



## Resolving Restrained Domination in Graphs

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**Abstract.** Let  $G$  be a connected graph. Brigham et al. [3] defined a *resolving dominating set* as a set  $S$  of vertices of a connected graph  $G$  that is both resolving and dominating. A set  $S \subseteq V(G)$  is a *resolving restrained dominating set* of  $G$  if  $S$  is a resolving dominating set of  $G$  and  $S = V(G)$  or  $\langle V(G) \setminus S \rangle$  has no isolated vertex. In this paper, we characterize the resolving restrained dominating sets in the join, corona and lexicographic product of graphs and determine the resolving restrained domination number of these graphs.

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### 1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [7].

Let  $G = (V(G), E(G))$  be a connected graph. The *open neighborhood* of  $v \in V(G)$  is  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . Any element  $u$  of  $N_G(v)$  is called a *neighbor* of  $v$ . The *closed neighborhood*  $v \in V(G)$  is  $N_G[v] = N_G(v) \cup \{v\}$ . Thus, the degree of  $v \in V(G)$  is given by  $deg_G(v) = |N_G(v)|$ . Customarily, for  $S \subseteq V(G)$ ,  $N_G(S) = \bigcup_{v \in S} N_G(v)$  and

$$N_G[S] = \bigcup_{v \in S} N_G[v].$$

A nonempty set  $S \subseteq V(G)$  is a *dominating set* in graph  $G$  if  $N_G[S] = V(G)$ . Otherwise, we say  $S$  is a *non-dominating set* of  $G$ . The *domination number* of a graph  $G$ , denoted

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by  $\gamma(G)$ , is given by  $\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$ . If  $S$  is a dominating set of  $G$  and if  $|S| = \gamma(G)$ , then  $S$  is called a *minimum dominating set* or a  $\gamma$ -set of  $G$ .

A vertex  $x$  of a connected graph  $G$  is said to *resolve two vertices*  $u$  and  $v$  of  $G$  if  $d_G(x, u) \neq d_G(x, v)$ . For an ordered set  $W = \{x_1, \dots, x_k\} \subseteq V(G)$  and a vertex  $v$  in  $G$ , the *k-vector*

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \dots, d_G(v, x_k))$$

is called the *representation* of  $v$  with respect to  $W$ . The set  $W$  is a *resolving set* for  $G$  if and only if no two vertices of  $G$  have the same representation with respect to  $W$ . The *metric dimension of  $G$* , denoted by  $\dim(G)$ , is the minimum cardinality over all resolving sets of  $G$ . A resolving set of cardinality  $\dim(G)$  is called a *basis*.

Brigham et al. [3] defined a *resolving dominating set* as a set  $S$  of vertices of a connected graph  $G$  that is both resolving and dominating. The cardinality of a minimum resolving dominating set is called the *resolving domination number* of  $G$  and is denoted by  $\gamma_R(G)$ . A resolving dominating set of cardinality  $\gamma_R(G)$  is called a  $\gamma_R$ -set of  $G$ .

Let  $G$  be a connected graph. A set  $S \subseteq V(G)$  is a *strictly resolving dominating set* of  $G$  if it is a resolving dominating set of  $G$  and  $N_G(u) \cap S \neq S$  for all  $u \in V(G) \setminus S$ . The *strictly resolving dominating number* of  $G$ , denoted by  $\gamma_{SR}(G)$ , is the smallest cardinality of a strictly resolving dominating set of  $G$ . A strictly resolving dominating set of  $G$  of cardinality  $\gamma_{SR}(G)$  is referred to as  $\gamma_{SR}$ -set of  $G$ .

Let  $G = (V(G), E(G))$  be a graph. A set  $S \subseteq V(G)$  is a *restrained dominating set* of  $G$  if  $S$  is a dominating set of  $G$  and for every  $v \in V(G) \setminus S$  there exists  $u \in (V(G) \setminus S) \cap N_G(v)$ . Equivalently, a dominating subset  $S$  of  $V(G)$  is a *restrained dominating set* of  $G$  if  $S = V(G)$  or  $\langle V(G) \setminus S \rangle$  has no isolated vertex. The *restrained domination number* of  $G$ , denoted by  $\gamma_r(G)$  is the minimum cardinality of a restrained dominating set of  $G$ . Any restrained dominating set of  $G$  of cardinality  $\gamma_r(G)$  is referred to as a  $\gamma_r$ -set of  $G$ .

Let  $G$  be a connected graph. A set  $S \subseteq V(G)$  is a *resolving restrained dominating set* of  $G$  if  $S$  is a resolving dominating set of  $G$  and  $S = V(G)$  or  $\langle V(G) \setminus S \rangle$  has no isolated vertex. The *resolving restrained domination number* of  $G$ , denoted by  $\gamma_{Rr}(G)$  is the smallest cardinality of a resolving restrained dominating set of  $G$ . Any resolving restrained dominating set of cardinality  $\gamma_{Rr}(G)$  is referred to as a  $\gamma_{Rr}$ -set of  $G$ .

Omega and Canoy [11] defined a *locating set* of  $G$  as a set  $S \subseteq V(G)$  if for every two distinct vertices  $u$  and  $v$  of  $V(G) \setminus S$ ,  $N_G(u) \cap S \neq N_G(v) \cap S$ . The *locating number* of  $G$ , denoted by  $ln(G)$ , is the smallest cardinality of a locating set of  $G$ . A locating set of  $G$  of cardinality  $ln(G)$  is referred to as *ln-set* of  $G$ .

