



Stable Locating-Dominating Sets in Graphs

Eman C. Ahmad^{1,*}, Gina A. Malacas¹, Sergio R. Canoy, Jr.¹

¹ *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra and Analysis-PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. A set $S \subseteq V(G)$ of a (simple) undirected graph G is a locating-dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $vw \in E(G)$ and $N_G(x) \cap S \neq N_G(y) \cap S$ for any distinct vertices x and y in $V(G) \setminus S$. S is a stable locating-dominating set of G if it is a locating-dominating set of G and $S \setminus \{v\}$ is a locating-dominating set of G for each $v \in S$. The minimum cardinality of a stable locating-dominating set of G , denoted by $\gamma_l^s(G)$, is called the stable locating-domination number of G . In this paper, we investigate this concept and the corresponding parameter for some graphs. Further, we introduce other related concepts and use them to characterize the stable locating-dominating sets in some graphs.

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1. Introduction

The standard concept of domination in a graph has been continuously modified to give rise to new domination parameters. Indeed, a lot of variations of domination have been introduced and studied at different angles and in many ways. One variation of domination which finds an interesting application in the location-determination problem of monitoring devices in a system to ensure its safety was defined and studied by Slater in [6] and [7]. This variant is called locating-domination, a combination of the concepts of locating and domination, and can be used to model a protection strategy that determines locations of monitoring devices (e.g. fire alarms or surveillance cameras) in such a way that the exact location of an intruder (e.g. fire, burglar) can be singled out when a problem (presence of an intruder or fire) at a facility or system arises. The papers in [1], [2], [3], [4], and [5] also dealt with the concept of locating-domination and some of its related concepts.

*Corresponding author.

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Email addresses: eman.ahmad@g.msuiit.edu.ph (E. Ahmad),

gina.malacas@g.msuiit.edu.ph (G. Malacas), sergio.canoy@g.msuiit.edu.ph (S. Canoy, Jr.)

In a system where monitoring devices are installed using the concept of locating-domination, an interesting issue to consider is when, at a given time, exactly one device unexpectedly becomes defective or non-functional. When this situation happens, the remaining number of devices may not necessarily function as intended, that is, presence of an intruder at a certain location in a system may not be precisely detected or identified. To address this specific problem, an additional condition can be imposed to the locating-domination concept to ensure stability and consistency of the devices installed using the modified concept. Thus, in this paper, we introduce the concept of stable locating-dominating set in a graph.

Let $G = (V(G), E(G))$ be a simple graph. The distance between two vertices u and v of G , denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v . Any path connecting u and v of length $d_G(u, v)$ is called a u - v geodesic. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v , denoted by $deg_G(v)$, is equal to $|N_G(v)|$. A vertex v of G is called *isolated* if $deg_G(v) = 0$. Vertex v is a *leaf* if $deg_G(v) = 1$ and the vertex $u \in V(G) \cap N_G(v)$ is called a *support vertex*. The *minimum degree* of G is $\delta(G) = \min\{deg_G(v) : v \in V(G)\}$ and its *maximum degree* is $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N_G[S] = V(G)$ (resp. $N_G(S) = V(G)$). The smallest cardinality of a dominating set of G , denoted by $\gamma(G)$, is called the *domination number* of G . A dominating set of G with cardinality $\gamma(G)$ is called a γ -set of G .

A subset S of G is a *locating set* in G if $N_G(u) \cap S \neq N_G(v) \cap S$ for every two distinct vertices u and v of $V(G) \setminus S$. Set S is said to be a *strictly locating set* if it is a locating set and $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. A locating (resp. strictly locating) subset S of $V(G)$ which is also a dominating set is called a *locating-dominating* (resp. *strictly locating-dominating*) set in G . The minimum cardinality of a locating-dominating (resp., strictly locating-dominating) set in G , denoted by $\gamma_l(G)$ (resp. $\gamma_{sl}(G)$) is called the *locating-domination* (resp. *strictly locating-domination*) number of G .

A locating set (resp. strictly locating set) S in G is a *stable locating set* (resp. *stable strictly locating set*) in G if $S_v = S \setminus \{v\}$ is a locating set (resp. strictly locating set) of G for each $v \in S$. A locating-dominating (strictly locating dominating) set S of G is a *stable locating-dominating set* (resp. *stable strictly locating-dominating set*) of G if $S_v = S \setminus \{v\}$ is a locating-dominating set (resp. strictly locating-dominating set) of G for each $v \in S$. The minimum cardinality of a stable locating dominating set (resp. stable strictly locating-dominating set) of G , denoted by $\gamma_l^s(G)$ (resp. $\gamma_{sl}^s(G)$), is called the *stable locating-domination* (resp. *stable strictly locating-domination*) number of G . A stable locating-dominating (resp. stable strictly locating-dominating) set of G with cardinality $\gamma_l^s(G)$ (resp. $\gamma_{sl}^s(G)$) is called a γ_l^s -set (resp. γ_{sl}^s -set) of G . Slater in [8] introduced and studied the concept of fault-tolerant locating-dominating set. He showed that every fault-tolerant locating-dominating set is also a stable locating-dominating set.

