**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 14, No. 3, 2021, 783-787
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Bessel type transform associated with Titchmarsh's Theorem

Balasaheb Bhagaji Waphare[1,*], Yashodha Sanjay Sindhe[1]

[1] *Mathematics Department, MAEER's MIT Arts, Commerce and Science College, Alandi, Pune-412105,Maharashtra, India*

**Abstract.** In this paper we have extended Titchmarsh's theorem for the Bessel transform for function on half-line $[0, \infty)$ in a weighted $L_p$ metric and is studied with the use of Bessel generalized translation.

**2020 Mathematics Subject Classifications**: 42A38, 42B37

**Key Words and Phrases**: Bessel operator, Bessel transform, Bessel generalized translation

## 1. Introduction and Preliminaries

In recent past, integral transforms are widely used to solve various problems in calculus, mechanics, Mathematical physics, engineering and computational mathematics (see [4, 6] the paper by R. Daher, M. EI Hamma and A. EI Houasni, Titehmarsh theorem for the Bessel Transform, MATEMATIKA, 2012, Vol.28, No.2, 127-131, motivated us to prepare this paper .
Titchmarsh ([2], Theorem 84) characterized the set of functions in $L^p(\mathbb{R})$ satisfying the estimate given in the following theorem.

**Theorem 1.** *Let $f(x) \in L^p(\mathbb{R})(1 < p \leq 2)$, and let*

$$\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^p dx = O\left(h^{\alpha p}\right)(0 < \alpha \leq 1)$$

*as $h \to 0$. Then $\mathcal{F}(f)(x) \in L^{\beta}(\mathbb{R})$ for*
$\frac{p}{p+\alpha p-1} < \beta < \frac{p}{p-1}$, *where $\mathcal{F}(f)$ stands for the Fourier transform of f.*

The main objective of this paper is to establish an analog of Theorem 1 in the Bessel type operators setting by means of the Bessel generalized translation. Let $\Delta = \Delta_{a,b} =$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.3986

*Email addresses:* balasahebwaphare@gmail.com (B.B.Waphare), ysindhe@gmail.com (Y.S. Sindhe)

 

$D_{tt}+\frac{a-b}{t}D_t$, be the Bessel type differential operator, where $D_t=\frac{d}{dt}$. By $j_{\frac{a-b-1}{2}}(t)$ denote the Bessel normed function of the first kind. i.e.

$$j_{\frac{a-b-1}{2}}(t)=\frac{2^{\frac{a-b-1}{2}}\Gamma\left(\frac{a-b+1}{2}\right)J_{\frac{a-b-1}{2}}(t)}{t^{\frac{a-b-1}{2}}},$$

where $J_\nu$ is Bessel function of the first kind with $\nu=\frac{a-b-1}{2}$ and $\Gamma(x)$ is the Gamma function (see [1]). The function $y=j_{\frac{a-b-1}{2}}(t)$ satisfies the differential equation $\Delta y+y=0$ with the initial conditions $y(0)=1, y'(0)=0$. The function $j_{\frac{a-b-1}{2}}(t)$ is infinitely differentiable, entire analytic.
Let $L^p_{a,b}(\mathbb{R}_+),(a-b)>0$ and $1<p\leq 2$ be the Banach space of measurable functions f(t) on $\mathbb{R}_+$ with the finite norm.

$$\|f\|=\|f\|_{p,a,b}=\left(\int_0^\infty |f(t)|^p t^{a-b}dt\right)^{1/p}.$$

Consider the Bessel generalized translation $T_h$ in $L^p_{a,b}(\mathbb{R}_+)$ (see [[4],p.121])

$$T_hf(x)=\frac{\Gamma\left(\frac{a-b+1}{2}\right)}{\Gamma(\frac{1}{2})\Gamma\left(\frac{a-b}{2}\right)}\int_0^\pi f\left(\sqrt{x^2+h^2-2xh\cos t}\right)\sin^{a-b-1}tdt, a-b>0, 0\leq h<1$$

which corresponds to the Bessel operator $\Delta_{a,b}$ It is not very difficult to see that

$$T_0f(x)=f(x).$$

If f(x) has a continuous first derivative, then

$$\frac{\partial}{\partial h}T_hf(x)|_{h=0}=0.$$

If it has a continuous second derivative, then $u(x,h)=T_hf(x)$ solves the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{a-b}{x}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial h^2}+\frac{a-b}{h}\frac{\partial u}{\partial h}$$