Canoy and Malacas [14] defined a locating (resp. strictly locating) subset  $S$  of  $V(G)$  which is also dominating is called a *locating-dominating* (resp. *strictly locating-dominating*) set in a connected graph  $G$ . The minimum cardinality of a locating-dominating (resp. strictly locating-dominating) set in  $G$ , denoted by  $\gamma_L(G)$  (resp.  $\gamma_{SL}(G)$ ), is called the *L-domination* (resp. *SL-domination*) number of  $G$ . Any *L-dominating* (resp. *SL-dominating*) set of cardinality  $\gamma_L(G)$  (resp.  $\gamma_{SL}(G)$ ) is then referred to as a  $\gamma_L$ -set ( $\gamma_{SL}$ -set) of  $G$ .

Let  $G$  be a connected graph. A set  $S \subseteq V(G)$  is a *strictly locating set* of  $G$  if it is a locating set of  $G$  and  $N_G(u) \cap S \neq S$  for all  $u \in V(G) \setminus S$ . The *strictly locating number* of

$G$ , denoted by  $sln(G)$ , is the smallest cardinality of a strictly locating set of  $G$ . A strictly locating set of  $G$  of cardinality  $sln(G)$  is referred to as a *sln-set* of  $G$ .

Let  $G$  be a connected graph. A set  $S \subseteq V(G)$  is a *restrained locating set* of  $G$  if  $S$  is a locating set of  $G$  and  $S = V(G)$  or  $\langle V(G) \setminus S \rangle$  has no isolated vertex. The *restrained locating number* of  $G$ , denoted  $rln(G)$ , is the smallest cardinality of a restrained locating set of  $G$ . A restrained locating set of cardinality  $rln(G)$  is then referred to as *rln-set* of  $G$ .

A connected graph  $G$  of order  $n \geq 3$  is *point distinguishing* if for any two distinct vertices  $u$  and  $v$  of  $G$ ,  $N_G[u] \neq N_G[v]$ . It is *totally point determining* if for any two distinct vertices  $u$  and  $v$  of  $G$ ,  $N_G(u) \neq N_G(v)$  and  $N_G[u] \neq N_G[v]$ .

In recent years, the concept of domination in graphs has been studied extensively and several research papers have been published on this topic. The said concept was not formally defined mathematically until the publications of the books by Claude Berge [2] in 1958 and Oystein Ore in 1962. In 1977, a survey paper by Cockayne and Hedetniemi [4] began to study the concept of domination.

On the other hand, the problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [12] brought in the notion of locating sets and its minimum cardinality as locating number. The same concept was also introduced by Harary and Melter [7] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. However, in recent studies, locating sets and resolving sets are defined differently. In 2013, Canoy and Malacas [13], defined a locating set as a subset  $S$  of  $V(G)$  in a connected graph  $G$  satisfying the condition that  $N_G(u) \cap S \neq N_G(v) \cap S$  for all  $u, v \in V(G) \setminus S$  with  $u \neq v$ . Meanwhile, in the same year, Bailey et al. [1] defined a resolving set as a set of vertices  $S$  in a graph  $G$  such that for any two distinct vertices  $u, v$ , there exists  $x \in S$  such that the distances  $d(u, x) \neq d(v, x)$ .

In 1999, Domke et al. [5] introduced and investigated the concept of restrained domination in graphs. In 2008, Hattingh et al. [8] investigated the same concept and obtained a Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs. Moreover, in 2015, Omega et al. [10] introduced and characterized the restrained locating-dominating sets of some graphs and determined the restrained  $L$ -domination numbers of these graphs.

Inspired by the above works, this study aims to define and characterize the resolving restrained dominating sets and determine the resolving restrained domination number in the join, corona and lexicographic product of two graphs.

## 2. Preliminary Results

**Remark 1.** For any connected graph  $G$  of order  $n \geq 2$ ,

$$\gamma_{Rr}(G) \in \{2, 3, 4, \dots, n-2, n\}.$$

**Remark 2.** Every resolving restrained dominating set of a connected graph  $G$  of order  $n \geq 2$  is a restrained dominating set of  $G$ . Hence,  $\gamma_r(G) \leq \gamma_{Rr}(G)$ .

**Theorem 1.** Let  $G$  be a connected graph of order  $n \geq 2$ . Then  $\gamma_{Rr}(G) = n$  if and only if  $G \cong K_n$  or  $G \cong K_{1,n-1}$ .

*Proof:* Suppose that  $\gamma_{Rr}(G) = n$ . If  $n = 2$ , then  $G = K_2$ . If  $n = 3$ , then  $G \cong K_3$  or  $G \cong K_{1,2}$ . Suppose that  $n \geq 4$  and suppose further that  $G \not\cong K_n$ . Let  $x \in V(G)$  with  $deg_G(x) = \Delta(G)$ . Suppose there exists  $y \in V(G) \setminus \{x\}$  such that  $xy \notin E(G)$ . Since  $G$  is connected,  $y$  can be chosen so that  $d_G(x, y) = 2$ . Let  $z \in N_G(x) \cap N_G(y)$ . If  $yp \in E(G)$  for all  $p \in N_G(x)$ , then choose  $W = V(G) \setminus \{y, z\}$ . Since  $deg_G(x) \geq deg_G(z) \geq 2$ , there exists  $u \in N_G(x) \setminus \{z\}$ . Let  $W = V(G) \setminus \{x, z\}$ . Since  $xz, xu, zy \in E(G)$ ,  $W$  is a restrained dominating set of  $G$ . Moreover, because  $y \in N_G(z) \setminus N_G(x)$ ,  $r_G(x/W) \neq r_G(z/W)$ . Thus,  $W$  is a resolving set of  $G$  and we have  $\gamma_{Rr}(G) \leq |W| = n - 2$ , a contradiction. Therefore,  $xy \in E(G)$  for all  $y \in V(G) \setminus \{x\}$ .

