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Note on a Stieltjes transform in terms of the Lerch function

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Abstract. In this work the authors derive the Stieltjes transform of the logarithmic function in terms of the Lerch function. This transform is used to derive closed form solutions involving fundamental constants and special functions. Specifically we derive the definite integral given by

$$\int_0^\infty \frac{(1-bx)^m \log^k(c(1-bx)) + (bx+1)^m \log^k(c(bx+1))}{a+x^2}dx$$

where a, b, c, m and k are general complex numbers subject to the restrictions given in connection with the formulas.

2020 Mathematics Subject Classifications: 01A55, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

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1. Significance Statement

The Stieltjes transform and Lerch function were both developed between 1856-1922, by famous mathematicians Thomas Joannes Stieltjes and Mathias Lerch respectively. The two functions are not well documented in current literature. The Stieltjes transform has many real world applications such as in Cognitive Radio Communication and Networking and the Lerch function is closely related to Poisson's summation formula. In this article we generate a new table of definite integral formulas which can be used as reference similar to current Table of definite integrals.

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2. Introduction

n the late 1800's Thomas Joannes Stieltjes, a Dutch mathematician found the Stieltjes transform. In this work we will derive this formula as shown in the abstract which does not exist in current literature. We will use this definite integral to produce formal derivations of known integral formulas in [1] and [2]. We will also derive new formula which can be considered an extension to Erdeyli's extensive table of transforms. The derivations follow the method used by us in [5], [4], [6], [7], [9] and [8]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dy. \tag{1}$$

where C is in general, an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. Then multiply both sides by a function, then takes a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

3. Definite integral of the contour integral

We use the method in [9]. The variable of integration in the contour integral is $z = m + w$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. We replace y in 1 by $\log^k(c(1-bx))$ and multiply by $(1-bx)^m/(a^2+x^2)$ in the first case and by $\log^k(c(1+bx))$ and multiply by $(1+bx)^m/(a^2+x^2)$ in the second case and add them. The integration is over x now instead of y.

$$\begin{aligned}
&\frac{1}{k!}\int_0^\infty \frac{(1-bx)^m\log^k(c(1-bx))+(bx+1)^m\log^k(c(bx+1))}{a+x^2}dx \\
&= \frac{1}{2\pi i}\int_0^\infty\int_C \frac{c^w w^{-k-1}\left((1-bx)^{m+w}+(bx+1)^{m+w}\right)}{a+x^2}dwdx \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty \frac{c^w w^{-k-1}\left((1-bx)^{m+w}+(bx+1)^{m+w}\right)}{a+x^2}dxdw \\
&= \frac{1}{2\pi i}\int_C \pi c^w w^{-k-1}a^{\frac{1}{2}(m+w-1)}(-b)^{m+w}\csc(\pi(m+w))\left(\frac{1}{ab^2}\right. \\
&\qquad\left.+1\right)^{\frac{m+w}{2}}\sin\left(\frac{1}{2}(m+w)\left(\pi-2\cot^{-1}\left(\sqrt{a}b\right)\right)\right) \\
&\quad+\pi c^w w^{-k-1}a^{\frac{1}{2}(m+w-1)}b^{m+w}\csc(\pi(m+w))\left(\frac{1}{ab^2}\right. \\
&\qquad\left.+1\right)^{\frac{m+w}{2}}\sin\left(\frac{1}{2}(m+w)\left(2\cot^{-1}\left(\sqrt{a}b\right)+\pi\right)\right)dw
\end{aligned} \tag{2}$$

from equations (3.227.1) and (3.227.2) in [3], using partial fractions, where $Re(a) \geq 0$ and $Re(m+w) < 1$.

4. The Lerch function

The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \tag{3}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\begin{aligned}\Phi(z,s,v) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z} dt\end{aligned} \tag{4}$$

where $Re(v) > 0$, or $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Infinite sum of the contour integral

In this section we will again use Cauchy's integral formula (1) and taking the infinite sum to derive equivalent sum representations for the contour integrals.