and

$$u|_{h=0}=f(x), \frac{\partial u}{\partial h}|_{h=0}=0.$$

The operator $T_h$ is linear, homogeneous and continuous. Below are some properties of this operator (see [[4], pp.124-125]):
(i) $T_hj_{\frac{a-b-1}{2}}(\lambda x)=j_{\frac{a-b-1}{2}}(\lambda h)j_{\frac{a-b-1}{2}}(\lambda x)$
(ii) $T_h$ is self-adjoint. If f(x) is continuous function such that $\int_0^\infty x^{a-b}|f(x)|dx<\infty$, and g(x) is continuous and bounded for all $x\geq 0$ then

$$\int_0^\infty (T_hf(x))g(x)x^{a-b}dx=\int_0^\infty f(x)(T_hg(x))x^{a-b}dx$$

(iii) $T_h f(x) = T_x f(h)$
(iv) $\|T_h f - f\| \to 0$ as $h \to 0$.
The Bessel transform defined by the formula (see [1, 3, 4])

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_{\frac{a-b-1}{2}}(\lambda t) t^{a-b} dt, \lambda \in \mathbb{R}_+.$$

The inverse Bessel transform is given by the formula

$$f(t) = \left(2^{\frac{a-b-1}{2}} \Gamma\left(\frac{a-b+1}{2}\right)\right)^{-2} \int_0^\infty \widehat{f}(\lambda) j_{\frac{a-b-1}{2}}(\lambda t) \lambda^{a-b} d\lambda.$$

The following relation connect the Bessel generalized translation, and the Bessel transform in [5], We have

$$\left(\widehat{T_h f}\right)(\lambda) = j_{\frac{a-b-1}{2}}(\lambda h) \widehat{f}(\lambda). \tag{1}$$

For $(a-b) > 0$ , we introduce the Bessel normalized function of the first kind $j_{\frac{a-b-1}{2}}$ defined by

$$j_{\frac{a-b-1}{2}}(x) = \Gamma\left(\frac{a-b+1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma\left(n + \frac{a-b+1}{2}\right)}. \tag{2}$$

Therefore from (2), we have

$$\lim_{x \to 0} \frac{\left(j_{\frac{a-b-1}{2}}(x) - 1\right)}{x^2} \neq 0.$$

By consequence, there exist $C > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Rightarrow |j_{\frac{a-b-1}{2}}(x) - 1| \geq C|x|^2. \tag{3}$$

## 2. An Analog of Titchmarsh's Theorem

In this section we give an analog of Titchmarsh's Theorem 1 (see [[2], Theorem 84]) for the Bessel transform.

**Theorem 2.** *Let $f(x) \in L^p{}_{a,b}(\mathbb{R}_+), (1 < p \leq 2)$, and let*

$$\int_0^\infty |T_h f(x) - f(x)|^p x^{a-b} dx = O(h^{\gamma p})(0 \leq \gamma \leq 2)$$

*as $h \to 0$. Then $\widehat{f}(x) \in L^\beta{}_{a,b}(\mathbb{R}_+)$, for $\frac{p(a-b+1)}{(a-b+1)(p-1)+\gamma p} < \beta \leq \frac{p}{p-1}$, where $0 < \gamma < 1$*

*Proof.* For a fixed h the Bessel transform of $T_h f(x)$ is $j_{\frac{a-b-1}{2}}(hx)\hat{f}(x)$.
Hence the Bessel transform of $T_h f(x) - f(x)$, as a function of x is $(j_{\frac{a-b-1}{2}}(hx) - 1)\hat{f}(x)$.
Thus

$$\int_0^\infty |j_{\frac{a-b-1}{2}}(hx) - 1|^{p'} |\hat{f}(x)|^{p'} x^{a-b} dx < k(p) \left(\int_0^\infty |T_h f(x) - f(x)|^p x^{a-b} dx\right)^{1/(p-1)} < k(p) h^{\gamma p'}$$

Now from (3), we have

$$\int_0^{\eta/h} |hx|^{2p'} |\hat{f}(x)|^{p'} x^{a-b} dx < k(p) h^{\gamma p'}, where\ p' = \frac{p}{p-1}$$