It remains to show that  $uv \notin E(G)$  for every distinct vertices  $u, v \in V(G) \setminus \{x\}$ . To this end, suppose there exist distinct vertices  $u$  and  $v$  in  $V(G) \setminus \{x\}$  such that  $uv \in E(G)$ . Since  $G \neq K_n$ , there exist  $a, b \in V(G)$  such that  $ab \notin E(G)$ . If  $av \in E(G)$  or  $bv \in E(G)$ , say  $av \in E(G)$ , then consider  $W = V(G) \setminus \{x, a\}$ . Note that  $xv, av \in E(G)$ . Thus,  $W$  is a dominating set of  $G$ . Since  $b \in N_G(x) \setminus N_G(a)$ ,  $r_G(a/W) \neq r_G(x/W)$ . Thus,  $W$  is a resolving dominating set of  $G$ . Since  $V(G) \setminus W = \{x, a\}$  and  $xa \in E(G)$ , it follows that  $W$  is a resolving restrained dominating set of  $G$ . If  $av, bv \notin E(G)$ , then take  $W = V(G) \setminus \{x, v\}$ . Again,  $W$  is a resolving restrained dominating set of  $G$ . In either case,  $\gamma_R(G) \leq |W| = n - 2$ , a contradiction to the assumption. Therefore,  $uv \notin E(G)$  for every two distinct vertices  $u, v \in V(G) \setminus \{x\}$ . Thus,  $G \cong K_{1,n-1}$ .

The converse is easy. □

**Theorem 2.** Let  $G$  be a connected graph of order  $n = 4$ . Then  $\gamma_{Rr} = 2$  if and only if  $G \notin \{K_4, K_{1,3}\}$ .

*Proof:* Suppose that  $\gamma_{Rr}(G) = 2$ . Then by Theorem 1,  $G \notin \{K_4, K_{1,3}\}$ .

For the converse, suppose that  $G \notin \{K_4, K_{1,3}\}$ . Since  $n = 4, \gamma_R(G) \geq 2$ . Choose  $x, y \in V(G)$  such that  $d_G(x, y) = 2$ . Let  $z \in N_G(x) \cap N_G(y)$  and  $p \in V(G) \setminus \{x, y, z\}$ . Consider the following cases:

**Case 1.** Suppose that  $p \in N_G(x) \cap N_G(y)$ .

Let  $W = \{x, z\}$ . Then  $W$  is a resolving restrained dominating set of  $G$ .

**Case 2.** Suppose that  $p \in N_G(x) \setminus N_G(y)$ .

Then  $W = \{y, p\}$ , thus  $W$  is a resolving restrained dominating set of  $G$ .

**Case 3.** Suppose that  $p \in N_G(y) \setminus N_G(x)$ .

Then  $W = \{x, p\}$ , hence  $W$  is a resolving restrained dominating set of  $G$ .

Therefore, in all cases,  $\gamma_{Rr}(G) = 2$ . □

**Theorem 3.** Let  $G$  be a connected graph of order  $n = 5$ . Then  $\gamma_{Rr}(G) = 2$  if and only if there exist distinct vertices  $x$  and  $y$  that dominate  $G$  such that  $|N_G(x) \cap N_G(y)| = 1, |N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$  and  $\langle V(G) \setminus \{x, y\} \rangle$  has no isolated vertex.

*Proof:* Suppose that  $\gamma_{Rr}(G) = 2$ . Then there exist distinct vertices  $x$  and  $y$  such that  $W = \{x, y\}$  is a minimum restrained resolving dominating set of  $G$ . Hence,  $|N_G(x) \cap N_G(y)| \leq 1$ . Suppose  $|N_G(x) \cap N_G(y)| = 0$ . Then one of  $x$  and  $y$ , say  $x$ , has at least two neighbors,  $u_1, u_2 \in (V(G) \setminus W) \setminus N_G(y)$ . Thus,  $r_G(u_1/W) = r_G(u_2/W)$ , a contradiction to the assumption. Hence,  $|N_G(x) \cap N_G(y)| = 1$ . Next, let  $z \in N_G(x) \cap N_G(y)$  and let  $b, c \in V(G) \setminus \{x, y, z\}$ . Then  $b, c \notin N_G(x) \cap N_G(y)$ . Since  $W$  is dominating,  $b \in N_G(x)$  or  $b \in N_G(y)$ . Suppose  $b \in N_G(x)$ . Then  $W$  is a resolving dominating set of  $G$  implies that  $c \in N_G(y) \setminus N_G(x)$ . Hence,  $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$ . Since  $W$  is a restrained resolving dominating set,  $(V(G) \setminus W)$  has no isolated vertex.

For the converse, suppose there exist distinct vertices  $x, y \in V(G)$  satisfying the given properties. Let  $W = \{x, y\}$ . Then  $W$  is a resolving restrained dominating set of  $G$ . Hence,  $\gamma_{Rr}(G) = 2$ .  $\square$

### 3. Resolving Restrained Domination in the Join of Graphs

**Theorem 4.** [6] *Let  $G$  and  $H$  be connected graphs. Then  $C \subseteq V(G + H)$  is a dominating set in  $G + H$  if and only if at least one of the following is true:*

- (i)  $C \cap V(G)$  is a dominating set in  $G$ .
- (ii)  $C \cap V(H)$  is a dominating set in  $H$ .
- (iii)  $C \cap V(G) \neq \emptyset$  and  $C \cap V(H) \neq \emptyset$ .

