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The double Laplace transform expressed in terms of the Lerch Transcendent

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Abstract. In this paper, the authors derive a formula for the double Laplace transform expressed in terms of the Lerch Transcendent. The log term mixes the variables so that the integral is not separable except for special values of k. The method of proof follows the method used by us to evaluate single integrals. This transform is then used to derive definite integrals in terms of fundamental constants, elementary and special functions. A summary of the results is produced in the form of a table of definite integrals for easy referencing by readers. The majority of the results in the work are new.

2020 Mathematics Subject Classifications: 01A35, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

Key Words and Phrases: Laplace transform, Lerch Transcendent, Contour integral, definite integral

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1. Introduction

In this paper we derive the double Laplace transform given by

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} \log^k \left(\frac{bx}{y}\right) e^{-(sx)^n-(ty)^n} dxdy \tag{1}$$

where the parameters a, k s and t are general complex numbers, $n > Re(p)$ and $Re(p) > 0$. The transform will be used to derive special cases in terms of special functions and fundamental constants. We then tabulate our results of these new integral formulae with respect the parameters of the transform. The table of results is only a subset of the actual domain and range for the transform as the number of variables are too many to compose a full table in this work. The authors however, used their transform to tabulate in their opinion, interesting integral forms and leave the readers to derive other forms if they wish. We present a formal derivation for a definite integral in [8]. In this work we derive

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the double Laplace transform in terms of the Lerch Transcendent [5]. The derivations follow the method used by us in [7]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

We then multiply both sides by a function of x and y, then take a definite double integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (2) by another function of x and y and take the infinite sums of both sides such that the contour integral of both equations are the same.

2. Definite Integral of the Contour Integral

We use the method in [7]. The variable of integration in the contour integral is $\alpha = p + w$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form two equations by replacing y by $\log\left(\frac{ax}{y}\right)$ and multiplying by $x^{p-1}y^{n-p-1}e^{-x^n-y^n}$ then taking the definite integral with respect $x \in [0,\infty)$ and $y \in [0,\infty)$ to get

$$\begin{aligned}
&\frac{1}{k!}\int_0^\infty\int_0^\infty x^{p-1}y^{n-p-1}e^{-x^n-y^n}\log^k\left(\frac{ax}{y}\right)dxdy \\
&= \frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_C w^{-k-1}x^{p-1}y^{n-p-1}e^{-x^n-y^n}\left(\frac{ax}{y}\right)^w dwdxdy \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty w^{-k-1}x^{p-1}y^{n-p-1}e^{-x^n-y^n}\left(\frac{ax}{y}\right)^w dxdydw \\
&= \frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}\csc\left(\frac{\pi(p+w)}{n}\right)}{n^2}dw
\end{aligned} \tag{3}$$

from equation (3.326.2) in [9] and using the reflection formula for Gamma functions. The condition on the left-hand side of equation (3) is $Re(p) > 0$ and $Re(n) > Re(p)$. We are able to switch the order of integration over α, x and y using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,\infty) \times [0,\infty)$.

3. Infinite Sum of the Contour Integral

Using equation (2) and replacing y by $\log(a) + \frac{i\pi(2y+1)}{n}$ then multiplying both sides by $-\frac{2i\pi e^{\frac{i\pi p(2y+1)}{n}}}{n^2}$ we get

$$-\frac{i(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k}e^{\frac{2i\pi py}{n}+\frac{i\pi p}{n}}\left(\frac{1}{2}(2y+1)-\frac{in\log(a)}{2\pi}\right)^{k}}{n^{2}k!}$$
$$=-\frac{1}{2\pi i}\int_{C}\frac{2i\pi w^{-k-1}\exp\left(w\left(\log(a)+\frac{i\pi(2y+1)}{n}\right)+\frac{i\pi p(2y+1)}{n}\right)}{n^{2}}dw \tag{4}$$

We then take the infinite sum over $y\in[0,\infty)$ to get

$$\frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1}e^{\frac{i\pi p}{n}}\Phi\left(e^{\frac{2ip\pi}{n}},-k,\frac{\pi-in\log(a)}{2\pi}\right)}{n^{3}k!}$$
$$=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_{C}\frac{2i\pi w^{-k-1}\exp\left(w\left(\log(a)+\frac{i\pi(2y+1)}{n}\right)+\frac{i\pi p(2y+1)}{n}\right)}{n^{2}}dw$$
$$=-\frac{1}{2\pi i}\int_{C}\sum_{y=0}^{\infty}\frac{2i\pi w^{-k-1}\exp\left(w\left(\log(a)+\frac{i\pi(2y+1)}{n}\right)+\frac{i\pi p(2y+1)}{n}\right)}{n^{2}}dw \tag{5}$$
$$=\frac{1}{2\pi i}\int_{C}\frac{\pi a^{w}w^{-k-1}\csc\left(\frac{\pi(p+w)}{n}\right)}{n^{2}}dw$$

from equation (1.232.3) in [9] and $Im(p+w)<0$.

4. The Lerch Function

We use (9.550) and (9.556) in [9] where $\Phi(z,s,v)$ is the Lerch function which is a generalization of the Hurwitz Zeta and Polylogarithm functions. The Lerch function has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^{n} \tag{6}$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^{t}-z}dt \tag{7}$$

where $Re(v)>0$, or $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$

5. Definite Integral in terms of the Lerch transcendent

Since the right-hand side of equation (3) is equal to equation (5) we may equate the left hand sides along with making the the substitutions $x=sx$, $y=ty$ and $a=bs/t$ and simplifying to get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} \log^k\left(\frac{bx}{y}\right) e^{-(sx)^n-(ty)^n} dxdy \\ = \frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1} e^{\frac{i\pi p}{n}} s^{-p} t^{p-n}}{n^3} \Phi\left(e^{\frac{2ip\pi}{n}}, -k, \frac{\pi - in\log\left(\frac{bt}{s}\right)}{2\pi}\right) \tag{8}$$

6. Derivation of Definite Integrals when $s = t = 1$

In this section we will derive various definite integrals referred to in [8] starting with (45) on p 245 using equation (8) in terms of special functions, fundamental constants and itemizing each as entries. Note in this section $a = b$.