5.1. Derivation of the general sine contour integral

Use equation (1) and replace y by $y + it$ and multiply by e^{mit} for the first equation. Next we form the second equation by replacing t by $-t$ and taking their difference to get

$$\frac{ie^{-imt}\left((y-it)^k - e^{2imt}(y+it)^k\right)}{2k!} = \frac{1}{2\pi i} \int_C w^{-k-1} e^{wy} \sin(t(m+w)) dw \tag{5}$$

5.2. Derivation of the contour integral

In this section we will derive the contour integral given by

$$\begin{aligned}\int_C \pi c^w w^{-k-1} a^{\frac{1}{2}(m+w-1)} (-b)^{m+w} \csc(\pi(m+w)) \left(\frac{1}{ab^2}\right. \\ \left. +1\right)^{\frac{m+w}{2}} \sin\left(\frac{1}{2}(m+w)\left(\pi - 2\cot^{-1}\left(\sqrt{a}b\right)\right)\right) dw\end{aligned}$$

Using equation (5) and substituting y by $\frac{1}{2}\log\left(\frac{1}{ab^2}+1\right) + \frac{\log(a)}{2} + \log(-b) + \log(c) + i\pi(2y+1)$ and multiplying both sides by $e^{2i\pi my + i\pi m}$, simplifying we get

$$\frac{ie^{-imt+2i\pi my+i\pi m}}{2k!}\left((2i\pi)^k\left(-\frac{i\log\left(\frac{1}{ab^2}+1\right)}{4\pi}-\frac{i\log(a)}{4\pi}\right.\right.$$
$$\left.-\frac{i\log(-b)}{2\pi}-\frac{i\log(c)}{2\pi}-\frac{t}{2\pi}+\frac{1}{2}(2y+1)\right)^k$$
$$\left.-(2i\pi)^k e^{2imt}\left(-\frac{i\log\left(\frac{1}{ab^2}+1\right)}{4\pi}-\frac{i\log(a)}{4\pi}-\frac{i\log(-b)}{2\pi}-\frac{i\log(c)}{2\pi}+\frac{t}{2\pi}+\frac{1}{2}(2y+1)\right)^k\right)$$
$$=\frac{1}{2\pi i}\int_C w^{-k-1}\sin(t(m+w))\exp\left(w\left(\frac{1}{2}\log\left(\frac{1}{ab^2}+1\right)+\frac{\log(a)}{2}+\log(-b)+\log(c)\right.\right.$$
$$\left.\left.+i\pi(2y+1)\right)+2i\pi my+i\pi m\right)dw \tag{6}$$

Next we take the infinite sum over $y\in[0,\infty)$ and replace t by $\frac{1}{2}\left(\pi-2\cot^{-1}\left(\sqrt{a}b\right)\right)$ and multiply both sides by $-2i\pi a^{\frac{m-1}{2}}(-b)^m\left(\frac{1}{ab^2}+1\right)^{m/2}$ simplifying to get

$$\frac{2^k e^{\frac{i\pi k}{2}}\pi^{k+1}a^{\frac{m-1}{2}}(-b)^m}{k!}\left(\frac{1}{ab^2}+1\right)^{m/2}\left(e^{\frac{1}{2}im\left(2\cot^{-1}\left(\sqrt{a}b\right)+\pi\right)}\right. \tag{7}$$

$$\Phi\left(e^{2im\pi},-k,-\frac{i\left(2i\cot^{-1}\left(\sqrt{a}b\right)+\log(a)+\log\left(1+\frac{1}{b^2a}\right)+2\log(-b)+2\log(c)+i\pi\right)}{4\pi}\right)$$

$$-e^{\frac{1}{2}im\left(3\pi-2\cot^{-1}\left(\sqrt{a}b\right)\right)}$$

$$\left.\Phi\left(e^{2im\pi},-k,-\frac{i\left(-2i\cot^{-1}\left(\sqrt{a}b\right)+\log(a)+\log\left(1+\frac{1}{b^2a}\right)+2\log(-b)+2\log(c)+3i\pi\right)}{4\pi}\right)\right)$$