**Theorem 5.** [14] *Let  $G$  and  $H$  be non-trivial connected graphs. A set  $W \subseteq V(G + H)$  is a locating-dominating set of  $G + H$  if and only if  $W = W_G \cup W_H$  where  $W_G \subseteq V(G)$  and  $W_H \subseteq V(H)$  are locating-dominating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set.*

**Theorem 6.** [9] *Let  $G$  and  $H$  be non-trivial connected graphs. A set  $W \subseteq V(G + H)$  is a resolving set of  $G + H$  if and only if  $W = W_G \cup W_H$  where  $W_G \subseteq V(G)$  and  $W_H \subseteq V(H)$  are locating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set.*

**Theorem 7.** *Let  $G$  and  $H$  be non-trivial connected graphs. A set  $W \subseteq V(G + H)$  is a resolving dominating set of  $G + H$  if and only if  $W$  is a locating-dominating set of  $G + H$ .*

*Proof:* Suppose that  $W$  is a resolving dominating set of  $G + H$ . Then  $W$  is a resolving set of  $G + H$ . By Theorem 6,  $W = W_G \cup W_H$  where  $W_G \subseteq V(G)$  and  $W_H \subseteq V(H)$  are locating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set. Since  $W$  is a dominating set of  $G + H$ ,  $W_G$  and  $W_H$  are also dominating sets of  $G$  and  $H$ , respectively. By Theorem 5,  $W$  is a locating-dominating set of  $G + H$ .

The converse immediately follows from Theorem 5 and Theorem 4(iii).  $\square$

The next result follows immediately from Theorem 5.

**Theorem 8.** *Let  $G$  and  $H$  be non-trivial connected graphs. A set  $W \subseteq V(G + H)$  is a resolving dominating set of  $G + H$  if and only if  $W = W_G \cup W_H$  where  $W_G = V(G) \cap W$  and  $W_H = V(H) \cap W$  are locating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set.*

**Theorem 9.** *Let  $G$  and  $H$  be nontrivial connected graphs. A set  $W \subseteq V(G + H)$  is a resolving restrained dominating set of  $G + H$  if and only if  $W = W_G \cup W_H$  and satisfies the following conditions:*

- (i)  $W_G$  and  $W_H$  are locating sets of  $G$  and  $H$ , respectively;
- (ii)  $W_G$  ( $W_H$ ) is a restrained locating set of  $G$  (resp.  $H$ ) whenever  $W_G = V(G)$  (resp.  $W_H = V(H)$ ); and
- (iii)  $W_G$  or  $W_H$  is a strictly locating set.

*Proof:* Let  $W \subseteq V(G + H)$  be a resolving restrained dominating set of  $G + H$ . Then by Theorem 8,  $W_G = V(G) \cap W$  and  $W_H = V(H) \cap W$  are locating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set. Hence, (i) and (iii) hold.

Suppose  $W_G = V(G)$  or  $W_H = V(H)$ . Since  $W = V(G + H)$  or  $V(G + H) \setminus W$  has no isolated vertex,  $W_H = V(H)$  or  $V(H) \setminus W_H$  has no isolated vertex. Hence,  $W_H$  is a restrained locating set of  $H$ . Similarly,  $W_G$  is a restrained locating set of  $G$  if  $W_H = V(H)$ . Thus, (ii) holds.

For the converse, suppose that  $W = W_G \cup W_H$  and satisfies (i), (ii), and (iii), where  $W_G$  and  $W_H$  are locating sets of  $G$  and  $H$ , respectively, and  $W_G$  or  $W_H$  is a strictly locating set. Then by Theorem 8,  $W$  is a resolving dominating set of  $G + H$ . By (ii) and the fact that  $W_G$  and  $W_H$  are non-empty,  $W$  is a restrained dominating set of  $G + H$ .  $\square$

**Lemma 1.** *Let  $G$  and  $H$  be non-trivial connected graphs such that  $sln(G) = |V(G)| = m$  and  $sln(H) \neq |V(H)|$ . Then  $m + rln(H) \geq sln(H) + ln(G)$ .*

*Proof:* If  $ln(H) = sln(H)$ , then  $m + rln(H) \geq ln(G) + sln(H)$ . Suppose  $ln(H) < sln(H)$ . Then  $ln(H) = sln(H) - 1 \leq rln(H)$ . Hence,

$$\begin{aligned} m + rln(H) &\geq m + sln(H) - 1 \\ &= sln(H) + (m - 1) \\ &= sln(H) + ln(G). \quad \square \end{aligned}$$

**Theorem 10.** [11] *Let  $G$  be a connected graph of order  $n \geq 2$ . If  $ln(G) < sln(G)$ , then  $1 + ln(G) = sln(G)$ .*

**Corollary 1.** *Let  $G$  and  $H$  be non-trivial connected graphs of order  $m$  and  $n$ , respectively. Then*

$$\gamma_{Rr}(G + H) = \begin{cases} m + n & , & \text{if } sln(G) = m \text{ and } sln(H) = n \\ sln(H) + ln(G) & , & \text{if } sln(G) = m \text{ and } sln(H) \neq n \\ sln(G) + ln(H) & , & \text{if } sln(G) \neq m \text{ and } sln(H) = n \\ \min \{sln(H) + ln(G), sln(G) + ln(H)\} & , & \text{if } sln(G) \neq m \text{ and } \\ & & sln(H) \neq n. \end{cases}$$

*Proof:* Consider the following cases:

**Case 1.** Suppose  $sln(G) = m$  and  $sln(H) = n$ .

Then  $V(G)$  and  $V(H)$  are the only strictly locating sets of  $G$  and  $H$ , respectively. Since  $ln(G) \leq m - 1$ , with  $1 + ln(G) = sln(G)$  by Theorem 10. Thus,  $ln(G) = m - 1$ . Since  $ln(G) \leq rln(G)$  and  $rln(G)$  cannot be equal to  $m - 1$ , it follows that  $rln(G) = m$ . Similarly,  $rln(H) = n$ . Thus,  $V(G)$  and  $V(H)$  are the only restrained locating sets of  $G$  and  $H$ , respectively. Hence, by Theorem 10, if  $W$  is a minimum restrained resolving dominating set of  $G + H$ , then  $W = V(G) \cup V(H)$ . Therefore,  $\gamma_{Rr}(G + H) = m + n$ .

**Case 2.** Suppose  $sln(G) = m$  and  $sln(H) \neq n$ .

Suppose first that  $sln(G) = m$  and  $sln(H) \neq n$ . Let  $W_G$  and  $W_H$  be minimum locating set and minimum strictly locating set of  $G$  and  $H$ , respectively. Then by Theorem 9,  $W = W_G \cup W_H$  is a restrained resolving dominating set of  $G + H$ . Thus,  $\gamma_{Rr}(G + H) \leq |W| = sln(H) + ln(G)$ .

**Case 3.** Suppose  $sln(G) \neq m$  and  $sln(H) = n$ .

Now, suppose that  $W'$  is a minimum resolving restrained dominating set of  $G + H$ . Then by Theorem 9,  $W' = W'_G \cup W'_H$ , where  $W'_G \neq V(G)$  is a locating set of  $G$  and  $W'_H \neq V(H)$  is a strictly locating set of  $H$ . Hence,  $\gamma_{Rr}(G + H) = |W'| \geq ln(G) + sln(H)$ . Therefore,  $\gamma_{Rr}(G + H) = ln(G) + sln(H)$ . Similarly, if  $sln(G) \neq m$  and  $sln(H) = n$ , then  $\gamma_{Rr}(G + H) = sln(G) + ln(H)$ .

**Case 4.** Suppose  $sln(G) \neq m$  and  $sln(H) \neq n$ .

Let  $W$  be a minimum resolving restrained dominating set of  $G + H$ . Let  $W_G = V(G) \cap W$  and  $W_H = V(H) \cap W$ . Then by Theorem 9,  $W_G$  and  $W_H$  are locating sets of  $G$  and  $H$ , respectively, where  $W_G$  or  $W_H$  is a strictly locating set. If  $W_G = V(G)$ , then by Theorem 10(i),  $W_H$  is a restrained locating set of  $H$ . Thus,

$$\gamma_{Rr}(G + H) = m + rln(H) \geq sln(G) + ln(H)$$

by Lemma 1. Similarly, by Theorem 9(i), if  $W_H = V(H)$ , then  $\gamma_{Rr}(G + H) = n + rln(G) \geq sln(H) + ln(G)$  by Lemma 1. Suppose that  $W_G \neq V(G)$  and  $W_H \neq V(H)$ . Assume first that  $W_G$  is a strictly locating set of  $G$ . Then

$$sln(G) + ln(H) \leq |W_G| + |W_H| = |W| = \gamma_{Rr}(G + H).$$

If  $W_H$  is a strictly locating set of  $H$ , then

$$sln(H) + ln(G) \leq |W_H| + |W_G| = |W| = \gamma_{Rr}(G + H).$$

Thus,  $\gamma_{Rr}(G + H) \geq \min \{sln(G) + ln(H), sln(H) + ln(G)\}$ .

Let  $W_G$  and  $W'_H$  be minimum strictly locating sets of  $G$  and  $H$ , respectively, and let  $W_H$  and  $W'_G$  be minimum locating sets of  $H$  and  $G$ , respectively. Then  $W = W_G \cup W_H$  and  $W' = W'_G \cup W'_H$  are resolving restrained dominating sets of  $G + H$  by Theorem 8. Thus,

$$\gamma_{Rr}(G + H) \leq |W| = |W_G| + |W_H| = sln(G) + ln(H)$$

and  $\gamma_{Rr}(G + H) \leq |W'| = ln(G) + sln(H)$ .

Therefore,  $\gamma_{Rr}(G + H) = \min \{sln(G) + ln(H), sln(H) + ln(G)\}$ .  $\square$

**Lemma 2.** *Let  $H$  be a non-trivial connected graph of order  $m$  and let  $K_n$  be a complete graph of order  $n \geq 3$ . Then  $n - 1 + sln(H) \leq n + rln(H)$ .*

*Proof:* Suppose  $ln(H) < sln(H)$ . Then  $ln(H) = sln(H) - 1$ . If  $ln(H) = rln(H)$ . Then  $rln(H) = sln(H) - 1$ . Hence,  $sln(H) + n - 1 = n + rln(H)$ . If  $ln(H) = sln(H)$ , then  $sln(H) \leq rln(H)$ . Hence,  $sln(H) - 1 < rln(H)$ , that is,  $n - 1 + sln(H) < n + rln(H)$ .  $\square$

The next result follows immediately from Corollary 1.