6.1. Derivation of entry 45

Using equation (8) and setting $a = 1$ and $k = 0$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} e^{-x^n-y^n} dxdy = \frac{\pi \csc\left(\frac{\pi p}{n}\right)}{n^2} \tag{9}$$

from entry (2) in Table below (64:12:7) in [6].

6.2. Derivation of new entry 46

Using equation (8) and setting $a = 1$ and $k = 1$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} e^{-x^n-y^n} \log\left(\frac{x}{y}\right) dxdy = -\frac{\pi^2 \cot\left(\frac{\pi p}{n}\right) \csc\left(\frac{\pi p}{n}\right)}{n^3} \tag{10}$$

from entry (3) in Table below (64:12:7) in [6].

6.3. Derivation of new entry 47

Using equation (8) and setting $a = 1$ and $k = 2$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} e^{-x^n-y^n} \log^2\left(\frac{x}{y}\right) dxdy = \frac{\pi^3\left(\cos\left(\frac{2\pi p}{n}\right)+3\right)\csc^3\left(\frac{\pi p}{n}\right)}{2n^4} \tag{11}$$

from entry (4) in Table below (64:12:7) in [6].

6.4. Derivation of new entry 48 in terms of the Hurwitz zeta function

Using equation (8) and setting $a = 1$, $p = 1$ and $n = 2$ and simplifying we get

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} \log^k\left(\frac{x}{y}\right) dxdy = 2^{k-1} e^{\frac{i\pi k}{2}} \pi^{k+1}\left(\zeta\left(-k, \frac{1}{4}\right) - \zeta\left(-k, \frac{3}{4}\right)\right) \tag{12}$$

from equation (64:13:3)in [6].

6.5. Derivation of new entry 49

Using equation (8) and setting $k = -1$, $n = 2$ and $a = -1$ rationalizing the denominator and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{\pi x^{p-1} y^{1-p} e^{-x^2-y^2}}{\log^2\left(\frac{x}{y}\right) + \pi^2} dxdy = \frac{1}{4}\left(4\sin\left(\frac{\pi p}{2}\right) + \pi\cos(\pi p) - 2\sin(\pi p)\log\left(\cot\left(\frac{\pi p}{4}\right)\right)\right) \tag{13}$$

and

$$\int_0^\infty \int_0^\infty \frac{x^{p-1} y^{1-p} e^{-x^2-y^2}\log\left(\frac{x}{y}\right)}{\log^2\left(\frac{x}{y}\right) + \pi^2} dxdy = -\frac{1}{4}\pi\sin(\pi p) + \cos\left(\frac{\pi p}{2}\right) - \frac{1}{2}\cos(\pi p)\log\left(\cot\left(\frac{\pi p}{4}\right)\right) \tag{14}$$

from entry (1) in Table below (64:12:7)

6.6. Derivation of new entry 50 in terms of the Polylogarithm Function

Using equation (8) and replacing a by $e^{\pi i/n}$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} e^{-x^n-y^n} \log^k\left(\frac{e^{\frac{i\pi}{n}} x}{y}\right) dxdy = \frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1} e^{-\frac{i\pi p}{n}} \mathrm{Li}_{-k}\left(e^{\frac{2ip\pi}{n}}\right)}{n^3} \tag{15}$$

from equation (64:12:2) in [6].

6.7. Derivation of new entry 51 in terms of Catalan's constant C

Using equation (8) and setting $a = k = 1$ and replacing p by $n/2$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{\frac{n}{2}-1} y^{\frac{n}{2}-1} e^{-x^n-y^n} \log\left(\frac{x}{y}\right)\log\left(\log\left(\frac{x}{y}\right)\right) dxdy = -\frac{4i\pi C}{n^3} \tag{16}$$

from equation (9.73) in [9].

6.8. Derivation of new entry 52

Using equation (8) and setting $k = 3$ and $a = -1$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{p-1} y^{n-p-1} e^{-x^n-y^n} \log^3\left(\frac{x}{y}\right) dxdy = -\frac{\pi^4\left(23\cos\left(\frac{\pi p}{n}\right) + \cos\left(\frac{3\pi p}{n}\right)\right)\csc^4\left(\frac{\pi p}{n}\right)}{4n^5} \tag{17}$$

6.9. Derivation of new entry 53 in terms of the Log-gamma Function

Using equation (8) and first replacing p by $n/2$ followed by taking the first partial derivative with respect to k and then setting $k = 0$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{\frac{n}{2}-1} y^{\frac{n}{2}-1} e^{-x^n-y^n} \log\left(\log\left(\frac{ax}{y}\right)\right) dxdy = \frac{2\pi \log\left(\frac{2\sqrt{\pi}\sqrt{\frac{i}{n}}\Gamma\left(\frac{3}{4}-\frac{in\log(a)}{4\pi}\right)}{\Gamma\left(\frac{\pi - in\log(a)}{4\pi}\right)}\right)}{n^2} \tag{18}$$

from equation (1.10.10) in [2].

6.10. Derivation of new entry 54

Using equation (8) and first replacing p by $n/2$ followed by taking the first partial derivative with respect to a and then setting $k = 2$ and $a = 1$ and simplifying we get

$$\int_0^\infty \int_0^\infty x^{\frac{n}{2}-1} y^{\frac{n}{2}-1} e^{-x^n-y^n} \log\left(\frac{x}{y}\right) dxdy = 0 \tag{19}$$

where $n \in \mathbb{C}$.

6.11. Derivation of new entry 55 in terms of Euler's constant

Using equation (8) and setting $p = 1/2$, $a = -1$ and $n = 1$ followed by taking the first partial derivative with respect to k and then setting $k = -1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y} \log\left(\log\left(-\frac{x}{y}\right)\right)}{\sqrt{x}\sqrt{y}\log\left(-\frac{x}{y}\right)} dxdy = \frac{1}{2}\log(2)\left(2i\gamma + \pi - i\log\left(8\pi^2\right)\right) \tag{20}$$

from equation (9.73) in [9].