2. Results

Given a graph G , we denote by $L(G)$ and $S(G)$ the sets of leaves and support vertices of G , respectively.

Proposition 1. *Let G be a non-trivial graph. Then G admits a stable locating-dominating set if and only if G has no isolated vertices. If G has no isolated vertices, then $2 \leq \gamma_l^s(G) \leq |V(G)|$. Moreover, the following statements hold:*

- (i) $\gamma_l^s(G) = 2$ if and only if $G = K_2$.
- (ii) If S is a stable locating-dominating set of G , then $L(G) \cup S(G) \subseteq S$.
- (iii) $\gamma_l^s(G) = |V(G)|$ if and only if for every $v \in V(G)$, $v \in L(G) \cup S(G)$ or there exists $w \in V(G) \setminus \{v\}$ such that $N_G(v) = N_G(w)$.

Proof. Suppose G has no isolated vertices. Let $S = V(G)$ and let $v \in S$. Clearly, $S \setminus \{v\}$ is locating set. Since v is not an isolated vertex, there exists $u \in S \setminus \{v\}$ such that $uv \in E(G)$. Hence, S is a dominating set, showing that S is a stable locating-dominating set of G .

For the converse, suppose that G admits a stable locating-dominating set, say S' . Suppose further that G has an isolated vertex, say z . Since S' is a dominating set, $z \in S'$. This implies that $S' \setminus \{z\}$ is not a dominating set, contrary to the assumption that S is a stable locating-dominating set of G . Therefore, G has no isolated vertices.

Next, suppose that G has no isolated vertices. Let S be a stable locating-dominating set of G . Clearly, $|S| \leq |V(G)|$. Since an empty set is not a dominating set, $|S| \geq 2$. Since S was an arbitrary stable locating-dominating set, it follows that $2 \leq \gamma_l^s(G) \leq |V(G)|$.

(i) Clearly, $\gamma_l^s(K_2) = 2$.

For the converse, suppose that $\gamma_l^s(G) = 2$, say $S = \{x, y\}$ is γ_l^s -set of G . Since $S_x = S \setminus \{x\} = \{y\}$ is a dominating set, $xy \in E(G)$. Suppose there exists $z \in V(G) \setminus S$. Then $N_G(x) \cap S_x = \{y\} = N_G(z) \cap S_x$, showing that S_x is not a locating set. Therefore, S is not a stable locating-dominating set, contrary to our assumption that S is γ_l^s -set. Therefore, $G = K_2$.

(ii) Let S be a stable locating-dominating set. Let $v \in L(G)$ and let $u \in N_G(v)$. Suppose $v \notin S$. Since S is a dominating set, $u \in S$. This, however, would mean that $S_u = S \setminus \{u\}$ is not a dominating set (there is no vertex in S_u that dominates v), contrary to the assumption that S is a stable locating-dominating set of G . Thus, $v \in S$ and, consequently, $L(G) \subseteq S$. Next, let $w \in S(G)$ and let $z \in L(G)$ be such that $zw \in E(G)$. Since $z \in S$ (by the first part) and $w \notin S$, $S_z = S \setminus \{z\}$ is not a dominating set (because $w \notin S_z$), a contradiction. This implies that $w \in S$, that is, $S(G) \subseteq S$. Therefore, $L(G) \cup S(G) \subseteq S$.

(iii) Suppose $\gamma_l^s(G) = |V(G)|$. Let v be vertex of G such that $v \notin L(G) \cup S(G)$. Suppose further that $N_G(v) \neq N_G(w)$ for all $w \in V(G) \setminus \{v\}$. Let $S = V(G) \setminus \{v\}$. Clearly, S is a locating-dominating set of G . Now let $z \in V(G) \setminus \{v\}$ and set $S_z = S \setminus \{z\}$. Suppose $vz \notin E(G)$. Choose any $x, y \in V(G)$ such that $xz, vy \in E(G)$. Then $x, y \in S_z$. Next, suppose

that $vz \in E(G)$. Since $v \notin S(G)$, $z \notin L(G)$. Hence, there exists a vertex $a \in S_z$ such that $az \in E(G)$. Further, since $v \notin L(G)$, there exists $b \in S_z$ such that $bv \in E(G)$. Therefore, S_z is a dominating set of G . By the additional assumption that $N_G(v) \neq N_G(w)$, it follows that S_z is a locating set in G . Since z was arbitrarily chosen from S , it follows that S is a stable locating-dominating set of G . Thus, $\gamma_l^s(G) \leq |S| = |V(G) - 1$, a contradiction. Consequently, there exists a $w \in V(G) \setminus \{v\}$ such that $N_G(v) = N_G(w)$.

