ has components greater than or equal to 3 and $r_{G\circ H}(y/S_v)$ has components less than or equal to 2. Since $|S_v| \geq 2$, $r_{G\circ H}(x/S)$ and $r_{G\circ H}(y/S)$ differ in at least 2 positions.

Subcase 2.2 $x = u$, $y \in V(v + H^v)$

Since $|S_u| \geq 2$, $r_{G\circ H}(x/S_u)$ and $r_{G\circ H}(y/S_u)$ differ in at least 2 positions. Hence, $r_{G\circ H}(x/S)$ and $r_{G\circ H}(y/S)$ differ in at least 2 positions.

Therefore, in any case, S is a 2-resolving set in $G \circ H$. □

Corollary 4. Let G and H be nontrivial connected graphs, where $|V(G)| = n$. Then $\dim_2(G \circ H) = n \cdot \dim_2(H)$.

Proof. Let S be a minimum 2-resolving set of $G \circ H$. Then by Theorem 5, $S = A \cup B$, where $A \subseteq V(G)$ and $B = \bigcup S_v$, $v \in V(G)$ and S_v is a 2-resolving set in H. Hence,

$$\begin{aligned}
\dim_2(G \circ H) &= |S| = |A| + |B| \\
&\geq |A| + |V(G)| \cdot \dim_2(H) \\
&= |A| + n.\dim_2(H) \\
&\geq n \cdot \dim_2(H).
\end{aligned}$$

Now, let C be a minimum 2-resolving set in H. For each $v \in V(G)$, choose $C_v \subseteq V(H^v)$ with $\langle C_v \rangle \cong \langle C \rangle$. Then $D = \bigcup_{v \in V(G)} C_v$ is a 2-resolving set in $G \circ H$ by Theorem 5. Hence,

$$\dim_2(G \circ H) \leq |D| = |\bigcup_{v \in V(G)} C_v| = n \cdot |C_v| = n \cdot |C| = n \cdot \dim_2(H).$$

Therefore, $\dim_2(G \circ H) = n \cdot \dim_2(H)$. $\square$

Example 14. For any integer $n \geq 2$ and $m \geq 5$,
$\dim_2(G \circ C_m) = \begin{cases} n\left(\lceil \frac{m}{2} \rceil\right), & \text{if } m \text{ is odd} \\ n\left(\frac{m}{2}\right), & \text{if } m \text{ is even} \end{cases}$

Example 15. For any integer $n, m \geq 2$,
$\dim_2(G \circ P_m) = \begin{cases} n\left(\lceil \frac{m}{2} \rceil\right), & \text{if } m \text{ is odd} \\ n\left[\left(\frac{m}{2}\right) + 1\right], & \text{if } m \text{ is even} \end{cases}$

References

[1] R Bailey and I Yero. Error-correcting codes from k-resolving sets. *Discussiones Mathematicae, Graph Theory*, 39:341–355, 2019.

[2] G Chartrand and P Zhang. *Graphs and Digraphs.* WMU,Kalamazoo, USA, sixth edition, 2016.

[3] F Harary. *Graph Theory.* Addison-Wesley Publishing Company, USA, 1969.

[4] F Harary and R Melter. On the metric dimension of a graph. *Ars Combinatoria.*, 2:191–195, 1976.

[5] G Monsanto and H Rara. Resolving restrained domination in graphs. *European Journal of Pure and Applied Mathematics.*, 2021(accepted).

[6] P Acal G Monsanto and H Rara. On strong resolving domination in the join and corona of graphs. *European Journal of Pure and Applied Mathematics.*, 13:170–179, 2020.

[7] A Estrada-Moreno J Rodriguez-Velasquez and I Yero. The k-metric dimension of a graph. *Applied Mathematics and Information Sciences*, 9:2829–2840, 2015.

[8] A Estrada-Moreno J Rodriguez-Velasquez and I Yero. The k-metric dimension of corona product graphs. *The Bulletin of the Malaysian Mathematical Society.*, 39:135–156, 2016.

[9] V Saenpholphat and P Zang. On connected resolvability of graphs. *Australian Journal of Combinatorics.*, 28:25–37, 2003.

[10] P Slater. *Congressus Numerantium*, 14:549–559, 1975.