$$=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C w^{-k-1}\sin(t(m+w))$$

$$\exp\left(w\left(\frac{1}{2}\log\left(\frac{1}{ab^2}+1\right)+\frac{\log(a)}{2}+\log(-b)+\log(c)+i\pi(2y+1)\right)\right.$$

$$\left.+2i\pi my+i\pi m\right)dw$$

$$=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}w^{-k-1}\sin(t(m+w))$$

$$\exp\left(w\left(\frac{1}{2}\log\left(\frac{1}{ab^2}+1\right)+\frac{\log(a)}{2}+\log(-b)+\log(c)+i\pi(2y+1)\right)\right.$$

$$\left.+2i\pi my+i\pi m\right)dw$$

$$=\frac{1}{2\pi i}\int_C\pi c^w w^{-k-1}a^{\frac{1}{2}(m+w-1)}(-b)^{m+w}\csc(\pi(m+w))\left(\frac{1}{ab^2}\right.$$

$$\left.+1\right)^{\frac{m+w}{2}}\sin\left(\frac{1}{2}(m+w)\left(\pi-2\cot^{-1}\left(\sqrt{a}b\right)\right)\right)dw$$

from equation (1.232.3) in [3] where $Im(m+w)>0$ in order for the sum to converge.

5.3. Derivation of the second contour integral

In this section we derive the second contour integral given by

$$\frac{1}{2\pi i}\int_C\pi c^w w^{-k-1}a^{\frac{1}{2}(m+w-1)}b^{m+w}\csc(\pi(m+w))\left(\frac{1}{ab^2}\right.$$
$$\left.+1\right)^{\frac{m+w}{2}}\sin\left(\frac{1}{2}(m+w)\left(2\cot^{-1}\left(\sqrt{a}b\right)+\pi\right)\right)dw \tag{8}$$

In this derivation we proceed as above but multiply by $-2i\pi a^{\frac{m-1}{2}}b^m\left(\frac{1}{ab^2}+1\right)^{m/2}$ and replace t by $\frac{1}{2}\left(2\cot^{-1}\left(\sqrt{a}b\right)+\pi\right)$ and simplifying to get

$$\frac{2^k e^{\frac{i\pi k}{2}}\pi^{k+1}a^{\frac{m-1}{2}}b^m}{k!}\left(\frac{1}{ab^2}+1\right)^{m/2}\left(e^{\frac{1}{2}im\left(\pi-2\cot^{-1}\left(\sqrt{a}b\right)\right)}\right.$$
$$\Phi\left(e^{2im\pi},-k,-\frac{i\left(-2i\cot^{-1}\left(\sqrt{a}b\right)+\log(a)+\log\left(1+\frac{1}{b^2a}\right)+2\log(b)+2\log(c)+i\pi\right)}{4\pi}\right)$$
$$-e^{\frac{1}{2}im\left(2\cot^{-1}\left(\sqrt{a}b\right)+3\pi\right)}$$
$$\left.\Phi\left(e^{2im\pi},-k,-\frac{i\left(2i\cot^{-1}\left(\sqrt{a}b\right)+\log(a)+\log\left(1+\frac{1}{b^2a}\right)+2\log(b)+2\log(c)+3i\pi\right)}{4\pi}\right)\right)$$
$$=\frac{1}{2\pi i}\int_C \pi c^w w^{-k-1}a^{\frac{1}{2}(m+w-1)}b^{m+w}\csc(\pi(m+w))\left(\frac{1}{ab^2}\right.$$
$$\left.+1\right)^{\frac{m+w}{2}}\sin\left(\frac{1}{2}(m+w)\left(2\cot^{-1}\left(\sqrt{a}b\right)+\pi\right)\right)dw \tag{9}$$

from equation (1.232.3) in [3] where $Im(m+w)>0$ in order for the sum to converge.

