EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 3, 2021, 895-904
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Introduction to Neutrosophic B-algebras

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Abstract. This paper introduces the notion of neutrosophic B-algebra. Several results on properties of neutrosophic B-algebras and neutrosophic subalgebras are presented and proved.

2020 Mathematics Subject Classifications: 03G25, 06F35

Key Words and Phrases: Neutrosophic B-algebra, neutrosophic subalgebra, neutrosophic algebraic structures

1. Introduction

In 1995, Smarandache [16] introduced the concept of neutrosophic logic as an extension of fuzzy logic in which indeterminacy is included. Indeterminacy means degrees of uncertainty, vagueness, imprecision, undefined, unknown, inconsistency or redundancy, for example, in tossing a die on irregular surface one can get $\{1, 2, 3, 4, 5, 6, indeterminacy\}$. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F.

Using neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache [11] in 2003 introduced a neutrosophic structure based on indeterminacy "I" only, which they called *I-Neutrosophic Algebraic Structures*, an algebraic structure based on sets of neutrosophic numbers of the form $N = a + bI$, where a, b are real (or complex) numbers, and I is called *literal indeterminacy*, which stands for unknown or non-determinate such that $I^2 = I$. Here, a is called the *determinate* part of N and bI is called the *indeterminate* part of N, with $mI + nI = (m + n)I$, $0 \cdot I = 0$. The indeterminacy I is different from the imaginary $i = \sqrt{-1}$. In general, one has $I^n = I$ if $n > 0$, and is undefined for $n \leq 0$.

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.3996

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In 2006, they introduced some neutrosophic algebraic structures like neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids, and neutrosophic rings [12, 17]. In 2015, A.A.A Agboola and B. Davvaz [1] introduced the concept of neutrosophic BCI/BCK.

In 2002, J. Neggers and H.S. Kim [14, 15] introduced the concept of B-algebra and established some properties of B-homomorphism[14]. From then on, several characterizations as to commutativity and center, cyclicity, isomorphism, direct product, Lagrange and Cauchy's Theorems, B-action and the Sylow Theorems for B-algebras as exemplified by the following literatures [2–5, 8–10, 13].

In this paper, we introduce the concepts of neutrosophic B-algebra and neutrosophic subalgebra. Some properties of neutrosophic B-algebras and neutrosophic subalgebras are presented and proved.

2. Preliminaries

For convenience, we view the neutrosophic number $N = a + bI$ as an ordered pair (a, bI).

Definition 2.1. [7] Let X be a nonempty set and let I be an indeterminate. A set $X(I) = \langle X, I\rangle = \{(x, yI) : x, y \in X\}$ is called a *neutrosophic set* generated by X and I.

A type $(2, 0)$ algebra is an algebra formed from a nonempty set X together with 2-ary operation $*$ and a 0-ary operation(with constant element 0).

Definition 2.2. [15] Let X be a nonempty set with a binary operation "$*$" on X and a constant 0. Then the algebra $(X; *, 0)$ of type $(2, 0)$ is called a *B-algebra* if it satisfies the following axioms: for all $x, y, z \in X$,
(B1) $x * x = 0$; (B2) $x * 0 = x$; (B3) $(x * y) * z = x * (z * (0 * y))$.

Example 2.3. The following are examples of B-algebra.

(i) [15] Let $X = \{0, 1, 2\}$. Define the operation "$*$" by the Cayley table shown below.

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

(ii) [15] Let $X = \{0, 1, 2, 3, 4, 5\}$. Define the operation "$*$" by the Cayley table shown below.

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

The following properties of B-algebra can be found in [15] and [18]. Let $(X;*,0)$ be a B-algebra. Then for any $x,y,z \in X$, (P1): $0*(0*x)=x$, (P2): $0*(x*y)=y*x$, (P3): $(x*z)*(y*z)=x*y$, and (P4): $x*y=0 \Longrightarrow x=y$.