For the converse, suppose that the given conditions hold in G and let S be a γ_l^s -set of G . Suppose there exists $v \in V(G) \setminus S$. By (ii), $v \notin L(G) \cup S(G) \subseteq S$. It follows from the assumption that there exists $w \in V(G) \setminus \{v\}$ such that $N_G(v) = N_G(w)$. Since S is a locating set, $w \in S$. However, the condition that $N_G(v) = N_G(w)$ would imply that $N_G(v) \cap S_w = N_G(w) \cap S_w$, where $S_w = S \setminus \{w\}$. Hence, S_w is not a locating set of G , contrary to the assumption that S is a stable locating-dominating set of G . Therefore, $S = V(G)$ and $\gamma_l^s(G) = |V(G)|$. \square

The *join* of two graphs G and H , denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 1. *Let G be a graph without isolated vertices and $K_1 = \langle v \rangle$. A set S is a stable locating-dominating set of $H = K_1 + G$ if and only if*

- (i) $v \in S$ and $S_G = S \setminus \{v\}$ is both a strictly locating-dominating and stable locating set of G or
- (ii) S is a stable strictly locating-dominating set of G .

Proof. Suppose that S is a stable locating-dominating set of H . Consider the following cases:

Case 1. $v \in S$

Since S is a stable locating-dominating set of H , $S_G = S \setminus \{v\}$ is a locating-dominating set of H . Hence, $N_H(v) \cap S_G = S_G \neq N_H(w) \cap S_G = N_G(w) \cap S_G$ for all $w \in V(G) \setminus S_G$. Also, for $x, y \in V(G) \setminus S_G$ with $x \neq y$, we have $[N_G(x) \cap S_G] \cup \{v\} = N_H(x) \cap S \neq N_H(y) \cap S = [N_G(y) \cap S_G] \cup \{v\}$. Thus, $N_G(x) \cap S_G \neq N_G(y) \cap S_G$, showing that S_G is a strictly locating-dominating set of G . Next, let $z \in S_G$ and let $S_G^z = S_G \setminus \{z\}$. Since $S_z = S \setminus \{z\}$ is a locating-dominating set in H , $N_H(p) \cap S_z \neq N_H(q) \cap S_z$ for all $p, q \in V(G) \setminus S_G^z$ with $p \neq q$. Since $S_z = (S_G \cup \{v\}) \setminus \{z\} = S_G^z \cup \{v\}$, $N_G(p) \cap S_G^z \neq N_G(q) \cap S_G^z$ for all $p, q \in V(G) \setminus S_G^z$ with $p \neq q$. This implies that S_G^z is a locating set in G . Therefore, S_G is a stable locating set in G . This shows that (i) holds.

Case 2. $v \notin S$

Clearly, S is a dominating set of G . Since S is a locating-dominating set of H , H and $v \notin S$, $N_G(x) \cap S = N_H(x) \cap S \neq N_H(y) \cap S = N_G(y) \cap S$ for all $x, y \in V(G) \setminus S$ with $x \neq y$ and $S = N_H(v) \cap S \neq N_H(z) \cap S = N_G(z) \cap S$ for all $z \in V(G) \setminus S$. Hence, S is a strictly locating-dominating set in G . Next, let $w \in S$ and let $S_w = S \setminus \{w\}$. Since S is a stable locating-dominating set of H , S_w is a locating-dominating set in H . This implies that $N_G(a) \cap S_w = N_H(a) \cap S_w \neq N_H(b) \cap S_w = N_G(b) \cap S_w$ for all $a, b \in V(G) \setminus S_w$

with $a \neq b$. Since $v \notin S_w$, $S_w = N_H(v) \cap S_w \neq N_H(z) \cap S_w = N_G(z) \cap S_w$ for all $z \in V(G) \setminus S_w$. Thus, S_w is a strictly locating-dominating set in G . Therefore, S is a stable strictly locating-dominating set of G , showing that (ii) holds.