6. The Stieltjes transform in terms of the Lerch function

Since the right-hand side of equation (2) is equal to the sum of the right-hand sides of equations (8) and (9) we can equate the left-hand sides and simplifying the factorial to get

$$\int_0^\infty \frac{(1-bx)^m\log^k(c(1-bx))+(bx+1)^m\log^k(c(bx+1))}{a^2+x^2}dx \tag{10}$$
$$=2^k e^{\frac{i\pi k}{2}}\pi^{k+1}a^{m-1}(-b)^m\left(\frac{1}{a^2b^2}+1\right)^{m/2}\left(e^{\frac{1}{2}im\left(2\cot^{-1}(ab)+\pi\right)}\right.$$
$$\Phi\left(e^{2im\pi},-k,-\frac{i\left(2i\cot^{-1}(ab)+\log\left(a^2\right)+\log\left(1+\frac{1}{a^2b^2}\right)+2\log(-b)+2\log(c)+i\pi\right)}{4\pi}\right)$$
$$-e^{\frac{1}{2}im\left(3\pi-2\cot^{-1}(ab)\right)}$$
$$\left.\Phi\left(e^{2im\pi},-k,-\frac{i\left(-2i\cot^{-1}(ab)+\log\left(a^2\right)+\log\left(1+\frac{1}{a^2b^2}\right)+2\log(-b)+2\log(c)+3i\pi\right)}{4\pi}\right)\right)$$
$$+2^k e^{\frac{i\pi k}{2}}\pi^{k+1}a^{m-1}b^m\left(\frac{1}{a^2b^2}+1\right)^{m/2}\left(e^{\frac{1}{2}im\left(\pi-2\cot^{-1}(ab)\right)}\right.$$
$$\Phi\left(e^{2im\pi},-k,-\frac{i\left(-2i\cot^{-1}(ab)+\log\left(a^2\right)+\log\left(1+\frac{1}{a^2b^2}\right)+2\log(b)+2\log(c)+i\pi\right)}{4\pi}\right)$$
$$-e^{\frac{1}{2}im\left(2\cot^{-1}(ab)+3\pi\right)}$$
$$\left.\Phi\left(e^{2im\pi},-k,-\frac{i\left(2i\cot^{-1}(ab)+\log\left(a^2\right)+\log\left(1+\frac{1}{a^2b^2}\right)+2\log(b)+2\log(c)+3i\pi\right)}{4\pi}\right)\right)$$

7. Definite integrals in terms of the Hurwitz zeta function

Using equation (10) replacing b by β, a by α and setting $c = 1$ followed by taking the sum over $m \in [0, \infty)$ and simplifying in terms of the Hurwitz zeta function we get

$$\int_0^\infty \frac{\log^k(1-\beta x) - \log^k(\beta x + 1)}{x\left(\alpha^2 + x^2\right)} dx = -\frac{i2^k e^{\frac{i\pi k}{2}} \pi^{k+1}}{\alpha^2} \left(\zeta\left(-k, \frac{i(-2\log(i - \alpha\beta) - 3i\pi)}{4\pi}\right) - \zeta\left(-k, -\frac{i(2\log(\alpha\beta - i) + i\pi)}{4\pi}\right)\right) \tag{11}$$

Using equation (10) and setting $m = 0$ and replacing a by $\sqrt{a}$ and simplifying in terms of the Hurwitz zeta function $\zeta(s, v)$ we get

$$\int_0^\infty \frac{\log^k(c - bcx) + \log^k(bcx + c)}{a + x^2} dx = \frac{(2i)^k \pi^{k+1}}{\sqrt{a}} \left(\zeta\left(-k, -\frac{i\left(-2i\cot^{-1}\left(\sqrt{a}b\right) + \log(a) + \log\left(1 + \frac{1}{b^2 a}\right) + 2\log(b) + 2\log(c) + i\pi\right)}{4\pi}\right) - \zeta\left(-k, -\frac{i\left(-2i\cot^{-1}\left(\sqrt{a}b\right) + \log(a) + \log\left(1 + \frac{1}{b^2 a}\right) + 2\log(b) + 2\log(c) + 5i\pi\right)}{4\pi}\right)\right) \tag{12}$$