Lemma 2.4. *Let X be a B-algebra. Then for any $x,y,z \in X$,*

(i) [6] (left cancellation law) $x*y=x*z$ *implies* $y=z$.

(ii) [15] $x*(y*z)=(x*(0*z))*y$.

Definition 2.5. [15] A B-algebra $(X;*,0)$ is said to be *commutative* if $a*(0*b)=b*(0*a)$ for any $a,b \in X$.

Lemma 2.6. [15] *Let X be a commutative B-algebra. Then for any $x,y,z \in X$, $x*(x*y)=y$.*

Definition 2.7. [14] Let $(X;*,0)$ be a B-algebra. A nonempty subset N of X is said to be a *subalgebra* of X if $a*b \in N$ for all $a,b \in N$. N is said to be *normal* if for any $x*y, a*b \in N$ implies $(x*a)*(y*b) \in N$.

Lemma 2.8. [9] *Let X be a B-algebra. If $\{N_\alpha : \alpha \in \mathscr{A}\}$ is any nonempty collection of subalgebras (resp., normal subalgebras) of X, then $\bigcap_{\alpha\in\mathscr{A}} N_\alpha$ is a subalgebra (resp., normal subalgebra) of X.*

3. Some Properties of Neutrosophic B-algebras and Neutrosophic Subalgebras

Definition 3.1. Let $(X;*,0)$ be any B-algebra. The set $X(I)=\{(x,yI) : x,y \in X\}$ is the neutrosophic set determined by X and I. Moreover, $(a,bI)=(c,dI)$ in $X(I)$ if and only if $a=c$ and $b=d$.

Definition 3.2. Let $(X;*,0)$ be any B-algebra. For any $x,y \in X$, we denote $x \wedge y = x*(x*y)$.

Lemma 3.3. *The mapping $\lambda : X(I) \times X(I) \to X(I)$ defined by $\lambda((a,bI),(c,dI)) = (a,bI)\cdot(c,dI) = \big(a*c, ((a*d \wedge b*c)\wedge b*d)I\big)$ for any $(a,bI),(c,dI) \in X(I)$ is well-defined.*

Proof: Let $(a,bI),(c,dI)$, $(x,yI),(u,vI) \in X(I)$ such that $(a,bI)=(x,yI)$ and $(c,dI)=(u,vI)$. Then $\lambda\big((a,bI),(c,dI)\big) = (a,bI)\cdot(c,dI) = \big(a*c, ((a*d \wedge b*c)\wedge b*d)I\big) = \big(x*u, ((x*v \wedge y*u)\wedge y*v)I\big) = (x,yI)\cdot(u,vI) = \lambda\big((x,yI),(u,vI)\big)$.

Hence, the λ is well-defined. □

Definition 3.4. The triple $(X(I);\cdot,(0,0I))$ is called a *neutrosophic B-algebra* determined by X and I with the binary operation $\cdot$ defined in Lemma 3.3 and $(0,0I)$ as its constant element.

Remark 3.5. *Every nonzero neutrosophic B-algebra $X(I)$ always contains the B-algebra $X'=\{(x,0I) : x \in X\}$ as a proper subset.*

Let $X(I)$ stand for a neutrosophic B-algebra $(X(I); \cdot, (0, 0I))$, unless otherwise stated.

Example 3.6. Consider the commutative B-algebra $X = \{0, 1, 2\}$ in Example 2.3(i). Then the neutrosophic B-algebra determined by X and I is given by $X(I) = \{(0, 0I), (0, I), (0, 2I), (1, 0I), (1, I), (1, 2I), (2, 0I), (2, I), (2, 2I)\}$.