For the converse, suppose first that (i) holds. Since $S_G = S \setminus \{v\}$ is a strictly locating-dominating set of G , S is a locating-dominating set of H . Let $x \in S$ and let $S_x = S \setminus \{x\}$. If $x = v$, then $S_x = S_G$ is a strictly locating-dominating set of G by assumption. Hence, $N_H(a) \cap S_x = N_G(a) \cap S_x \neq N_G(b) \cap S_w = N_H(b) \cap S_x$ for all $a, b \in V(H) \setminus (S_x \cup \{v\})$ with $a \neq b$, and $N_H(x) \cap S_x = S_x \neq N_G(d) \cap S_x = N_H(d) \cap S_x$ for all $d \in V(H) \setminus (S_x \cup \{v\})$. This implies that S_x is an locating-dominating set of H . Suppose $x \neq v$. Then $x \in S_G$. Since S_G is an stable locating set of G , $S_G^x = S_G \setminus \{x\}$ is a locating set of G . Since $S_x = S_G^x \cup \{v\}$, S_x is a locating set of H . Further, because $v \in S_x$, S_x is a locating-dominating set in H . Therefore, S is a stable locating -dominating set of H .

Next, suppose that (ii) holds. Since S is strictly locating-dominating set in G , it is a locating-dominating set of H . Let $z \in S$. By assumption, $S_z = S \setminus \{z\}$ is a strictly locating-dominating set G . Therefore, S_z is a locating-dominating set of H . This shows that S is a stable locating-dominating set of H . □

Corollary 1. *Let G be a graph without isolated vertices. Then*

$$\gamma_l^s(K_1 + G) = \begin{cases} \rho(G), & \text{if } \rho(G) = \gamma_{sl}^s(G) \\ \rho(G) + 1, & \text{if } \rho(G) < \gamma_{sl}^s(G), \text{ where} \end{cases}$$

$\rho(G) = \min\{|S| : S \text{ is both a strictly locating-dominating and a stable locating set of } G\}$.

Proof. Let S be a γ_{sl}^s -set of G . Then S is a strictly locating-dominating set of G . In particular, S is a locating set of G . Since S is a a stable strictly locating-dominating set of G , $S \setminus \{z\}$ is a strictly locating-dominating set of G for each $z \in S$. This implies that $S \setminus \{z\}$ is a locating set of G for each $z \in S$. Thus, S is a stable locating set of G , showing that $\rho(G) \leq |S| = \gamma_{sl}^s(G)$.

Now, suppose that S_0 is a γ_l^s -set of $K_1 + G$. Suppose that $\rho(G) = \gamma_{sl}^s(G)$. Then $\gamma_{sl}^s(G) < \rho(G) + 1$. By Theorem 1, S_0 must be a γ_{sl}^s -set of G . Hence, $\gamma_l^s(K_1 + G) = |S_0| = \gamma_{sl}^s(G)$. If $\rho(G) < \gamma_{sl}^s(G)$, then $\rho(G) + 1 \leq \gamma_{sl}^s(G)$. By Theorem 1, $v \in S_0$ and $S_0 \setminus \{v\}$ must be both a strictly locating-dominating and stable locating set of G and $|S_0 \setminus \{v\}| = \rho(G)$. Therefore, $\gamma_l^s(K_1 + G) = |S_0| = |S_0 \setminus \{v\}| + 1 = \rho(G) + 1$. This proves the assertion. □

Theorem 2. *Let G and H be non-trivial graphs. A set S is a stable locating-dominating set of $G + H$ if and only if $S = S_G \cup S_H$ and S_G and S_H are stable locating sets of G and H , respectively, and at least one of them is a stable strictly locating set or both of them are strictly locating sets.*

Proof. Suppose S is a stable locating-dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Suppose $S_G = \emptyset$. Then $S = S_H$. Pick $x, y \in V(G) \subset V(G + H) \setminus S$ with $x \neq y$ (these vertices exist because G is non-trivial). Then $N_{G+H}(x) \cap S = S = N_{G+H}(y) \cap S$, a contradiction to the fact that S is a locating set. Therefore, $S_G \neq \emptyset$.

Similarly, $S_H \neq \emptyset$. Now, let $a, b \in V(G) \setminus S_G$ with $a \neq b$. Since S is a locating set of $G + H$, $[N_G(a) \cap S_G] \cup S_H = N_{G+H}(a) \cap S \neq N_{G+H}(b) \cap S = [N_G(b) \cap S_G] \cup S_H$. Hence, $N_G(a) \cap S_G \neq N_G(b) \cap S_G$, showing that S_G is a locating set of G . Next, let $z \in S_G$ and let $S_G^z = S_G \setminus \{z\}$. Since S is a stable locating-dominating set of $G + H$, $S_z = S \setminus \{z\} = S_G^z \cup S_H$ is a locating-dominating in $G + H$. It follows that for any two distinct vertices $p, q \in V(G) \setminus S_G^z$, $[N_G(p) \cap S_G^z] \cup S_H = N_{G+H}(p) \cap S_z \neq N_{G+H}(q) \cap S_z = [N_G(q) \cap S_G^z] \cup S_H$. This implies that $N_G(p) \cap S_G^z \neq N_G(q) \cap S_G^z$. Thus, S_G^z is a locating set in G , showing that S_G is a stable locating set of G . Similarly, S_H is a stable locating set of H .