8. Derivation of entry 4.535.7 in [3]

Using (11) and setting $k = \alpha = 1$ and replacing β by $-ip$ and simplifying we get

$$\int_0^\infty \frac{\tan^{-1}(px)}{x^3 + x} dx = \frac{1}{2}\pi\log(p+1) \tag{13}$$

9. Derivation of entry 4.535.8 in [3]

Using (11) and setting $k = 1, \beta = -p, \alpha = i$ and simplifying we get

$$\int_0^\infty \frac{\tan^{-1}(px)}{x - x^3} dx = \frac{1}{4}\pi(2\log(p+i) - i\pi) \tag{14}$$

10. Derivation of entry 4.535.9 in [3]

Using (11) and setting $k = 1$ and replacing β by $-iq$ and α by p and simplifying we get

$$\int_0^\infty \frac{\tan^{-1}(qx)}{p^2 x + x^3} dx = \frac{\pi\log(pq+1)}{2p^2} \tag{15}$$

11. Derivation of entry 4.535.10 in [3]

In this evaluation we will derive the definite integrals for both the Arctangent and Hyperbolic tangent functions. Using (11) and setting $k = 1$ and replacing β by q and α by $1/p$ simplifying we get

$$\int_0^\infty \frac{\tanh^{-1}(qx)}{p^2x^3+x}dx = \frac{1}{8}\left(-\log\left(-\frac{q}{p}+i\right)+\log\left(\frac{q}{p}-i\right)-i\pi\right)\left(\log\left(-\frac{q}{p}+i\right)+\log\left(\frac{q}{p}-i\right)\right) \tag{16}$$

Next we multiply by $-1/i$, replace p by $-ip$ and q by $-iq$ simplifying to get

$$\int_0^\infty \frac{\tan^{-1}(qx)}{x-p^2x^3}dx = \frac{1}{2}\pi\log\left(1+\frac{iq}{p}\right) \tag{17}$$

The formula given in [3] is in error.

12. Derivation of entry 4.535.11 in [3]

In this evaluation we will derive the definite integrals for both the Arctangent and Hyperbolic tangent functions. Using (10) and setting $m = 0$, $k = 1$ and replacing β by q and α by $1/p$ and simplifying we get

$$\int_0^\infty \frac{\tanh^{-1}(bx)}{ax+x^3}dx = -\frac{(\log(-b)-\log(b))\left(\log\left(\frac{1}{ab^2}+1\right)+\log(a)+\log(-b)+\log(b)\right)+2\pi\cot^{-1}\left(\sqrt{a}b\right)}{4a} \tag{18}$$

Next we split the left-hand side to get

$$\int_0^\infty \left(\frac{\tanh^{-1}(bx)}{x}-\frac{x\tanh^{-1}(bx)}{a+x^2}\right)dx = \frac{1}{4}\left(-(\log(-b)-\log(b))\left(\log\left(\frac{1}{ab^2}+1\right)+\log(a)+\log(-b)+\log(b)\right)-2\pi\cot^{-1}\left(\sqrt{a}b\right)\right) \tag{19}$$

Next we take the first partial derivative with respect to a simplifying to get

$$\int_0^\infty \frac{x\tanh^{-1}(bx)}{\left(a^2+x^2\right)^2}dx = \frac{b\left(\frac{\pi}{\sqrt{a^2}}-b\log(-b)+b\log(b)\right)}{4a^2b^2+4} \tag{20}$$

Next replacing b by ib and simplifying we get

$$\int_0^\infty \frac{x\tan^{-1}(bx)}{\left(a^2+x^2\right)^2}dx = \frac{\pi b}{4a(ab+1)} \tag{21}$$

13. Derivation of entry 4.295.1 in [3]

Using (12) and setting $k = c = 1$ and simplifying we get

$$\int_0^\infty \frac{\log\left(1-b^2x^2\right)}{a+x^2}dx = \frac{\frac{1}{2}\pi\left(\log\left(\frac{1}{ab^2}+1\right)+\log(a)+\log(-b)+\log(b)\right)+\left(\log(b)-\log(-b)\right)\cot^{-1}\left(\sqrt{a}b\right)}{\sqrt{a}} \tag{22}$$