Lemma 3.7. *Let X be a B-algebra. Then for any $x, y \in X$,*
(i) $x \wedge x = x$, (iv) $x * y \wedge y * x = y * x$,
(ii) $x \wedge 0 = 0$, (v) $x \wedge y = 0$ *if and only if* $y = 0$.
(iii) $0 \wedge x = x$,

Proof: (i) By (B1) and (B2), $x \wedge x = x * (x * x) = x * 0 = x$; (ii) By (B2) and (B1), $x \wedge 0 = x * (x * 0) = x * x = 0$; (iii) By (P1), $0 \wedge x = 0 * (0 * x) = x$; (iv) By Lemma 2.4(ii), (B1) and (P3), $x * y \wedge y * x = (x * y) * [(x * y) * (y * x)] = [(x * y) * [0 * (y * x)]] * (x * y) = [(x * y) * (x * y)] * (x * y) = 0 * (x * y) = y * x$; (v) $x \wedge y = 0$ implies that $x * (x * y) = 0$. By (P4), $x = x * y$ which can be written as $x * 0 = x * y$. Hence, by Lemma 2.4(i), $y = 0$. The converse follows directly from (ii). $\square$

Lemma 3.8. *If $X(I)$ is a neutrosophic B-algebra, then for any $(a, bI), (c, dI) \in X(I)$,*

(i) $(a, bI) \cdot (0, 0I) = \big(a, ((a \wedge b) \wedge b)I\big)$,

(ii) $(a, cI) \cdot (b, cI) = (a * b, 0I)$,

(iii) $(a, aI) \cdot (b, bI) = (a * b, (a * b)I)$,

(iv) $(a, bI) \cdot (c, dI) = (0, 0I)$ *if and only if* $(a, bI) = (c, dI)$.

Proof: (i) By Definition 3.4 and (B2), $(a, bI) \cdot (0, 0I) = (a * 0, ((a * 0 \wedge b * 0) \wedge b * 0)I) = (a, ((a \wedge b) \wedge b)I)$; (ii) By Definition 3.4, (B1), (B2) and Lemma 3.7(ii), $(a, cI) \cdot (b, cI) = (a*b, ((a*c \wedge c*b) \wedge c*c)I) = (a*b, ((a*c \wedge c*b) \wedge 0)I) = (a*b, 0I)$; (iii) Follows directly from Definition 3.4 and Lemma 3.7(i); (iv) By Definition 3.4, $(a, bI) \cdot (c, dI) = (0, 0I)$ implies that $\big(a*c, ((a*d \wedge b*c) \wedge b*d)I\big) = (0, 0I)$. That is, $a*c = 0$ and $(a*d \wedge b*c) \wedge b*d = 0$. By (P4), $a = c$ and by Lemma 3.7(v), $b*d = 0$. Thus, by (P4), $b = d$. Hence, $(a, bI) = (c, dI)$. Conversely, let $(a, bI) = (c, dI)$. Then $a = c$ and $b = d$. Thus, by Definition 3.4, (B1) and Lemma 3.7(ii and iv), $(a, bI) \cdot (c, dI) = (a * c, ((a * d \wedge b * c) \wedge b * d)I) = (a * a, ((a * b \wedge b * a) \wedge b * b)I) = (0, ((b * a) \wedge 0)I) = (0, 0I)$. $\square$

Lemma 3.9. *If X is a commutative B-algebra, then for any $x, y, z \in X$,*
(i) $x \wedge y = y$, (ii) $(x \wedge y) \wedge z = x \wedge (y \wedge z) = z$.