Suppose that S_G and S_H are not stable strictly locating sets. Suppose that S_G is not a strictly locating set of G . Then there exists $v \in V(G) \setminus S_G$ such that $N_G(v) \cap S_G = S_G$. Suppose S_H is not a strictly locating set. Then there exists $w \in V(H) \setminus S_H$ such that $N_H(w) \cap S_H = S_H$. Consequently, $N_{G+H}(v) \cap S = S = N_{G+H}(w) \cap S$, contrary to the fact that S is a locating set in $G + H$. Thus, S_H is a strictly locating set of H . Now, let $y \in S_H$ and set $S_H^y = S_H \setminus \{y\}$. Since S_H is a stable locating set of H , it follows that S_H^y is a locating set of H . From the assumption that S is a stable locating-dominating set of $G + H$, the set $S_y = S \setminus \{y\} = S_G \cup S_H^y$ is a locating set of $G + H$. This implies that $[N_G(v) \cap S_G] \cup S_H^y = S_G \cup S_H^y = N_{G+H}(v) \cap S_y \neq N_{G+H}(u) \cap S_y = S_G \cup [N_H(u) \cap S_H^y]$ for all $u \in V(H) \setminus S_H^y$. Hence, $N_H(u) \cap S_H^y \neq S_H^y$ for all $u \in V(H) \setminus S_H^y$. This shows that S_H^y is a strictly locating set of H . Therefore, S_H is a stable strictly locating set of H , a contradiction. Therefore, S_G is a strictly locating set of G . Similarly, S_H is a strictly locating set of H .

Conversely, suppose that $S = S_G \cup S_H$ and S_G and S_H are stable locating sets of G and H , respectively, such that at least one of them is a stable strictly locating set or both of them are strictly locating sets. Then S is a dominating set of $G + H$. Suppose first that one of S_G or S_H , say S_G is a stable strictly locating set. Let $a, b \in V(G + H) \setminus S$ where $a \neq b$. Since S_G and S_H are locating sets, $N_{G+H}(a) \cap S \neq N_{G+H}(b) \cap S$ if $a, b \in V(G) \setminus S_G$ or $a, b \in V(H) \setminus S_H$. Suppose $a \in V(G) \setminus S_G$ and $b \in V(H) \setminus S_H$. Since S_G is strictly locating, $N_G(a) \cap S_G \neq S_G$. Hence, $N_{G+H}(a) \cap S = [N_G(a) \cap S_G] \cup S_H \neq S_G \cup [N_H(b) \cap S_H] = N_{G+H}(b) \cap S$. Therefore, S is a locating set of $G + H$. Next, let $w \in S$ and let $S_w = S \setminus \{w\}$. Suppose $w \in S_G$ and let $S_G^w = S_G \setminus \{w\}$. Then S_G^w is strictly locating set because S_G is a stable strictly locating set of G . Following an earlier argument, we find that $S_w = S_G^w \cup S_H$ is a locating-dominating set of $G + H$. If $w \in S_H$, then $S_w = S \setminus \{w\} = S_G \cup S_H^w$, where $S_H^w = S_H \setminus \{w\}$ is a locating set of H by an assumption. These and the fact that S_G is strictly locating will imply that S_w is a locating-dominating set of $G + H$. Therefore, S is a stable locating-dominating set of G .

Next, suppose that S_G and S_H are both strictly locating sets. Then S is a locating-dominating set of $G + H$. Let $v \in S$. Suppose that $v \in S_G$ and let $S_v = S \setminus \{v\} = (S_G \setminus \{v\}) \cup S_H$. Since S_G is a stable locating set of G , $S_G^v = S_G \setminus \{v\}$ is a locating set of G . Clearly, S_v is a dominating set of G . Let $x, y \in V(G + H) \setminus S_v$ with $x \neq y$. Consider the following cases:

Case 1. $x, y \in V(G)$ Then $x, y \in V(G) \setminus S_G^v$, where $S_G^v = S_G \setminus \{v\}$. Since S_G^v is a locating set in G , $N_{G+H}(x) \cap S_v = [N_G(x) \cap S_G^v] \cup S_H \neq [N_G(y) \cap S_G^v] \cup S_H = N_{G+H}(y) \cap S_v$.