6.12. Derivation of new entry 56 in terms of the derivative of the Hurwitz zeta Function

Using equation (8) and setting $p = 1$, $a = 1$ and $n = 2$ followed by taking the first partial derivative with respect to k and then setting $k = 2$ and simplifying we get

$$\begin{aligned}\int_0^\infty \int_0^\infty & e^{-x^2-y^2} \log^2\left(\frac{x}{y}\right) \log\left(\log\left(\frac{x}{y}\right)\right) dxdy \\ &= \frac{1}{32}\pi^3\left(64\left(\zeta'\left(-2,\frac{1}{4}\right) - \zeta'\left(-2,\frac{3}{4}\right)\right) + i\pi + \log\left(4\pi^2\right)\right)\end{aligned} \tag{21}$$

from equation (64:10:1) in [6].

6.13. Derivation of new entry 57 in terms of the Hurwitz zeta Function

Using equation (8) and setting $k = 1/2$, $a = p = 1$ and $n = 2$ and simplifying we get

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\sqrt{\log\left(\frac{x}{y}\right)}dxdy = \left(\frac{1}{2}+\frac{i}{2}\right)\pi^{3/2}\left(\zeta\left(-\frac{1}{2}\frac{1}{4}\right)-\zeta\left(-\frac{1}{2},\frac{3}{4}\right)\right) \tag{22}$$

from equation (64:13:3) in [6].

6.14. Derivation of new entry 58 in terms of $\log(2)$, $\zeta(3)$, Glaisher's constant A and π

Using equation (8) and setting $p = 1/2$, $n = 1$ and $a = -1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\log^k\left(-\frac{x}{y}\right)}{\sqrt{x}\sqrt{y}}dxdy = -i^k\left(2^{k+1}-1\right)(2\pi)^{k+1}\zeta(-k) \tag{23}$$

from entry (3) in Table below (64:12:7), equation (64:13:4) and entry (2) in Table below (64:7) in [6].

Next we apply L'Hopital's rule to the right-hand side of equation (23) as $k \to -1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}}{\sqrt{x}\sqrt{y}\left(\log^2\left(\frac{x}{y}\right)+\pi^2\right)}dxdy = \frac{\log(2)}{\pi} \tag{24}$$

Next using equation (23) and taking the first partial derivative with respect to k and setting $k = 2$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\log^2\left(-\frac{x}{y}\right)\log\left(\log\left(-\frac{x}{y}\right)\right)}{\sqrt{x}\sqrt{y}}dxdy = 14\pi\zeta(3) \tag{25}$$

Next using equation (23) and taking the first partial derivative with respect to k and setting $k = 1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\log\left(-\frac{x}{y}\right)\log\left(\log\left(-\frac{x}{y}\right)\right)}{\sqrt{x}\sqrt{y}}dxdy = i\pi^2\log\left(\frac{4i\sqrt[3]{2}e\pi}{A^{12}}\right) \tag{26}$$

6.15. Derivation of new entry 59 in terms of digamma function

Using equation (8) and setting $p = n/2$, $k = -1$ and $a = e^{ai}$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{n}{2}-1}y^{\frac{n}{2}-1}e^{-x^n-y^n}}{a^2+\log^2\left(\frac{x}{y}\right)}dxdy = \frac{\psi^{(0)}\left(\frac{an+3\pi}{4\pi}\right)-\psi^{(0)}\left(\frac{an+\pi}{4\pi}\right)}{2an} \tag{27}$$

7. Derivation of Definite Integrals when $s \neq t$

The main related functions of the Lerch function and transcendent are the Hurwitz zeta function $\zeta(s,a)$ from equation (64:12:1) in [6], Jonquiére's function $\phi(z,s)$ from equation (64:12:2) in [6], and the Dirichlet L-functions $L(s,\chi)$ from entry (3) in Table below (64:12:7) in [6]. In this section we will derive definite integrals in terms of these special functions along with a few other examples.

7.1. Derivation entry 60 in terms of the Hurwitz zeta function $\zeta(k,z)$

Using equation (8) and replacing n by $2p$ and simplifying we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty x^{p-1}y^{p-1}\log^k\left(\frac{bx}{y}\right)e^{-(sx)^{2p}-(ty)^{2p}}dxdy\\&=\frac{2^{k-1}\pi^{k+1}\left(\frac{i}{p}\right)^k s^{-p}t^{-p}\left(\zeta\left(-k,\frac{\pi-2ip\log\left(\frac{bt}{s}\right)}{4\pi}\right)-\zeta\left(-k,\frac{3}{4}-\frac{ip\log\left(\frac{bt}{s}\right)}{2\pi}\right)\right)}{p^2}\end{aligned} \tag{28}$$

from equation (64:13:3) in [6].

7.2. Derivation of entry 61 in terms of the Polylogarithm function Li_{-k}

Using equation (8) and replacing b by $\frac{e^{\frac{i\pi}{n}}s}{t}$ and simplifying we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty x^{p-1}y^{n-p-1}e^{-(sx)^n-(ty)^n}\log^k\left(\frac{e^{\frac{i\pi}{n}}sx}{ty}\right)dxdy\\&=\frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1}e^{-\frac{i\pi p}{n}}s^{-p}t^{p-n}\mathrm{Li}_{-k}\left(e^{\frac{2ip\pi}{n}}\right)}{n^3}\end{aligned} \tag{29}$$

from equation (64:12:2) in [6].