Proof: Let $x, y, z \in X$. (i) By Definition 3.4 and Lemma 2.6, $x \wedge y = x * (x * y) = y$; (ii) By (i), $(x \wedge y) \wedge z = y \wedge z = z = x \wedge z = x \wedge (y \wedge z)$. $\square$

Theorem 3.10. *If X is commutative, then $X(I)$ is a B-algebra.*

Proof: Let $(a, bI), (c, dI) \in X(I)$. By Lemma 3.9(ii), the binary operation in $X(I)$ is $(a, bI) \cdot (c, dI) = \big(a * c, ((a * d \wedge b * c) \wedge b * d)I\big) = \big(a * c, (b * d)I\big)$. This coincides with the binary operation of $X \times X$ as a B-algebra. Therefore, $X(I)$ is a B-algebra. $\square$

Remark 3.11. *By* Theorem 3.10*, the neutrosophic B-algebra $X(I)$ in* Example 3.6 *is a B-algebra. However, a neutrosophic B-algebra is not a B-algebra in general as shown in the following example.*

Example 3.12. Consider the non-commutative B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.3(ii). Then the set $X(I) = \{(0, 0I), (1, 0I), ..., (5, 0I), (0, I), (1, I), ..., (5, I), (0, 2I), (1, 2I), ..., (5, 2I), (0, 3I), (1, 3I), ..., (5, 3I), (0, 4I), (1, 4I), ..., (5, 4I), (0, 5I), (1, 5I), ..., (5, 5I)\}$ is the neutrosophic B-algebra determined by X and I. $X(I)$ is not a B-algebra since by Lemma 3.8(i), $(3, 4I) \cdot (0, 0I) = (3, ((3 \wedge 4) \wedge 4)I) = (3, (5 \wedge 4)I) = (3, 3I) \neq (3, 4I)$.

Definition 3.13. A neutrosophic B-algebra $X(I)$ is said to be *commutative* if $(a, bI) \cdot [(0, 0I) \cdot (c, dI)] = (c, dI) \cdot [(0, 0I) \cdot (a, bI)]$ for any $(a, bI), (c, dI) \in X(I)$.

Remark 3.14. *If $X(I)$ is a commutative neutrosophic B-algebra, then $X' = \{(x, 0I) : x \in X\}$ is a commutative B-algebra.*

In Example 3.12, $X(I)$ is not commutative since $(2, 2I), (5, 5I) \in X(I)$ but by Lemma 3.8(iii), $(2, 2I) \cdot [(0, 0I) \cdot (5, 5I)] = (4, 4I) \neq (3, 3I) = (5, 5I) \cdot [(0, 0I) \cdot (2, 2I)]$. In Example 3.6, X is a commutative B-algebra and it can be verified that $X(I)$ is also commutative. This observation is generalized in the following result.

Theorem 3.15. *X is commutative if and only if $X(I)$ is commutative.*

Proof: Let $(a, bI), (c, dI) \in X(I)$. Suppose that X is commutative. Then by Definition 3.4 and Lemma 3.9 (ii),

$$\begin{aligned}
(a, bI) \cdot [(0, 0I) \cdot (c, dI)] &= (a, bI) \cdot \big(0 * c, ((0 * d \wedge 0 * c) \wedge 0 * d)I\big) \\
&= (a, bI) \cdot \big(0 * c, (0 * d)I\big) \\
&= \big(a * (0 * c), \big[\big(a * (0 * d) \wedge b * (0 * c)\big) \wedge b * (0 * d)\big]I\big) \\
&= \big(a * (0 * c), [b * (0 * d)]I\big) \\
&= \big(c * (0 * a), [d * (0 * b)]I\big) \\
&= \big(c * (0 * a), \big[\big(c * (0 * b) \wedge d * (0 * a)\big) \wedge d * (0 * b)\big]I\big) \\
&= (c, dI) \cdot \big(0 * a, (0 * b)I\big) \\
&= (c, dI) \cdot \big(0 * a, ((0 * b \wedge 0 * a) \wedge 0 * b)I\big) \\
&= (c, dI) \cdot [(0, 0I) \cdot (a, bI)].
\end{aligned}$$

Therefore, $X(I)$ is commutative. Conversely, suppose that $X(I)$ is commutative. Then by Remark 3.14, X' is commutative so that for any $(x, 0I), (y, 0I) \in X'$, $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$. Hence, by Definition 2.5, X is commutative. □

Corollary 3.16. *If $X(I)$ is commutative, then $X(I)$ is a B-algebra.*

The notion of neutrosophic subalgebra of a neutrosophic B-algebra will now be introduced.