Case 2. $x, y \in V(H)$ Then $x, y \in V(H) \setminus S_H$. Since S_H is a locating set in H ,

$$N_{G+H}(x) \cap S_v = S_G^v \cup (N_H(x) \cup S_H) \neq S_G^z \cup (N_H(y) \cap S_H) = N_{G+H}(y) \cap S_v.$$

Case 3. $x \in V(G)$ and $y \in V(H)$ (or $y \in V(G)$ and $x \in V(H)$)

Then $x \in V(G) \setminus S_G^v$, $y \in V(H) \setminus S_H$, $N_{G+H}(x) \cap S_v = [N_G(x) \cap S_G^v] \cup S_H$, and $N_{G+H}(y) \cap S_v = S_G^v \cup [N_H(y) \cap S_H]$. Since S_H is a strictly locating set of H , $N_H(y) \cap S_H \neq S_H$. Therefore, $N_{G+H}(x) \cap S_v \neq N_{G+H}(y) \cap S_v$.

Since S_G is also strictly locating, it follows that $N_{G+H}(x) \cap S_v \neq N_{G+H}(y) \cap S_v$ when $v \in S_H$ and $S_v = S \setminus \{v\} = S_G \cup (S_H \setminus \{v\})$. Accordingly, S is a stable locating-dominating set of $G + H$. □

Lemma 1. *Let G be a non-trivial connected graph. If $\gamma(G) = 1$, then every nonempty set $S \subseteq V(G)$ is not a stable strictly locating set of G . In particular, G does not admit a stable strictly locating set.*

Proof. Suppose $\emptyset \neq S \subseteq V(G)$ and let $\{v\}$ be a dominating set of G . If $v \notin S$, then $N_G(v) \cap S = S$. Hence, S is not a strictly locating set. Suppose $v \in S$. Then $N_G(v) \cap S_v = S_v$, where $S_v = S \setminus \{v\}$. This implies that S_v cannot be a strictly locating set. Therefore, S is not a stable strictly locating set of G . □

Theorem 3. *Let G be a graph without isolated vertices. Then G has a stable strictly locating set if and only if $\gamma(G) \neq 1$.*

Proof. Suppose G has a stable strictly locating set. Then $\gamma(G) \neq 1$ by Lemma 1.

For the converse, suppose that $\gamma(G) \neq 1$. Let $S = V(G)$. Then clearly, S is a strictly locating set. Let $w \in S$ and let $S_w = S \setminus \{w\}$. Then S_w is a locating set of G . Moreover, since $\{w\}$ is not a dominating set of G , $N_G(w) \cap S_w \neq S_w$. This implies that S_w is a strictly locating set of G . Therefore, $S = V(G)$ is a stable strictly locating set of G . □

Theorem 4. *Let G be a connected graph with $\Delta(G) = n - 2$, where n is the order G . If S is a stable strictly locating set, then $v, w \in S$ for every pair of vertices v and w with $deg_G(v) = n - 2$ and $vw \notin E(G)$.*

Proof. Let $v, w \in V(G)$ with $deg_G(v) = n - 2$ and $vw \notin E(G)$. Suppose $v \notin V(G)$. Since S is a strictly locating set of G , $w \in S$. By the assumption that S is a stable strictly locating set, it follows that $S_w = S \setminus \{w\}$ is a strictly locating set. This, however, is not possible because $N_G(v) \cap S_w = S_w$. Therefore, $v \in S$. Next, suppose that $w \notin S$. Then $N_G(v) \cap S_v = S_v = S \setminus \{v\}$. Hence, S is not a stable strictly locating set of G , a contradiction. Thus, $w \in S$, proving our assertion. □

Given a graph G without isolated vertices, we will use the following notations:

$$\eta_{sls}(G) = \min\{|S| : S \text{ is a stable locating set of } G\}$$

$$\eta_{ssls}(G) = \min\{|S| : S \text{ is a stable strictly locating set of } G\}$$

$$\eta_{slsls}(G) = \min\{|S| : S \text{ is a strictly locating set and a stable locating set of } G\}$$

Corollary 2. *Let G and H be graphs without isolated vertices.*

(i) *If $\gamma(G) = \gamma(H) = 1$, then*

$$\gamma_l^s(G + H) = \eta_{slsls}(G) + \eta_{slsls}(H).$$

(ii) *If $\gamma(G) \neq 1$ and $\gamma(H) = 1$, then*

$$\gamma_l^s(G + H) = \min\{\eta_{slsls}(G) + \eta_{slsls}(H), \eta_{ssls}(G) + \eta_{sls}(H)\}.$$