7.3. Derivation of entry 62 in terms of the Hypergeometric function ${}_2F_1(1,v;v+1;z)$

The relationship between the Lerch function and the Hypergeometric function is given by

$$\Phi(z,1,v)=\frac{{}_2F_1(1,v;v+1;z)}{v},\quad |z|<1 \tag{30}$$

Using equation (8) and setting $k=-1$ and simplifying we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{x^{p-1}y^{n-p-1}e^{-(sx)^n-(ty)^n}}{\log\left(\frac{bx}{y}\right)}dxdy \\
&= -\frac{e^{\frac{i\pi p}{n}}s^{-p}t^{p-n}\Phi\left(e^{\frac{2ip\pi}{n}},1,\frac{\pi-in\log\left(\frac{bt}{s}\right)}{2\pi}\right)}{n} \\
&= -\frac{2\pi e^{\frac{i\pi p}{n}}s^{-p}t^{p-n}\,{}_2F_1\left(1,\frac{\pi-in\log\left(\frac{bt}{s}\right)}{2\pi};\frac{3}{2}-\frac{in\log\left(\frac{bt}{s}\right)}{2\pi};e^{\frac{2ip\pi}{n}}\right)}{n\left(\pi-in\log\left(\frac{bt}{s}\right)\right)}
\end{aligned} \tag{31}$$

from equation (1.11.10) in [2], where $Im(p/n) > 0$.

7.4. Derivation of entry 63 in terms of the Harmonic number function H_n

Using equation (8) and replacing n by $2p$ and setting $k = -1$ and simplifying we get

$$\int_0^\infty\int_0^\infty \frac{x^{p-1}y^{p-1}e^{-(sx)^{2p}-(ty)^{2p}}}{\log\left(\frac{bx}{y}\right)}dxdy = \frac{is^{-p}t^{-p}\left(H_{\frac{-2ip\log\left(\frac{bt}{s}\right)-3\pi}{4\pi}}-H_{-\frac{2ip\log\left(\frac{bt}{s}\right)+\pi}{4\pi}}\right)}{4p} \tag{32}$$

from equations (1.8.6) and (1.11.10) in [2].

7.5. Derivation of entry 64 in terms of the zeta function of Riemann $\zeta(s)$ and Hurwitz zeta $\zeta(s,a)$

Using equation (28) and replacing b by $\frac{e^{\frac{3i\pi}{2p}}s}{t}$ and simplifying we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{p-1}y^{p-1}e^{-(sx)^{2p}-(ty)^{2p}}\log^k\left(\frac{e^{\frac{3i\pi}{2p}}sx}{ty}\right)dxdy \\
&= \frac{2^{k-1}\pi^{k+1}\left(\frac{i}{p}\right)^k s^{-p}t^{-p}\left(\zeta(-k)-\zeta\left(-k,\frac{3}{2}\right)\right)}{p^2}
\end{aligned} \tag{33}$$

from equations (1.10.1) and (1.12.1) in [2].

7.6. Derivation of entry 65 in terms of the zeta function of Riemann $\zeta(s)$

Using equation (28) and replacing n by $2p$ and b by $\frac{e^{\frac{i\pi}{2p}}s}{t}$ and simplifying we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{p-1}y^{p-1}e^{-(sx)^{2p}-(ty)^{2p}}\log^k\left(\frac{e^{\frac{i\pi}{2p}}sx}{ty}\right)dxdy \\
&= -\frac{1}{2p^2}\left(2^{k+1}-1\right)\pi^{k+1}\left(\frac{i}{p}\right)^k\zeta(-k)s^{-p}t^{-p}
\end{aligned} \tag{34}$$

from equations (1.12.1) in [2].

8. Derivation of definite integrals with $\log\left(\frac{x}{y}\right)$ in the denominator

We will form two equations and take their difference. Firstly, using equation (8) we replace p by $p+q$ to form the first equation. For the second, again using equation (8) we replace p by $p-q$. Next we take the difference of these two equations setting $k=-1$, $b=1$ and simplifying to get

$$\int_0^\infty \int_0^\infty \frac{x^{p-q-1}\left(y^{2q}-x^{2q}\right) y^{n-p-q-1} e^{-(sx)^n-(ty)^n}}{\log\left(\frac{x}{y}\right)} dxdy$$
$$= \frac{s^{-p-q} t^{-n+p-q}}{n} \left(t^{2q} e^{\frac{i\pi(p+q)}{n}} \Phi\left(e^{\frac{2i\pi(p+q)}{n}}, 1, \frac{\pi - in\log\left(\frac{t}{s}\right)}{2\pi}\right) - s^{2q} e^{\frac{i\pi(p-q)}{n}} \Phi\left(e^{\frac{2i\pi(p-q)}{n}}, 1, \frac{\pi - in\log\left(\frac{t}{s}\right)}{2\pi}\right)\right) \tag{35}$$

from entry (1) in Table below (64:12:7) and equations (58:4:4) and (58:12:2) in [6].

8.1. Derivation of entry 66 in terms of the logarithmic function

Using equation (35) and setting $s=1, t=1, p=1/2, q=1/3, n=1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6} y^{5/6} \log\left(\frac{x}{y}\right)} dxdy = \log\left(7+4\sqrt{3}\right) \tag{36}$$

from Table (18-1) in [6].

8.2. Derivation of entry 67 in terms of the hyperbolic cotangent function

Using equation (35) and setting $s=1, t=1, p=1/2, q=1/3, n=2$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{\sqrt[6]{y} e^{-x^2-y^2}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6} \log\left(\frac{x}{y}\right)} dxdy = \coth^{-1}\left(\sqrt{2}\right) \tag{37}$$

from Table (18-1) in [6].

8.3. Derivation of entry 68 in terms of the hyperbolic cotangent function

Using equation (35) and setting $s=1, t=1, p=1/2, q=1/4, n=1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(\sqrt{x}-\sqrt{y}\right)}{x^{3/4}y^{3/4}\log\left(\frac{x}{y}\right)}dxdy = 2\coth^{-1}\left(\sqrt{2}\right) \tag{38}$$

from Table (18-1) in [6].