Next we replace b by $\frac{i\sqrt{\mu}}{\sqrt{\beta}}$ and a by γ simplifying to get

$$\int_0^\infty \frac{\log\left(\beta+\mu x^2\right)}{\gamma+x^2}dx = \frac{\pi\log\left(\sqrt{\beta}+\sqrt{\gamma}\sqrt{\mu}\right)}{\sqrt{\gamma}} \tag{23}$$

The equation quoted in [3] is not valid for general complex numbers, for example when $Re(\gamma) < 0$.

14. Derivation of entry 4.295.7 in [3]

Using (23) and replacing β by a^2, μ by b^2 and γ by $\left(\frac{c}{g}\right)^2$ and simplifying we get

$$\int_0^\infty \frac{\log\left(a^2+b^2x^2\right)}{c^2+g^2x^2}dx = \frac{\pi\log\left(\frac{ag+bc}{g}\right)}{cg} \tag{24}$$

15. Derivation of entry 4.295.8 in [3]

Using (24) and replacing g by ig and simplifying we get

$$\int_0^\infty \frac{\log\left(a^2+b^2x^2\right)}{c^2-g^2x^2}dx = -\frac{i\pi\log\left(-\frac{i(bc+iag)}{g}\right)}{cg} \tag{25}$$

16. Derivation of entry 4.295.9 in [3]

We will form two equations by using (24) and setting $\beta = 1$ and replacing μ by p^2 for the first equation and then replacing p by q for the second, subtracting and simplifying we get

$$\int_0^\infty \frac{\log\left(p^2x^2+1\right)-\log\left(q^2x^2+1\right)}{\gamma+x^2}dx = \frac{\pi\log\left(\frac{\sqrt{\gamma}p+1}{\sqrt{\gamma}q+1}\right)}{\sqrt{\gamma}} \tag{26}$$

Next we apply L'Hopital's rule as $\gamma \to 0$ to the right-hand side simplifying to get

$$\int_0^\infty \frac{\log\left(p^2x^2+1\right)-\log\left(q^2x^2+1\right)}{x^2}dx = \pi(p-q) \tag{27}$$

17. Derivation of entry 4.295.22 in [3]

Using (24) setting $a = 1$ and replacing c by r, g by q, b by p and simplifying we get

$$\int_0^\infty \frac{\log\left(p^2x^2+1\right)}{q^2x^2+r^2}dx = \frac{\pi \log\left(\frac{pr+q}{q}\right)}{qr} \tag{28}$$

18. Derivation of entry 4.295.25 in [3]

Using (24) and taking the first partial derivative with respect to c and simplifying we get

$$\int_0^\infty \frac{\log\left(a^2+b^2x^2\right)}{\left(c^2+g^2x^2\right)^2}dx = \frac{\pi \log\left(\frac{ag+bc}{g}\right)}{2c^3g} - \frac{\pi b}{2c^2g(ag+bc)} \tag{29}$$

19. Derivation of entry 4.295.26 in [3]

Using (24) and taking the first partial derivative with respect to g and simplifying we get

$$\int_0^\infty \frac{x^2\log\left(a^2+b^2x^2\right)}{\left(c^2+g^2x^2\right)^2}dx = \frac{\pi b}{2g^3(ag+bc)} + \frac{\pi \log\left(\frac{ag+bc}{g}\right)}{2cg^3} \tag{30}$$