Then

$$\int_0^{\eta/h} x^{2p'} |\hat{f}(x)|^{p'} x^{a-b} dx < k(p) h^{(\gamma-2)p'},$$

Let

$$\varphi(\xi) = \int_1^{\xi} |x^2 \hat{f}(x)|^{\beta} x^{(a-b)p'/\beta} dx.$$

Then, if $\beta < p'$

$$\begin{aligned}
\varphi(\xi) &\leq \left( \int_1^{\xi} |x^2 \hat{f}(x)|^{p'} x^{a-b} dx \right)^{\beta/p'} \left( \int_1^{\xi} dx \right)^{1-\beta/p'} \\
&= O\left( \xi^{(2-\gamma)p' \frac{\beta}{p'}} \xi^{1-\frac{\beta}{p'}} \right) \\
&= O\left( \xi^{2\beta - \gamma\beta + 1 - \frac{\beta}{p'}} \right)
\end{aligned}$$

Hence

$$\begin{aligned}
\int_1^{\xi} |\hat{f}(\xi)|^{\beta} x^{a-b} dx &= \int_1^{\xi} x^{-2\beta-(a-b)\frac{\beta}{p'}} \varphi'(x) x^{a-b} dx \\
&= \xi^{-2\beta-(a-b)\frac{\beta}{p'}} \xi^{a-b} \varphi(\xi) + (2\beta + a - b)\frac{\beta}{p'} - (a-b) \int_1^{\xi} x^{-2\beta-(a-b)\frac{\beta}{p'}+(a-b-1)} \varphi(x) dx \\
&= O\left( \xi^{-2\beta-(a-b)\frac{\beta}{p'}+a-b+1-\gamma\beta+\beta\left(\frac{p+1}{p}\right)} \right) + O\left( \int_1^{\infty} x^{-2\beta-(a-b)\frac{\beta}{p'}+(a-b-1)} x^{1-\gamma\beta+\beta\left(\frac{p+1}{p}\right)} dx \right) \\
&= O\left( \xi^{-2\beta-(a-b)\frac{\beta}{p'}+a-b+1-\gamma\beta+\beta\left(\frac{p+1}{p}\right)} \right)
\end{aligned}$$

and this is bounded as $\xi \to \infty$ if $-2\beta - (a-b)\frac{\beta}{p'} + a - b + 1 - \gamma\beta + \beta\left(\frac{p+1}{p}\right) < 0$
i.e. if

$$\beta > \frac{p(a-b+1)}{(a-b+1)(p-1) + \gamma p}\ .$$

Thus theorem is proved.

## 3. Conclusion

There are many theorems known about to classical Fourier transform can be generalized for the Bessel type transform, among them is Titchmarsh's theorem. We have succussfully generelised this Titchmarsh's Theorem for the Bessel transform in this space $L^p_{a,b}\mathbb{R}_+$

**Remarks:**

1. If we take $a = \alpha + \frac{3}{4},\ b = -\alpha - \frac{1}{4}$ throughout this paper then we obtain the results studied by R. Daher, M. EI Hamma and A.EI Houashi, published in Matematika, 2012, Vol. 28, No.2, 127-131.

2. Authors claim that results studied in this paper are stronger than that of R. Daher, M.EI Hamma and A.EI Houasni.

## References

[1] Levitan B.M. Expansion in fourier series and integrals over bessel functions. *Usperchi. Mat. Nauk.*, 1951.

[2] Titchmarsh E.C.(2005). Introduction to the Theory of Fourier Integrals. *MOSCOW: Clarendon. Oxford,Kormkniga, Moscow.*, 1948.

[3] Trimeche K. Generalized Harmonic Analysis and Wavelet packets. Amsterdam: Gordon and Breach Science publ. *Amsterdam: Gordon and Breach Science publ.*, 2000.

[4] Kipriyanov I. V. singular Elliptic Boundary value problems. *MOSCOW:Nauka.*, 1997.

[5] Abilov V.A., Abilova F.V., and Kerimov M.K. Estimates for the fourier- bessel integral transform in the $l_2(\mathbb{R}_+)$.. *Zh. Vychis. Mat. Mat. Fiz*, 49(7):1158–1166, 2009.

[6] Zhitomirskii Y.A. Cauchy's problem for systems of linear partial differential equations with differential operators of bessel type. *Mat. Sb.*, 36(2):299–310, 1955.