8.4. Derivation of entry 69 in terms of the logarithmic function

Using equation (35) and setting $s=1, t=1, p=1/2, q=-1/3, n=1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(y^{2/3}-x^{2/3}\right)}{x^{5/6}y^{5/6}\log\left(\frac{x}{y}\right)}dxdy = -2\log\left(2+\sqrt{3}\right) \tag{39}$$

from Table (18-1) in [6].

8.5. Derivation of entry 70 in terms of the logarithmic function

Using equation (35) and setting $s=1, t=1, p=1/2, q=1/4, n=3$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{y^{5/4}e^{-x^3-y^3}\left(\sqrt{x}-\sqrt{y}\right)}{x^{3/4}\log\left(\frac{x}{y}\right)}dxdy = -\frac{1}{6}\log\left(5-2\sqrt{6}\right) \tag{40}$$

8.6. Derivation of entry 71 in terms of the hyperbolic tangent function

Using equation (35) and setting $s=1, t=1, p=1/4, q=i/3, n=1$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)}{x^{13/12}y^{7/12}\log\left(\frac{x}{y}\right)}dxdy = 2\tanh^{-1}\left(\sqrt{\frac{3}{2}}\right) \tag{41}$$

from Table (18-1) in [6].

8.7. Derivation of entry 72 in terms of the hypergeometric function

Using equation (35) and setting $s=1, t=1, p=1/4, q=1/3, n=3/2$ and simplifying we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{e^{-x^{3/2}-y^{3/2}}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6}\sqrt[3]{y}\log\left(\frac{x}{y}\right)}dxdy\\&=-\frac{4}{3}\left((-1)^{5/9}\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};e^{-\frac{8i\pi}{9}}\right)-\sqrt[9]{-1}\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};e^{\frac{2i\pi}{9}}\right)\right)\end{aligned} \tag{42}$$

from equations (58:4:4) and (58:12:2) in [6].

9. The general case of the difference of the double Laplace transform

We will form two equations and take their difference. Firstly, using equation (8) we replace p by $p+q$ to form the first equation. For the second, again using equation (8) we replace p by $p-q$. Next we take the difference of these two equations and simplifying to get

$$\begin{aligned}&\int_0^\infty\int_0^\infty x^{p-q-1}\left(y^{2q}-x^{2q}\right)y^{n-p-q-1}\log^k\left(\frac{bx}{y}\right)e^{-(sx)^n-(ty)^n}dxdy\\&=\frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1}s^{-p-q}t^{-n+p-q}}{n^3}\left(s^{2q}e^{\frac{i\pi(p-q)}{n}}\Phi\left(e^{\frac{2i\pi(p-q)}{n}},-k,\frac{\pi-in\log\left(\frac{bt}{s}\right)}{2\pi}\right)\right.\\&\qquad\left.-t^{2q}e^{\frac{i\pi(p+q)}{n}}\Phi\left(e^{\frac{2i\pi(p+q)}{n}},-k,\frac{\pi-in\log\left(\frac{bt}{s}\right)}{2\pi}\right)\right)\end{aligned} \tag{43}$$

9.1. Derivation of entry 73 in terms of the Lerch transcendent

Using equation (43) and setting $s=1,t=1,n=1,p=1/2,q=1/3,b=1,k=-1/2$ and simplifying we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6}y^{5/6}\sqrt{\log\left(\frac{x}{y}\right)}}dxdy\\&=\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi}\left(\left(\sqrt{3}-i\right)\Phi\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},\frac{1}{2},\frac{1}{2}\right)+\left(\sqrt{3}+i\right)\Phi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},\frac{1}{2},\frac{1}{2}\right)\right)\end{aligned} \tag{44}$$

9.2. Derivation of entry 74 in terms of the Lerch transcendent

Using equation (43) and setting $s=1,t=1,p=1/2,q=1/3,n=1,k=1/2,b=1$ and simplifying we get

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)\sqrt{\log\left(\frac{x}{y}\right)}}{x^{5/6}y^{5/6}}dxdy$$
$$= (1+i)\pi^{3/2}\left(\left(1+i\sqrt{3}\right)\Phi\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},-\frac{1}{2},\frac{1}{2}\right)+i\left(\sqrt{3}+i\right)\Phi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},-\frac{1}{2},\frac{1}{2}\right)\right) \tag{45}$$

9.3. Derivation of entry 75 in terms of the Lerch transcendent

Using equation (43) and setting $s=1,t=1,p=1/2,q=1/3,n=2,k=2,b=2$ and simplifying we get

$$\int_0^\infty\int_0^\infty \frac{\sqrt[6]{y}e^{-x^2-y^2}\left(x^{2/3}-y^{2/3}\right)\log^2\left(\frac{2x}{y}\right)}{x^{5/6}}dxdy = -\frac{\pi\left(39\pi^2+4\log^2(2)-20\pi\log(2)\right)}{4\sqrt{2}} \tag{46}$$

from (64:12:4) in [6].

9.4. Derivation of entry 76 in terms of the Lerch transcendent

Using equation (43) and setting $s=1,t=1,p=1/2,q=1/3,n=1,b=1$ followed by taking the first partial derivative with respect to k then setting $k=0$ and simplifying we get

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}y^{5/6}}dxdy$$
$$= \pi\left(\left(-1-i\sqrt{3}\right)\Phi'\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},0,\frac{1}{2}\right)+\left(1-i\sqrt{3}\right)\Phi'\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},0,\frac{1}{2}\right)\right) \tag{47}$$

9.5. Derivation of entry 77 in terms of the Lerch transcendent

Using equation (43) and setting $s=1,t=1,p=1/2,q=1/3,n=1,b=1$ followed by taking the first partial derivative with respect to k then setting $k=1$ and simplifying we get

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)\log\left(\frac{x}{y}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}y^{5/6}}dxdy$$
$$= 2\pi^2\left(\left(\sqrt{3}-i\right)\Phi'\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},-1,\frac{1}{2}\right)+\left(\sqrt{3}+i\right)\Phi'\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},-1,\frac{1}{2}\right)\right. \tag{48}$$
$$\left.+\sqrt{3}\log(4)+2\sqrt{3}\log(i\pi)\right)$$

9.6. Derivation of entry 78 in terms of the Lerch transcendent

Using equation (43) and setting $s = t, p = 1/2, q = 1/3, n = 1, b = 1, k = 1$ followed by taking the integral with respect to $t \in [1, 2]$ and simplifying we get

$$\int_0^\infty \int_0^\infty \frac{e^{-2(x+y)}\left(e^{x+y}-1\right)\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\log\left(\frac{x}{y}\right)}{x^{5/6}y^{5/6}\left(x^{2/3}-\sqrt[3]{x}\sqrt[3]{y}+y^{2/3}\right)}dxdy = \sqrt{3}\pi^2\log(16) \tag{49}$$

from entry (3) in Table below (64:12:7) in [6].