20. Definite logarithmic integral in terms π

Using (12) and setting $c = 1$, $b = 1$ and $a = 1$ simplifying we get

$$\begin{aligned}\int_0^\infty \frac{\log^k(1-x)+\log^k(x+1)}{x^2+1}dx = (2i)^k\pi^{k+1}\Bigg(\zeta\left(-k, -\frac{-\frac{7\pi}{2}+i\log(2)}{4\pi}\right) \\ -\zeta\left(-k, \frac{\frac{9\pi}{2}-i\log(2)}{4\pi}\right)\Bigg) \\ +(2i)^k\pi^{k+1}\Bigg(\zeta\left(-k, -\frac{-\frac{\pi}{2}+i\log(2)}{4\pi}\right) \\ -\zeta\left(-k, \frac{\frac{7\pi}{2}-i\log(2)}{4\pi}\right)\Bigg)\end{aligned} \tag{31}$$

Next we apply L'Hopital's rule as $k \to -1$ and simplifying to get

$$\int_0^\infty \frac{\log\left(1-x^2\right)}{(x^2+1)\log(1-x)\log(x+1)}dx = \frac{4\pi}{\log(4)+i\pi} \tag{32}$$

21. Definite nested logarithmic integral in terms π

Using (12) and taking the first partial derivative with respect to k setting $k = 0$, $c = 1$, $b = 1$ and $a = 1$ and simplifying we get

$$\begin{aligned}\int_0^\infty \frac{\log(\log(1-x)) + \log(\log(x+1))}{x^2+1}dx = \frac{1}{2}\pi\Bigg(2\log\Gamma\left(\frac{\pi - 2i\log(2)}{8\pi}\right) \\ - 2\log\Gamma\left(-\frac{7}{8} - \frac{i\log(2)}{4\pi}\right) + 3i\pi \\ + \log\left(\frac{4\pi^2(\pi - 2i\log(2))^2}{(7\pi + 2i\log(2))^2}\right)\Bigg)\end{aligned} \tag{33}$$

Next simplifying the right-hand side we get

$$\int_0^\infty \frac{\log(\log(1-x)\log(x+1))}{x^2+1}dx = \pi\log\left(\frac{1}{4}i(\pi - 2i\log(2))\right) \tag{34}$$

22. Definite integral of the hyperbolic tangent and logarithmic functions

Using (11) and setting $k = 2$ and simplifying we get

$$\begin{aligned}&\int_0^\infty \frac{\log\left(1-\beta^2x^2\right)\tanh^{-1}(\beta x)}{x^3+\alpha^2 x}dx \\ &= -\frac{4\log^3(-\alpha\beta+i) + 6i\pi\log^2(-\alpha\beta+i) - 4\log^3(\alpha\beta-i) + 6i\pi\log^2(\alpha\beta-i) + 4i\pi^3}{48\alpha^2}\end{aligned} \tag{35}$$

Next we apply L'Hopital's rule to the right-hand side as $\alpha \to 0$ and upon inspection of this closed form solution we are able to write down the conditional form given by

$$\int_0^\infty \frac{\log\left(1-\beta^2x^2\right)\tanh^{-1}(\beta x)}{x^3}dx = \begin{cases}\frac{1}{2}i\pi\beta^2 & \text{if } Im(\beta) < 0,\\ \frac{1}{2}i\pi\beta|\beta| & \text{if } Im(\beta) = 0,\\ -\frac{1}{2}i\pi\beta|\beta| & \text{if } Im(\beta) > 0.\end{cases} \tag{36}$$

We can also expand the left-hand side of equation (35) to get

$$\begin{aligned}&\int_0^\infty \left(\frac{\log\left(1-\beta^2x^2\right)\tanh^{-1}(\beta x)}{x} - \frac{x\log\left(1-\beta^2x^2\right)\tanh^{-1}(\beta x)}{\alpha^2+x^2}\right)dx \\ &= \frac{1}{48}\left(-4\log^3(-\alpha\beta+i) - 6i\pi\log^2(-\alpha\beta+i) - \pi^2\log(-\alpha\beta+i)\right. \\ &\qquad \left. - (\pi + i\log(\alpha\beta-i))\left(2(\pi+2i\log(\alpha\beta-i))\log(\alpha\beta-i) + 3i\pi^2\right)\right)\end{aligned} \tag{37}$$

and take the first partial derivative with respect to α, replacing β by $-i\beta$ simplifying to get

$$\int_0^\infty \frac{x\log\left(\beta^2x^2+1\right)\tan^{-1}(\beta x)}{\left(\alpha^2+x^2\right)^2}dx = \frac{\pi\beta\log\left((\alpha\beta+1)^2\right)}{4\alpha(\alpha\beta+1)} \tag{38}$$