Definition 3.17. Let $X(I)$ be a neutrosophic B-algebra. A subset $P(I)$ of $X(I)$ is said to be a *proper subset* of $X(I)$ if $P(I) \neq X(I)$.

Definition 3.18. A nonempty subset $S(I)$ of a neutrosophic B-algebra $X(I)$ is said to be a *neutrosophic subalgebra* of $X(I)$ if the following conditions hold:

(i) $(a, bI) \cdot (c, dI) \in S(I)$ for all $(a, bI), (c, dI) \in S(I)$, and

(ii) $S(I)$ contains a proper subset which is a B-algebra.

In view of Lemma 3.8(iv), $(0, 0I)$ is an element of any neutrosophic subalgebra $S(I)$ of $X(I)$.

Remark 3.19. *Let $X(I)$ be a nonzero neutrosophic B-algebra.*

(i) *Then $X(I)$ is a neutrosophic subalgebra of itself. However, $\{(0, 0I)\}$ is not a neutrosophic subalgebra of $X(I)$ since it does contain a proper subset which is a B-algebra but $\{(0, 0I)\}$ is a B-algebra.*

(ii) *If $S(I)$ is a neutrosophic subalgebra of $X(I)$, then $S(I)$ is a neutrosophic B-algebra in its own right.*

Theorem 3.20. *Let $X(I)$ be a nonzero neutrosophic B-algebra. Then*

(i) $X' = \{(x, 0I) : x \in X\}$ *is a neutrosophic subalgebra of $X(I)$.*

(ii) $X'' = \{(0, xI) : x \in X\}$ *is a neutrosophic subalgebra of $X(I)$.*

(iii) $X_\omega(I) = \{(a, aI) : a \in X\}$ *is a neutrosophic subalgebra of $X(I)$.*

Proof: (i) Clearly, $(0, 0I) \in X'$. Suppose that$(a, 0I), (b, 0I) \in X'$. Then by Remark 3.5, $(a, 0I) \cdot (b, 0I) \in X'$. Moreover, $\{(0, 0I)\} \subsetneq X'$ is a B-algebra. Hence, X' is a neutrosophic subalgebra of $X(I)$. (ii) Clearly, $(0, 0I) \in X''$. Suppose that$(0, aI), (0, bI) \in X''$. Then $a, b \in X$ and $(0, aI) \cdot (0, bI) = (0, ((0 * b \wedge a * 0) \wedge a * b)I)$. Since X is a B-algebra, $(0 * b \wedge a * 0) \wedge a * b \in X$ and so $(0, aI) \cdot (0, bI) = (0, ((0 * b \wedge a * 0) \wedge a * b)I) \in X''$. Moreover, $\{(0, 0I)\} \subsetneq X''$ is a B-algebra. Therefore, X'' is a neutrosophic subalgebra of $X(I)$. (iii) Clearly, $(0, 0I) \in X_\omega(I)$ and $\{(0, 0I)\} \subsetneq X_\omega(I)$. Let $(a, aI), (b, bI) \in X_\omega(I)$. Then by Lemma 3.8(iii), $(a, aI) \cdot (b, bI) = (a * b, (a * b)I) \in X_\omega(I)$. Therefore, $X_\omega(I)$ is a neutrosophic subalgebra of $X(I)$. □

Definition 3.21. A neutrosophic subalgebra $N(I)$ of $X(I)$ is *normal* if for any $(a, bI) \cdot (c, dI), (x, yI) \cdot (u, vI) \in N(I)$, $[(a, bI) \cdot (x, yI)] \cdot [(c, dI) \cdot (u, vI)] \in N(I)$.