9.7. Derivation of entry 79 in terms of the Lerch transcendent

Using equation (43) and setting $s = 1/3, t = 1/2, p = 1/2, q = 1/3, n = 1, b = 1, k = 1/2$ and simplifying we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{\frac{1}{6}(-2x-3y)}\left(x^{2/3}-y^{2/3}\right)\sqrt{\log\left(\frac{x}{y}\right)}}{x^{5/6}y^{5/6}}dxdy \\ &= (2-2i)\sqrt[6]{-6}\pi^{3/2}\left((-3)^{2/3}\Phi\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},-\frac{1}{2},\frac{\pi-i\log\left(\frac{3}{2}\right)}{2\pi}\right)-2^{2/3}\Phi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},-\frac{1}{2},\frac{\pi-i\log\left(\frac{3}{2}\right)}{2\pi}\right)\right)\end{aligned} \tag{50}$$

9.8. Derivation of entry 80 in terms of the Lerch transcendent

Using equation (43) and setting $s = 1/3, t = 1/2, p = 1/2, q = 1/3, n = 1, b = 1, k = -1/2$ and simplifying we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{\frac{1}{6}(-2x-3y)}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6}y^{5/6}\sqrt{\log\left(\frac{x}{y}\right)}}dxdy \\ &= (1+i)\sqrt[6]{-6}\sqrt{\pi}\left(2^{2/3}\Phi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},\frac{1}{2},\frac{\pi-i\log\left(\frac{3}{2}\right)}{2\pi}\right)-(-3)^{2/3}\Phi\left(\frac{1}{2}-\frac{i\sqrt{3}}{2},\frac{1}{2},\frac{\pi-i\log\left(\frac{3}{2}\right)}{2\pi}\right)\right)\end{aligned} \tag{51}$$

10. Some special cases of the double Laplace transform involving the nested logarithm function

In this section we will use a special case the double Laplace transform to derive definite integrals involving the special functions. We proceed by using equation (43) and replacing p by $2q$ and setting $n = s = t = 1$ and simplifying to get

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{q-1}y^{-3q}e^{-x-y}\left(y^{2q}-x^{2q}\right)\log^k\left(\frac{bx}{y}\right)dxdy \\ &= (2i\pi)^{k+1}e^{i\pi q}\left(e^{2i\pi q}\Phi\left(e^{6i\pi q},-k,\frac{\pi-i\log(b)}{2\pi}\right)-\Phi\left(e^{2i\pi q},-k,\frac{\pi-i\log(b)}{2\pi}\right)\right)\end{aligned} \tag{52}$$

10.1. Derivation of entry 81 involving the Log-gamma function

Using equation (52) setting $b = 1$ and $q = 1/6$ followed by taking the first partial derivative by k then setting $k = 0$ and simplifying to get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}\sqrt{y}}dxdy\\&=\frac{1}{2}\pi\left(4(-1)^{2/3}\Phi'\left(\sqrt[3]{-1},0,\frac{1}{2}\right)+i\pi+\log\left(\frac{81\pi^2\Gamma\left(-\frac{3}{4}\right)^4}{\Gamma\left(-\frac{1}{4}\right)^4}\right)\right)\end{aligned} \tag{53}$$

from equation (1.10.10) in [2].

10.2. Derivation of entry 82 involving the Stieltjes constant (γ_n)

Using equation (52) setting $b = 1$ and $q = 1/6$ followed by taking the first partial derivative by k and simplifying to get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log\left(\log\left(\frac{x}{y}\right)\right)\log^k\left(\frac{x}{y}\right)}{x^{5/6}\sqrt{y}}dxdy\\&=2^k e^{\frac{i\pi k}{2}}\pi^{k+1}\left(2\left((-1)^{2/3}\Phi'\left(\sqrt[3]{-1},-k,\frac{1}{2}\right)+2^k\left(\zeta'\left(-k,\frac{1}{4}\right)-\zeta'\left(-k,\frac{3}{4}\right)\right)\right)\right.\\&\left.+\sqrt[6]{-1}(\pi-2i\log(2\pi))\Phi\left(\sqrt[3]{-1},-k,\frac{1}{2}\right)-i2^k(\pi-2i\log(4\pi))\left(\zeta\left(-k,\frac{1}{4}\right)-\zeta\left(-k,\frac{3}{4}\right)\right)\right)\end{aligned} \tag{54}$$

Next we apply L'Hopitals rule to the right-hand side as $k \to -1$ and simplifying to get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy\\&=-\frac{1}{2}i\left(2(-1)^{2/3}\Phi'\left(\sqrt[3]{-1},1,\frac{1}{2}\right)+2\sqrt[6]{-1}\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};\sqrt[3]{-1}\right)(\pi-2i\log(2\pi))\right.\\&\left.-\gamma_1\left(\frac{1}{4}\right)+\gamma_1\left(\frac{3}{4}\right)+\frac{1}{2}(2\log(4\pi)+i\pi)\left(\psi^{(0)}\left(\frac{1}{4}\right)-\psi^{(0)}\left(\frac{3}{4}\right)\right)\right)\end{aligned} \tag{55}$$

Next we simplify the right-hand side in terms of the Stieltjes constant (γ_n) and simplifying to get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy\\&=\frac{1}{4}\left(4\sqrt[6]{-1}\Phi'\left(\sqrt[3]{-1},1,\frac{1}{2}\right)+2i\gamma_1\left(\frac{1}{4}\right)-2i\gamma_1\left(\frac{3}{4}\right)\right.\\&\left.-\pi^2+2i\pi\log(4\pi)-4i\pi\tanh^{-1}\left(\sqrt[6]{-1}\right)-8\log(2\pi)\tanh^{-1}\left(\sqrt[6]{-1}\right)\right)\end{aligned}\tag{56}$$

from equation (7.6) in [1].