**Corollary 2.** *Let  $H$  be a non-trivial connected graph of order  $m$  and let  $K_n$  be a complete graph of order  $n \geq 3$ . Then*

$$\gamma_{Rr}(H + K_n) = \begin{cases} sln(H) + n - 1, & \text{if } sln(H) \neq m \\ m + n & , \text{ otherwise.} \end{cases}$$

**Theorem 11.** [14] *Let  $H$  be a non-trivial connected graph and let  $K_1 = \langle v \rangle$ . Then  $W \subseteq V(H)$  is a locating-dominating set of  $H + K_1$  if and only if either  $v \notin W$  and  $W$  is a strictly locating-dominating set of  $H$  or  $W = \{v\} \cup W_1$ , where  $W_1$  is a locating set of  $H$ .*

The next result follows immediately from Theorem 11.

**Theorem 12.** *Let  $H$  be a non-trivial connected graph and let  $K_1 = \langle v \rangle$ . Then  $W \subseteq V(H)$  is a resolving dominating set of  $H + K_1$  if and only if either  $v \notin W$  and  $W$  is a strictly resolving dominating set of  $H$  or  $W = \{v\} \cup W_H$ , where  $W_H$  is a locating set of  $H$ .*

**Theorem 13.** *Let  $H$  be a non-trivial connected graph and let  $K_1 = \langle v \rangle$ . Then  $W \subseteq V(H + K_1)$  is a resolving restrained dominating set of  $H + K_1$  if and only if either  $v \notin W$  and  $W$  is a strictly resolving dominating set of  $H$  with  $V(H) \neq W$  or  $W = \{v\} \cup W_H$ , where  $W_H$  is a restrained locating set of  $H$ .*

*Proof:* Suppose that  $W$  is a resolving restrained dominating set of  $H + K_1$  and let  $W_H = V(H) \cap W$ . Then by Theorem 12,  $W = W_H \cup \{v\}$ , where  $W_H$  is a locating set of  $H$  or  $v \notin W$  and  $W$  is a strictly resolving dominating set of  $H$ . Suppose that  $v \notin W$ . Since  $\langle V(H + K_1) \setminus W \rangle = \langle \{v\} \cup (V(H) \setminus W) \rangle$  has no isolated vertex, it follows that  $W \neq V(H)$ . Next, suppose that  $W = W_H \cup \{v\}$ . Again, since  $\langle V(H + K_1) \setminus W \rangle = \langle V(H) \setminus W_H \rangle$ , and  $W$  is a restrained locating set, it follows that  $W_H = V(H)$  or  $\langle V(H) \setminus W_H \rangle$  has no isolated vertex. Hence,  $W_H$  is a restrained locating set of  $H$ .

Conversely, assume first that  $v \notin W$  and  $W$  is a strictly resolving dominating set of  $H$  with  $W \neq V(H)$ . By Theorem 12,  $W$  is a resolving dominating set of  $H + K_1$ . Since  $\langle V(H + K_1) \setminus W \rangle$  has no isolated vertex. Thus,  $W$  is a restrained dominating set of  $H + K_1$ . Finally, suppose that  $W = W_H \cup \{v\}$ , where  $W_H$  is a restrained locating set of  $H$ . By Theorem 12,  $W$  is a resolving dominating set of  $H + K_1$ . Consequently,  $W$  is a restrained dominating set of  $H + K_1$ . Therefore,  $W$  is a resolving restrained dominating set of  $H + K_1$ .  $\square$



**Corollary 3.** Let  $G$  be a non-trivial connected graph of order  $m$ . Then

$$\gamma_{Rr}(G + K_1) = \begin{cases} m & , \text{ if } \gamma_{SR}(G) = m \text{ and } rln(G) = m; \\ rln(G) + 1 & , \text{ if } \gamma_{SR}(G) = m \text{ and } rln(G) \neq m; \\ \min \{ \gamma_{SR}(G), rln(G) + 1 \} & , \text{ if } \gamma_{SR}(G) \neq m \text{ and } rln(G) \neq m. \end{cases}$$

#### 4. Resolving Restrained Domination in the Corona of Graphs

**Theorem 14.** [9] Let  $G$  and  $H$  be non-trivial connected graphs. Then  $W \subseteq V(G \circ H)$  is a resolving set of  $G \circ H$  if and only if  $W \cap V(H^v) \neq \emptyset$  for every  $v \in V(G)$  and  $W = A \cup B$ , where  $A \subseteq V(G)$  and

$$B = \cup \{ B_v : v \in A \text{ and } B_v \text{ is a locating set of } H^v \}.$$

**Theorem 15.** Let  $G$  and  $H$  be non-trivial connected graphs. Then  $W \subseteq V(G \circ H)$  is a resolving dominating set of  $G \circ H$  if and only if  $W \cap V(H^v) \neq \emptyset$  for every  $v \in V(G)$  and  $W = A \cup B \cup D$ , where  $A \subseteq V(G)$ ,

$$B = \cup \{ B_v : v \in A \text{ and } B_v \text{ is a locating set of } H^v \} \text{ and}$$

$$D = \cup \{ D_u : u \notin A \text{ and } D_u \text{ is a locating-dominating set of } H^u \}.$$

*Proof:* Let  $W$  be a resolving dominating set of  $G \circ H$ . Then by Theorem 14,  $W \cap V(H^v) \neq \emptyset$  for any  $v \in V(G)$ . Since  $W$  is a resolving set,  $G = A \cup B^*$ , where  $A \subseteq V(G)$  and  $B^* = \cup \{ B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v \}$  by Theorem 14. Let  $B = \cup \{ B_v : v \in A \}$  and  $D = \cup \{ B_u : u \in V(G) \setminus A \}$ . Since  $W$  is a dominating set, it follows that  $B_u$  is a dominating set for each  $u \in V(G) \setminus A$ .