(iii) *If $\gamma(G) = 1$ and $\gamma(H) \neq 1$, then*

$$\gamma_l^s(G + H) = \min\{\eta_{slsls}(G) + \eta_{slsls}(H), \eta_{sls}(G) + \eta_{ssls}(H)\}.$$

(iv) *If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$, then*

$$\gamma_l^s(G + H) = \min\{\eta_{slsls}(G) + \eta_{slsls}(H), \eta_{sls}(G) + \eta_{ssls}(H), \eta_{ssls}(G) + \eta_{sls}(H)\}.$$

Proof. Let S be a γ_l^s -set of $G + H$. By Theorem 2, $S = S_G \cup S_H$ and S_G and S_H are stable locating sets of G and H , respectively.

(i) If $\gamma(G) = \gamma(H) = 1$, then G and H do not admit stable strictly locating sets by Theorem 3. By Theorem 2, S_G and S_H are both strictly locating sets. Thus,

$$\gamma_l^s(G + H) = \eta_{slsls}(G) + \eta_{slsls}(H).$$

(ii) Suppose $\gamma(G) \neq 1$ and $\gamma(H) = 1$. Then only H does not admit a stable strictly locating set. By Theorem 2 and by combining all possible pairings, we have

$$\gamma_l^s(G + H) = \min\{\eta_{slsls}(G) + \eta_{slsls}(H), \eta_{ssls}(G) + \eta_{sls}(H)\}.$$

(iii) This is similar to (ii).

(iv) Since each of the graphs G and H admits a stable strictly locating set, it follows from Theorem 2 that

$$\gamma_l^s(G + H) = \min\{\eta_{slsls}(G) + \eta_{slsls}(H), \eta_{ssls}(G) + \eta_{sls}(H), \eta_{ssls}(G) + \eta_{sls}(H)\}.$$

These prove our assertions. □

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H , and then forming the join $\langle\{v}\rangle + H^v = v + H^v$, where H^v is a copy of H , for each $v \in V(G)$.

Theorem 5. *Let G be a connected non-trivial graph and let H be any graph without isolated vertices. Then $S \subseteq V(G \circ H)$ is a stable locating-dominating set of $G \circ H$ if and only if $S = A \cup [\cup_{v \in V(G)} D_v]$ and satisfies the following properties:*

(i) $A \subseteq V(G)$.

- (ii) D_v is a stable locating-dominating set of H^v for each $v \in (V(G) \setminus A)$, and, in addition, strictly locating when $|N_G(v) \cap A| = 1$.
- (iii) D_v is a stable strictly locating-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$.
- (iv) D_v is a dominating stable locating set for each $v \in A$ and, in addition, strictly locating when $N_G(v) \cap A = \emptyset$.

Proof. Suppose that S is a stable locating-dominating set of $G \circ H$. Let $A = S \cap V(G)$ and let $D_v = S \cap V(H^v)$ for each $v \in V(G)$. Then (i) holds and $S = A \cup [\cup_{v \in V(G)} D_v]$. Suppose $D_v = \emptyset$ for some $v \in V(G)$. Since S is a dominating set, $v \in A$. As S is a stable locating-dominating set, this would imply that $S_v = S \setminus \{v\}$ is a locating-dominating set of $G \circ H$. This, however, is impossible because the $V(H^v) \cap N_{G \circ H}[S_v] = \emptyset$. Thus, $S_v \neq \emptyset$ for each $v \in V(G)$.

Let $v \in V(G) \setminus A$. Since S is a stable locating-dominating set of $G \circ H$ and $v \notin A$, D_v is must be a stable locating dominating set of H^v . Suppose $|N_G(v) \cap A| = 1$, say $w \in N_G(v) \cap A$. Again, since S is a stable locating-dominating set of $G \circ H$, $S_w = S \setminus \{w\}$ is a locating-dominating set of $G \circ H$. We also find that $N_{G \circ H}(v) \cap S_w = D_v$. Therefore, since S_w is a locating set, $N_{G \circ H}(a) \cap S_w = N_{H^v}(a) \cap D_v \neq D_v$ for all $a \in V(H^v) \setminus D_v$. Thus, D_v is a strictly locating set of H^v . This shows that (ii) holds. Suppose now that $v \in V(G) \setminus N_G[A]$. Then, by Theorem 1(ii), D_v is a stable strictly locating-dominating set of H^v , showing that (iii) holds.