10.3. Derivation of entry 83 involving the Catalan's constant (C)

Using equation (52) setting $b=1$ and $q=1/6$ followed by taking the first partial derivative by k and setting $k=1$ and simplifying to get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log\left(\frac{x}{y}\right)\log\left(\log\left(\frac{x}{y}\right)\right)}{x^{5/6}\sqrt{y}}dxdy\\&=\pi\left(\pi\left(-2\left(\sqrt{3}+i\right)\Phi'\left(\sqrt[3]{-1},-1,\frac{1}{2}\right)-i\sqrt{3}\pi-2\sqrt{3}\log(2\pi)\right)+4iC\right)\end{aligned}\tag{57}$$

from (9.73) in [9].

10.4. Derivation of entry 84 involving the Polylogarithm function

Using equation (52) setting $b=-1$ and $q=1/6$ followed by taking the first partial derivative by k and setting $k=1$ and simplifying to get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)\log^k\left(-\frac{x}{y}\right)}{x^{5/6}\sqrt{y}}dxdy\\&=i^{k+2}(2\pi)^{k+1}\left(\left(1-2^{k+1}\right)\zeta(-k)-e^{-\frac{2i\pi}{3}}\mathrm{Li}_{-k}\left(e^{\frac{i\pi}{3}}\right)\right)\end{aligned}\tag{58}$$

Next we use L'Hopital's rule on the right hand-side as $k\to -1$ and simplifying to get

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\left(\sqrt[3]{y}-\sqrt[3]{x}\right)}{x^{5/6}\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy=\frac{1}{2}i\left(\pi-4\tan^{-1}\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\right)\tag{59}$$

11. A general case of definite integrals involving $\log\left(\frac{x}{y}\right)$ in the denominator

In this section we will derive the double Laplace transform and derive a few examples illustrating this form. This form was derived by Gröbner and Hofreiter [3] and Bierens de Haan [4]. Using equation (52) and setting $k = -1, b = s = t = 1$ and simplifying we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{e^{-x^n-y^n}x^{p-q-1}\left(y^{2q}-x^{2q}\right)y^{n-p-q-1}}{\log\left(\frac{x}{y}\right)}dxdy \\
&= \frac{2}{n}\left(\tanh^{-1}\left(e^{\frac{i\pi(p-iq)}{n}}\right)-\tanh^{-1}\left(e^{\frac{i\pi(p+iq)}{n}}\right)\right) \\
&= \frac{1}{n}\log\left(\tan\left(\frac{\pi p}{2n}-\frac{\pi q}{2n}\right)\cot\left(\frac{\pi p}{2n}+\frac{\pi q}{2n}\right)\right)
\end{aligned} \tag{60}$$

11.1. Derivation of entry 85 involving the logarithm, cotangent and tangent functions

Using equation (60) setting $p = 1, q = 1/2, n = 2$ and simplifying to get

$$\int_0^\infty\int_0^\infty \frac{e^{-x^2-y^2}(x-y)}{\sqrt{x}\sqrt{y}\log\left(\frac{x}{y}\right)}dxdy = \log\left(\cot\left(\frac{\pi}{8}\right)\right) \tag{61}$$

11.2. Derivation of entry 86 involving the logarithm, cotangent and tangent functions

Using equation (60) setting $p = 1, q = 1/3, n = 2$ and simplifying to get

$$\int_0^\infty\int_0^\infty \frac{e^{-x^2-y^2}\left(x^{2/3}-y^{2/3}\right)}{\sqrt[3]{x}\sqrt[3]{y}\log\left(\frac{x}{y}\right)}dxdy = \frac{\log(3)}{2} \tag{62}$$

11.3. Derivation of entry 87 involving the logarithm, cotangent and tangent functions

Using equation (60) setting $p = 1/2, q = 1/6, n = 3$ and simplifying to get

$$\int_0^\infty\int_0^\infty \frac{y^{4/3}e^{-x^3-y^3}\left(\sqrt[3]{x}-\sqrt[3]{y}\right)}{x^{2/3}\log\left(\frac{x}{y}\right)}dxdy = \frac{1}{3}\log\left(\tan\left(\frac{\pi}{9}\right)\cot\left(\frac{\pi}{18}\right)\right) \tag{63}$$

11.4. Derivation of entry 88 involving the hyperbolic, cotangent and tangent functions

Using equation (52) setting $k = -1, b = 1$, replacing s by t and simplifying to get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{x^{q-1} y^{n-3q-1} \left(y^{2q} - x^{2q}\right) e^{-(tx)^n-(ty)^n}}{\log\left(\frac{x}{y}\right)} dxdy \\ &= -\frac{2t^{-n}}{n}\left(\tanh^{-1}\left(e^{\frac{i\pi q}{n}}\right) - \tanh^{-1}\left(e^{\frac{3i\pi q}{n}}\right)\right)\end{aligned} \tag{64}$$

Next setting $t = 8, n = 3/2, q = 1/4$, comparing real and imaginary parts simplifying we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{e^{-16\sqrt{2}\left(x^{3/2}+y^{3/2}\right)} \left(\sqrt{y} - \sqrt{x}\right)}{x^{3/4}\sqrt[4]{y}\log\left(\frac{x}{y}\right)} dxdy \\ &= \frac{i\left(\pi - 4\cot^{-1}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right)}{48\sqrt{2}} \\ &= -\frac{\log\left(2+\sqrt{3}\right)}{24\sqrt{2}}\end{aligned} \tag{65}$$

12. Table of Definite Integrals

In this section we create a table to summarize our results in sections (6), (7) and (8). A table of integrals is a more compact way of showcasing our results and makes the list of integral formulae easier to digest from the readers' perspective.