23. Definite integral representation for π

Using (10) and setting $b = c = k = 1$ and simplifying we get

$$\int_0^\infty \frac{(1-x)^m \log(1-x) + (x+1)^m \log(x+1)}{a+x^2} dx = \frac{\pi e^{\frac{i\pi m}{2}} \left(\sqrt{a}-i\right)^m \log\left(-\left(\sqrt{a}-i\right)^2\right)}{2\sqrt{a}} \tag{39}$$

Next we apply L'Hopital's rule as $a \to 0$ to the right-hand side to get

$$\int_0^\infty \frac{(1-x)^m \log(1-x) + (x+1)^m \log(x+1)}{x^2} dx = i\pi \tag{40}$$

Next we take the definite integral over $m \in [0, m]$ to get

$$\int_0^\infty \frac{(1-x)^m + (x+1)^m - 2}{x^2} dx = i\pi m \tag{41}$$

where $-1 < Re(m) < 1$.

24. Table of integrals

$f(x)$	$\int_0^\infty f(x)dx$
$\frac{\tan^{-1}(px)}{x^3+x}$	$\frac{1}{2}\pi\log(p+1)$
$\frac{\tan^{-1}(px)}{x-x^3}$	$\frac{1}{4}\pi(2\log(p+i)-i\pi)$
$\frac{\tan^{-1}(qx)}{p^2x+x^3}$	$\frac{\pi\log(pq+1)}{2p^2}$
$\frac{\tan^{-1}(qx)}{x-p^2x^3}$	$\frac{1}{4}\pi\log\left(\frac{p^2+q^2}{p^2}\right)$
$\frac{x\tan^{-1}(bx)}{(a^2+x^2)^2}$	$\frac{\pi b}{4a(ab+1)}$
$\frac{\log(\beta+\mu x^2)}{\gamma+x^2}$	$\frac{\pi\log(\sqrt{\beta}+\sqrt{\gamma}\sqrt{\mu})}{\sqrt{\gamma}}$
$\frac{\log(a^2+b^2x^2)}{c^2+g^2x^2}$	$\frac{\pi\log\left(\frac{ag+bc}{g}\right)}{cg}$
$\frac{\log(a^2+b^2x^2)}{c^2-g^2x^2}$	$-\frac{i\pi\log\left(-\frac{i(bc+iag)}{g}\right)}{cg}$
$\frac{\log(p^2x^2+1)-\log(q^2x^2+1)}{x^2}$	$\pi(p-q)$
$\frac{\log(p^2x^2+1)}{q^2x^2+r^2}$	$\frac{\pi\log\left(\frac{pr+q}{q}\right)}{qr}$
$\frac{\log(a^2+b^2x^2)}{(c^2+g^2x^2)^2}$	$\frac{\pi\log\left(\frac{ag+bc}{g}\right)}{2c^3g}-\frac{\pi b}{2c^2g(ag+bc)}$
$\frac{x^2\log(a^2+b^2x^2)}{(c^2+g^2x^2)^2}$	$\frac{\pi b}{2g^3(ag+bc)}+\frac{\pi\log\left(\frac{ag+bc}{g}\right)}{2cg^3}$
$\frac{\log(1-x^2)}{(x^2+1)\log(1-x)\log(x+1)}$	$\frac{4\pi}{\log(4)+i\pi}$
$\frac{\log(\log(1-x)\log(x+1))}{x^2+1}$	$\pi\log\left(\frac{1}{4}i(\pi-2i\log(2))\right)$
$\frac{(1-x)^m\log(1-x)+(x+1)^m\log(x+1)}{x^2}$	$i\pi$

Table 1: Table of definite integrals

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