For the converse, suppose  $W = A \cup B \cup D$ , where  $A, B$  and  $D$  are the sets possessing the properties described. Then by Theorem 14,  $W$  is a resolving set of  $G \circ H$ . Since  $D_u$  is a dominating set of  $H^u$  for each  $u \notin W$ ,  $W$  is a resolving dominating set of  $G \circ H$ .  $\square$

**Theorem 16.** Let  $G$  and  $H$  be non-trivial connected graphs. Then  $W \subseteq V(G \circ H)$  is a resolving restrained dominating set of  $G \circ H$  if and only if  $W \cap V(H^v) \neq \emptyset$  for every  $v \in V(G)$  and  $W = A \cup B \cup D$ , where  $A \subseteq V(G)$ ,

$$B = \cup \{ B_v : v \in A \text{ and } B_v \text{ is a restrained locating set of } H^v \} \text{ and}$$

$$D = \cup \{ D_u : u \notin A \text{ and } D_u \text{ is a locating-dominating set of } H^u \}$$

where  $u \in N_G(V(G) \setminus A)$  if  $D_u = V(H^u)$ .

*Proof:* Suppose  $W$  is a resolving restrained set of  $G \circ H$ . By Theorem 15,  $W \cap V(H^v) \neq \emptyset$  for all  $v \in V(G)$  and  $W = A \cup B \cup D$  where  $A, B$  and  $D$  are sets described in Theorem 15. Since  $W$  is a restrained dominating set of  $G \circ H$ ,  $B_v$  is a restrained dominating set of  $H^v$  for each  $v \in A$ , and  $u \in N_G(V(G) \setminus A)$  for each  $u \in V(G) \setminus A$  with  $D_u = V(H^u)$ . Therefore,

$$B = \cup \{ B_v : v \in A \text{ and } B_v \text{ is a restrained locating set of } H^v \} \text{ and}$$

$$D = \cup\{D_u : u \notin A \text{ and } D_u \text{ is a locating dominating set of } H^u\}$$

where  $u \in N_G(V(G) \setminus A)$  if  $D_u = V(H^u)$ .

For the converse, suppose that  $W \cap V(H^v) \neq \emptyset$  for every  $v \in V(G)$  and  $W = A \cup B \cup D$  where  $A, B$  and  $D$  are as described above. By Theorem 15,  $W$  is a resolving dominating set of  $G \circ H$ . Further, because of the additional properties of the sets in  $B$  and  $D$ ,  $W$  is a restrained dominating set of  $G \circ H$ . □

**Remark 3.** [11] *For any connected graph  $G$ ,  $ln(G) \leq \gamma_L(G) \leq \gamma_{SL}(G)$ .*

**Theorem 17.** [11] *Let  $G$  be a connected graph of order  $n \geq 2$*

(i) *If  $ln(G) < sln(G)$ , then  $1 + ln(G) = sln(G)$ .*

(ii) *If  $ln(G) < \gamma_L(G)$ , then  $1 + ln(G) = \gamma_L(G)$ .*

(iii) *If  $sln(G) < \gamma_{SL}(G)$ , then  $1 + sln(G) = \gamma_{SL}(G)$ .*

**Corollary 4.** *Let  $G$  and  $H$  be non-trivial connected graphs with  $|V(G)| = n$ . Then  $\gamma_{Rr}(G \circ H) = n \cdot \gamma_L(H)$ .*

*Proof:* Let  $W$  be a minimum resolving restrained dominating of  $G \circ H$ . Then  $W = A \cup B \cup D$  are the sets described in Theorem 16. By Remark 3 and Theorem 17, it follows that

$$\begin{aligned} \gamma_{Rr}(G \circ H) &= |W| \\ &= |A| + |B| + |D| \\ &\geq |A| + |A| \cdot rln(H) + (n - |A|) \cdot \gamma_L(H) \\ &\geq |A| + |A| \cdot ln(H) + (n - |A|) \cdot \gamma_L(H) \\ &= |A|(1 + ln(H)) + (n - |A|) \cdot \gamma_L(H) \\ &\geq |A| \cdot \gamma_L(H) + (n - |A|) \cdot \gamma_L(H) \\ &= n \cdot \gamma_L(H). \end{aligned}$$

Now, let  $F$  be a minimum locating-dominating set of  $H$ . For each  $v \in V(G)$ , pick  $F_v \subseteq V(H^v)$  with  $\langle F_v \rangle \cong \langle F \rangle$ . Then  $W = \bigcup_{v \in V(G)} F_v$  is a resolving restrained dominating set of  $G \circ H$  by Theorem 16. Hence,

$$\gamma_{Rr}(G \circ H) \leq |W| = n \cdot \gamma_L(H).$$

Therefore,  $\gamma_{Rr}(G \circ H) = n \cdot \gamma_L(H)$ . □

### 5. Resolving Restrained Domination in the Lexicographic Product of Graphs

**Theorem 18.** [9] *Let  $G$  and  $H$  be non-trivial connected graphs with  $\Delta(H) \leq |V(H)| - 2$ . Then  $W = \bigcup_{x \in S} [\{x\} \times T_x]$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ , is a resolving set of  $G[H]$  if and only if  $W$  is a locating set of  $G[H]$ .*

**Theorem 19.** [6] *Let  $G$  and  $H$  be non-trivial connected graphs. Then  $C \subseteq V(G[H])$  is a dominating set in  $G[H]$  if and only if  $C = \bigcup_{x \in S} [\{x\} \times T_x]$  and either*

- (i)  $S$  is a total dominating set in  $G$  or
- (ii)  $S$  is a dominating set in  $G$  and  $T_x$  is a dominating set in  $H$  for every  $x \in S \setminus N_G(S)$ .