Table 1: Table of definite integrals

$f(x,y)$	$\int_0^\infty \int_0^\infty f(x,y)dxdy$
$x^{p-1}y^{n-p-1}e^{-x^n-y^n}$	$\frac{\pi \csc\left(\frac{\pi p}{n}\right)}{n^2}$
$x^{p-1}y^{n-p-1}e^{-x^n-y^n} \log\left(\frac{x}{y}\right)$	$-\frac{\pi^2 \cot\left(\frac{\pi p}{n}\right)\csc\left(\frac{\pi p}{n}\right)}{n^3}$
$x^{p-1}y^{n-p-1}e^{-x^n-y^n} \log^2\left(\frac{x}{y}\right)$	$\frac{\pi^3\left(\cos\left(\frac{2\pi p}{n}\right)+3\right)\csc^3\left(\frac{\pi p}{n}\right)}{2n^4}$
$e^{-x^2-y^2}\log^k\left(\frac{x}{y}\right)$	$2^{k-1}e^{\frac{i\pi k}{2}}\pi^{k+1}\left(\zeta\left(-k,\frac{1}{4}\right)-\zeta\left(-k,\frac{3}{4}\right)\right)$
$\frac{\pi x^{p-1}y^{1-p}e^{-x^2-y^2}}{\log^2\left(\frac{x}{y}\right)+\pi^2}$	$\frac{1}{4}\left(4\sin\left(\frac{\pi p}{2}\right)+\pi\cos(\pi p)-2\sin(\pi p)\log\left(\cot\left(\frac{\pi p}{4}\right)\right)\right)$
$\frac{x^{p-1}y^{1-p}e^{-x^2-y^2}\log\left(\frac{x}{y}\right)}{\log^2\left(\frac{x}{y}\right)+\pi^2}$	$-\frac{1}{4}\pi\sin(\pi p)+\cos\left(\frac{\pi p}{2}\right)-\frac{1}{2}\cos(\pi p)\log\left(\cot\left(\frac{\pi p}{4}\right)\right)$
$x^{p-1}y^{n-p-1}e^{-x^n-y^n}\log^k\left(\frac{e^{\frac{i\pi}{n}}x}{y}\right)$	$\frac{(2\pi)^{k+1}\left(\frac{i}{n}\right)^{k-1}e^{-\frac{i\pi p}{n}}\text{Li}_{-k}\left(e^{\frac{2ip\pi}{n}}\right)}{n^3}$
$x^{\frac{n}{2}-1}y^{\frac{n}{2}-1}e^{-x^n-y^n}\log\left(\frac{x}{y}\right)\log\left(\log\left(\frac{x}{y}\right)\right)$	$-\frac{4i\pi C}{n^3}$
$x^{p-1}y^{n-p-1}e^{-x^n-y^n}\log^3\left(\frac{x}{y}\right)$	$-\frac{\pi^4\left(23\cos\left(\frac{\pi p}{n}\right)+\cos\left(\frac{3\pi p}{n}\right)\right)\csc^4\left(\frac{\pi p}{n}\right)}{4n^5}$
$x^{\frac{n}{2}-1}y^{\frac{n}{2}-1}e^{-x^n-y^n}\log\left(\log\left(\frac{ax}{y}\right)\right)$	$\frac{2\pi\log\left(\frac{2\sqrt{\pi}\sqrt{\frac{i}{n}}\Gamma\left(\frac{3}{4}-\frac{in\log(a)}{4\pi}\right)}{\Gamma\left(\frac{\pi-in\log(a)}{4\pi}\right)}\right)}{n^2}$
$x^{\frac{n}{2}-1}y^{\frac{n}{2}-1}e^{-x^n-y^n}\log\left(\frac{x}{y}\right)$	0
$\frac{e^{-x-y}\log\left(\log\left(-\frac{x}{y}\right)\right)}{\sqrt{x}\sqrt{y}\log\left(-\frac{x}{y}\right)}$	$\frac{1}{2}\log(2)\left(2i\gamma+\pi-i\log\left(8\pi^2\right)\right)$
$e^{-x^2-y^2}\log^2\left(\frac{x}{y}\right)\log\left(\log\left(\frac{x}{y}\right)\right)$	$\frac{1}{32}\pi^3\left(64\left(\zeta'\left(-2,\frac{1}{4}\right)-\zeta'\left(-2,\frac{3}{4}\right)\right)+i\pi+\log\left(4\pi^2\right)\right)$
$\frac{e^{-x-y}\log^k\left(-\frac{x}{y}\right)}{\sqrt{x}\sqrt{y}}$	$-i^k\left(2^{k+1}-1\right)(2\pi)^{k+1}\zeta(-k)$
$\frac{e^{-x-y}}{\sqrt{x}\sqrt{y}\left(\log^2\left(\frac{x}{y}\right)+\pi^2\right)}$	$\frac{\log(2)}{\pi}$
$\frac{e^{-x-y}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6}y^{5/6}\log\left(\frac{x}{y}\right)}$	$\log\left(7+4\sqrt{3}\right)$
$\frac{\sqrt[6]{y}e^{-x^2-y^2}\left(x^{2/3}-y^{2/3}\right)}{x^{5/6}\log\left(\frac{x}{y}\right)}$	$\coth^{-1}\left(\sqrt{2}\right)$

13. Conclusion

In this work the authors used their contour integral method to derive a double integral in terms of the Lerch function. By deriving such an integral transform the authors produced new closed form solutions for double integral formula not present in current literature. This formulae derived in this work showcases a new mathematical method which could be used derive other integral formulae. The authors will be using this method for future work to produce more tables of definite integrals.

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