Example 3.22. Consider the neutrosophic B-algebra $X(I)$ in Example 3.12 and its subset $N(I) = \{(0, 0I), (0, I), (0, 2I), (1, 0I), (1, I), (1, 2I), (2, 0I), (2, I), (2, 2I)\}$ determined by $N = \{0, 1, 2\}$ and I. It can be verified that $N(I)$ is a normal neutrosophic subalgebra of $X(I)$ with $\{(0, 0I), (1, 0I), (2, 0I)\}$ as its proper subset which is a B-algebra.. It can also be verified that N is a normal subalgebra of X.

The observation in the preceding example are generalized in the next theorem.

Theorem 3.23. *Let $S(I) = \{(a, bI) \in X(I) : a, b \in S, S \subseteq X\}$. Then $S(I)$ is a neutrosophic subalgebra of $X(I)$ if and only if S is a nonzero subalgebra of X. Moreover, if $S(I)$ is normal in $X(I)$, then S is normal in X.*

Proof: Let S be a nonzero subalgebra of X. Let $(a, bI), (c, dI) \in S(I)$. Then $a, b, c, d \in S$ and $(a, bI) \cdot (c, dI) = \big(a * c, ((a * d \wedge b * c) \wedge b * d)I\big)$. Since S is a subalgebra of X, $a * c$, $a * d$, $b * c$, $b * d \in S$ so that $(a * d \wedge b * c) \wedge b * d \in S$. Thus, $(a, bI) \cdot (c, dI) \in S(I)$. By Definition 3.4, $S(I)$ is a neutrosophic B-algebra determined by S and I. Hence, by Remark 3.5, $S(I)$ contains the B-algebra $S' = \{(x, 0I) : x \in S\}$ as a proper subset. Therefore, $S(I)$ is a neutrosophic subalgebra of $X(I)$. Conversely, let $S(I)$ be a neutrosophic subalgebra of $X(I)$. Then $S \neq \varnothing$ and $S \neq \{0\}$. By Remark 3.19(ii), $S(I)$ is a neutrosophic B-algebra. Thus, S is a B-algebra by Definition 3.4. Hence, S is a nonzero subalgebra of X.

Moreover, let $a * b, c * d \in S$. By definition of $S(I)$ and Lemma 3.8(ii), $(a * b, 0I) = (a, 0I) \cdot (b, 0I), (c * d, 0I) = (c, 0I) \cdot (d, 0I) \in S(I)$. By normality of $S(I)$ and Lemma 3.8(ii),

$$\begin{aligned}[(a, 0I) \cdot (c, 0I)] \cdot [(b, 0I) \cdot (d, 0I)] &= \big[(a * c, 0I) \cdot (b * d, 0I)\big] \\ &= \big([(a * c) * (b * d)], 0I\big) \in S(I).\end{aligned}$$

By definition of $S(I)$, $[(a * c) * (b * d)] \in S$ and hence S is normal in X. □

Since a neutrosophic subalgebra is also a neutrosophic B-algebra contained in a given neutrosophic B-algebra, the following remark follows.

Remark 3.24. *If $N(I)$ is a normal neutrosophic subalgebra of $X(I)$, then $N(I)$ is normal in every neutrosophic subalgebra of $X(I)$ containing $N(I)$.*

Example 3.25. In the neutrosophic B-algebra $X(I)$ in Example 3.6, $X(I)$ has three neutrosophic subalgebras: $\{(0, 0I), (0, I), (0, 2I)\}$, $\{(0, 0I), (1, 0I), (2, 0I)\}$ and itself. The first two have $\{(0, 0I)\}$ as their proper subset which is a B-algebra and the latter has $\{(0, 0I), (1, 0I), (2, 0I)\}$. In the neutrosophic B-algebra $X(I)$ in Example 3.12, the following are some of its neutrosophic subalgebras:

$$\begin{aligned}S(I)_1 =& X(I) = \{(0, 0I), (1, 0I), (2, 0I), (3, 0I), (4, 0I), (5, 0I), (0, I), (1, I),\\ & (2, I), (3, I), (4, I), (5, I), (0, 2I), (1, 2I), (2, 2I), (3, 2I), (4, 2I), (5, 2I),\\ & (0, 3I), (1, 3I), (2, 3I), (3, 3I), (4, 3I), (5, 3I), (0, 4I), (1, 4I), (2, 4I),\\ & (3, 4I), (4, 4I), (5, 4I), (0, 5I), (1, 5I), (2, 5I), (3, 5I), (4, 5I), (5, 5I)\}\\ S(I)_2 =& \{(0, 0I), (0, I), (0, 2I), (0, 3I), (0, 4I), (0, 5I), (1, 0I), (1, I), (1, 2I),\\ & (1, 3I), (1, 4I), (1, 5I), (2, 0I), (2, I), (2, 2I), (2, 3I), (2, 4I), (2, 5I)\},\\ S(I)_3 =& \{(0, 0I), (0, I), (0, 2I), (0, 3I), (0, 4I), (0, 5I), (3, 0I), (3, I), (3, 2I),\\ & (3, 3I), (3, 4I), (3, 5I)\},\\ S(I)_4 =& \{(0, 0I), (0, I), (0, 2I), (0, 3I), (0, 4I), (0, 5I), (4, 0I), (4, I), (4, 2I),\\ & (4, 3I), (4, 4I), (4, 5I)\},\end{aligned}$$

$$S(I)_5 = \{(0,0I),(0,I),(0,2I),(0,3I),(0,4I),(0,5I),(5,0I),(5,I),(5,2I),(5,3I),(5,4I),(5,5I)\}$$
$$S(I)_6 = \{(0,0I),(1,I),(2,2I),(3,3I),(4,4I),(5,5I)\}$$

Aside from $\{(0,0I)\}$, sets X', $\{(0,0I),(1,0I),(2,0I)\}$, $\{(0,0I),(3,3I)\}$, $\{(0,0I),(4,4I)\}$, $\{(0,0I),(5,5I)\}$ and $\{(0,0I),(1,I),(2,2I)\}$ are proper subsets of $S(I)_i$, $i = 1,2,3,4,5,6$, respectively, which are B-algbras.

Notice that the neutrosophic B-algebra in Example 3.6 is a neutrosophic subalgebra of the neutrosophic B-algebra in Example 3.12.

In a B-algebra, the intersection of any collection of subalgebras is also a subalgebra by Lemma 2.8. However, this is not always the case for neutrosophic B-algebra. Consider the following example.

Example 3.26. Consider the neutrosophic B-algebra in Example 3.12 and a collection of its neutrosophic subalgebras in Example 3.25. Clearly, $\bigcap_{i=1}^{6} S(I)_i = \{(0,0I)\}$ is not a neutrosophic subalgebra of $X(I)$ by Remark 3.19. However, $\bigcap_{i=1}^{5} S(I)_i = \{(0,0I),(0,I),(0,2I)\}$ is a neutrosophic subalgebra of $X(I)$ with $\{(0,0I)\}$ as its proper subset which is a B-algebra.

The following theorem provides a necessary and sufficient condition for the intersection of neutrosophic subalgebras to be a neutrosophic subalgebra.

Theorem 3.27. *Let $\{S(I)_\alpha : \alpha \in \mathscr{A}\}$ be any nonempty collection of neutrosophic subalgebras (resp., normal neutrosophic subalgebras) of a neutrosophic B-algebra $X(I)$. If $\bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha \neq \{(0,0I)\}$, then $\bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$ is a neutrosophic subalgebra (resp., normal neutrosophic subalgebra) of $X(I)$.*