**Theorem 20.** [14] *Let  $G$  and  $H$  be non-trivial connected graphs with  $\Delta(H) \leq |V(H)| - 2$ . Then  $W = \bigcup_{x \in S} [\{x\} \times T_x]$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ , is a locating-dominating set of  $G[H]$  if and only if*

- (i)  $S = V(G)$ ;
- (ii)  $T_x$  is a locating set of  $H$  for every  $x \in V(G)$ ;
- (iii)  $T_x$  or  $T_y$  is strictly locating set of  $H$  whenever  $x$  and  $y$  are adjacent vertices of  $G$  with  $N_G[x] = N_G[y]$ ; and
- (iv)  $T_x$  or  $T_y$  is (locating) dominating set of  $H$  whenever  $x$  and  $y$  are nonadjacent vertices of  $G$  with  $N_G(x) = N_G(y)$ .

**Theorem 21.** *Let  $G$  and  $H$  be non-trivial connected graphs with  $\Delta(H) \leq |V(H)| - 2$ . Then  $W = \bigcup_{x \in S} [\{x\} \times T_x]$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ , is a resolving dominating set of  $G[H]$  if and only if  $W$  is a locating-dominating set of  $G[H]$ .*

*Proof:* Suppose  $W$  is a resolving dominating set of  $G[H]$ . Then by Theorem 20,  $W$  is a locating-dominating set of  $G[H]$ .

The converse follows from Theorem 20 and Theorem 19.  $\square$

The next result follows immediately from Theorem 20 and Theorem 21.

**Theorem 22.** *Let  $G$  and  $H$  be non-trivial connected graphs with  $\Delta(H) \leq |V(H)| - 2$ . Then  $W = \bigcup_{x \in S} [\{x\} \times T_x]$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ , is a resolving dominating set of  $G[H]$  if and only if*

- (i)  $S = V(G)$ ;
- (ii)  $T_x$  is a locating set of  $H$  for every  $x \in V(G)$ ;
- (iii)  $T_x$  or  $T_y$  is strictly locating set of  $H$  whenever  $x$  and  $y$  are adjacent vertices of  $G$  with  $N_G[x] = N_G[y]$ ; and
- (iv)  $T_x$  or  $T_y$  is locating-dominating set of  $H$  whenever  $x$  and  $y$  are nonadjacent vertices of  $G$  with  $N_G(x) = N_G(y)$ .

**Theorem 23.** *Let  $G$  and  $H$  be non-trivial connected graphs with  $\Delta(H) \leq |V(H)| - 2$ . Then  $W = \bigcup_{x \in S} [\{x\} \times T_x]$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ , is a resolving restrained dominating set of  $G[H]$  if and only if it is a resolving dominating set.*

*Proof:* Let  $W = \bigcup_{x \in S} [\{x\} \times T_x]$  where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$  be a resolving restrained dominating set of  $G[H]$ . Then by Theorem 22, (i), (iii) and (iv) hold and  $T_x$  is a locating set of  $H$  for every  $x \in V(G)$ . Now, let  $S' = \{y \in V(G) : T_y \neq V(H)\}$  and let  $x \in S' \setminus N_G(S')$ . Suppose that  $T_x$  is not a restrained locating set. Then  $\langle V(H) \setminus T_x \rangle$  has an isolated vertex, say  $u$ . Then  $(x, u)$  is an isolated vertex in  $V(G[H]) \setminus W$ , contrary to the assumption that  $W$  is a resolving restrained dominating set of  $G[H]$ . Hence,  $T_x$  is a restrained locating set of  $H$  for all  $x \in S' \setminus N_G(S')$ . Therefore,  $W$  is a resolving dominating set of  $G[H]$ .

For the converse, suppose that  $W = \bigcup_{x \in S} [\{x\} \times T_x]$  is a resolving dominating set of  $G[H]$ . Suppose that  $V(G[H]) = W$ . Then  $W$  is a resolving restrained dominating set of  $G[H]$ . Suppose that  $V(G[H]) \neq W$ . Let  $(y, u) \in V(G[H]) \setminus W$ . Then  $u \notin T_y$ . Hence,  $y \in S'$ . If  $y \in N_G(S')$ , then there exists  $z \in S' \cap N_G(y)$ . Pick any  $v \in V(H) \setminus T_z$ . Then  $(y, u)(z, v) \in E(G[H])$ . If  $y \notin N_G(S')$ , then by (ii), there exists  $p \in (V(H) \setminus T_y) \cap N_G(u)$ . Thus,  $(y, u)(y, p) \in E(G[H])$ . Hence  $\langle V(G[H]) \setminus W \rangle$  has no isolated vertex. Therefore,  $W$  is a resolving restrained dominating set of  $G[H]$ .  $\square$

**Corollary 5.** *Let  $G$  be a connected totally point determining graph of order  $n \geq 3$  and let  $H$  be any non-trivial connected graph. Then*

$$\gamma_{Rr}(G[H]) \leq |V(G)| \cdot \ln(H).$$

*Proof:* Let  $D$  be a minimum locating set of  $H$  and let  $T_x = D$  for each  $x \in V(G)$ . Then  $T_x \neq V(H)$  for all  $x \in V(G)$ , that is,  $S' = V(G)$ . By Theorem 23 and hypothesis,  $W = \bigcup_{x \in V(G)} [\{x\} \times T_x]$  is a resolving restrained dominating set of  $G[H]$ . Thus,  $\gamma_{Rr}(G[H]) \leq |W| = |V(G)| \cdot \ln(H)$ .  $\square$

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