Proof: Since $(0,0I) \in S(I)_\alpha$ for every $\alpha \in \mathscr{A}$, $(0,0I) \in \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$ so that $\bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha \neq \varnothing$. Since $\bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha \neq \{(0,0I)\}$, there exists $(a,bI) \in \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$ such that $(a,bI) \neq (0,0I)$. Thus, $\{(0,0I)\} \subsetneq \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$ which is a B-algebra. Let $(a,bI),(c,dI) \in \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$. Then $(a,bI),(c,dI) \in S(I)_\alpha$ for every $\alpha \in \mathscr{A}$. Since for every $\alpha \in \mathscr{A}$, $S(I)_\alpha$ is a neutrosophic subalgebra of $X(I)$, $(a,bI)\cdot(c,dI) \in S(I)_\alpha$ for every $\alpha \in \mathscr{A}$. Thus, $(a,bI)\cdot(c,dI) \in \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$. Hence, $\bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$ is a neutrosophic subalgebra of $X(I)$. Moreover, let $\{S(I)_\alpha : \alpha \in \mathscr{A}\}$ be any nonempty collection of normal neutrosophic subalgebras of a $X(I)$ and $(a,bI)\cdot(c,dI),(x,yI)\cdot(u,vI) \in \bigcap_{\alpha\in\mathscr{A}} S(I)_\alpha$. Then $(a,bI)\cdot(c,dI),(x,yI)\cdot(u,vI) \in S(I)_\alpha$ for every $\alpha \in \mathscr{A}$. Since $S(I)_\alpha$ is normal for every $\alpha \in \mathscr{A}$, $[(a,bI)\cdot(x,yI)]\cdot[(c,dI)\cdot(u,vI)] \in$

$S(I)_\alpha$ for every $\alpha \in \mathscr{A}$. Hence, $[(a, bI) \cdot (x, yI)] \cdot [(c, dI) \cdot (u, vI)] \in \bigcap_{\alpha \in \mathscr{A}} S(I)_\alpha$. Therefore, $\bigcap_{\alpha \in \mathscr{A}} S(I)_\alpha$ is a normal neutrosophic subalgebra of $X(I)$. □

Example 3.28. Consider the neutrosophic B-algebra in Example 3.12 and a collection of its neutrosophic subalgebras in Example 3.25. The union $\bigcup_{i=4}^{6} S(I)_i$ is not a neutrosophic subalgebra of $X(I)$ since $(1, I), (0, 5I) \in \bigcup_{i=4}^{6} S(I)_i$ but $(1, I) \cdot (0, 5I) = (1, 4I) \notin \bigcup_{i=4}^{6} S(I)_i$.

Theorem 3.29. *Let $\{S(I)_i : i \in \mathscr{I}\}$ be any nonempty collection of neutrosophic subalgebras of a neutrosophic B-algebra $X(I)$ such that $S(I)_1 \subseteq S(I)_2 \subseteq S(I)_3 \subseteq \cdots$. Then $\bigcup_{i \in \mathscr{I}} S(I)_i$ is a neutrosophic subalgebra of $X(I)$.*

Proof:
Clearly, $\bigcup_{i \in \mathscr{I}} S(I)_i \neq \varnothing$. Let $(a, bI), (c, dI) \in \bigcup_{i \in \mathscr{I}} S(I)_i$. Then for some $i \in \mathscr{I}$, $(a, bI), (c, dI) \in S(I)_i$ and $(a, bI) \cdot (c, dI) \in S(I)_i$. Thus, $(a, bI) \cdot (c, dI) \in \bigcup_{i \in \mathscr{I}} S(I)_i$. Let $P(I)_i$ be a proper subset of $S(I)_i$, for every $i \in \mathscr{I}$ which is a B-algebra. Then for any $i \in \mathscr{I}$, $P(I)_i \subsetneq \bigcup_{i \in \mathscr{I}} S(I)_i$. Therefore, $\bigcup_{i \in \mathscr{I}} S(I)_i$ is a neutrosophic subalgebra of $X(I)$. □

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