Next, suppose that $v \in A$. Since S is a locating set of $G \circ H$ and $v \in A$, $[N_{H^v}(p) \cap D_v] \cup \{v\} = N_{G \circ H}(p) \cap S \neq N_{G \circ H}(q) \cap S = N_{H^v}(q) \cap D_v] \cup \{v\}$ for all $p, q \in V(H^v) \setminus D_v$ with $p \neq q$. Therefore, $N_{H^v}(p) \cap D_v \neq N_{H^v}(q) \cap D_v$ for all $p, q \in V(H^v) \setminus D_v$ with $p \neq q$, showing that D_v is a locating set of H^v . Now, since S is a stable locating-dominating set, it follows that $S \setminus \{v\}$ is a locating-dominating set of $G \circ H$. This implies that D_v is a dominating set of H^v . Let $x \in D_v$ and set $D_v^x = D_v \setminus \{x\}$. Since $S_x = S \setminus \{x\}$ is a locating set of $G \circ H$, it follows that D_v^x is a locating set of H^v . Therefore, D_v is a stable locating set of H^v . If $N_G(v) \cap A = \emptyset$, then $D_v \cup \{v\}$ is a stable locating-dominating set of $v + H^v$. By Theorem 1(i), D_v is a strictly locating set of H^v . This shows that (iv) holds.

For the converse, suppose that S has the given form and satisfies properties (i)-(iv). Then clearly, S is a dominating set of $G \circ H$. Let $x, y \in V(G \circ H)$ with $x \neq y$ and let $v, w \in V(G)$ such that $x \in V(v + H^v)$ and $y \in V(w + H^w)$. Consider the following cases:

Case 1. $v = w$

Suppose first that $x = v$ and $y \in V(H^v) \setminus D_v$. If $N_G(x) \cap A \neq \emptyset$, then $N_{G \circ H}(x) \cap S = (N_G(x) \cap A) \cup D_v \neq N_{H^v}(y) \cap D_v = N_{G \circ H}(y) \cap S$. Suppose $N_G(x) \cap A = \emptyset$. Then by (iii), D_v is a strictly locating set of H^v . Hence, $N_{G \circ H}(y) \cap S = N_{H^v}(y) \cap D_v \neq D_v = N_{G \circ H}(x) \cap S$. Next, suppose that $x, y \in V(H^v) \setminus D_v$. Since D_v is a locating set by assumption, it follows that $N_{H^v}(x) \cap D_v \neq N_{H^v}(y) \cap D_v$. Hence, whether or not v is in A , we have $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$.

Case 2. $v \neq w$

Suppose $x = v$ or $y = w$. Since $D_v \subseteq N_{G \circ H}(x) \cap S$ and $D_w \subseteq N_{G \circ H}(y) \cap S$, $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$. Suppose $x \in V(H^v) \setminus D_v$ and $y \in V(H^w) \setminus D_w$. Since

D_v and D_w are dominating sets of H^v and H^w , respectively, $N_{H^v}(x) \cap D_v \neq \emptyset$ and $N_{H^w}(y) \cap D_w \neq \emptyset$. Hence, $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$.

Therefore, S is a locating set of $G \circ H$. Accordingly, S is a locating-dominating set of $G \circ H$.

Next, let $z \in S$ and let $u \in V(G)$ such that $z \in V(u + H^u)$. Let $S_z = S \setminus \{z\}$. Suppose $z = u \in A$. Since D_u is a dominating set of H^u according to (iv), it follows that S_z is a dominating set of $G \circ H$. By assumption, D_u is a locating set of H^u and strictly locating if $N_G(u) \cap A = \emptyset$. Using this and the assumption, it is routine to show that S_z is a locating set of $G \circ H$.

Lastly, suppose that $z \neq u$. Then $z \in D_u$. Note that $D_u \setminus \{z\}$ is a locating-dominating set of H^u if $u \in (V(G) \setminus A)$ by (ii); a strictly locating set if $u \in (V(G) \setminus A) \setminus N_G(A)$ by (iii); and a locating set if $v \in A$ by (iv). Using this and the assumption, it can be shown that S_z is a locating set of $G \circ H$.

Accordingly, S is a stable locating-dominating set of $G \circ H$. □

Let H be a graph without isolated vertices. We shall be using the following notations in our next results.

$$\gamma_i^{sls}(H) = \min\{|D| : D \text{ is a strictly locating and stable locating-dominating set of } H\},$$

$$\gamma_{SL}^s(H) = \min\{|D| : D \text{ is a strictly locating-dominating and stable locating set of } H\}.$$

Any strictly locating and stable locating-dominating set (strictly locating-dominating and stable locating set) of H with cardinality $\gamma_i^{sls}(H)$ (resp. $\gamma_{SL}^s(H)$) is called a γ_i^{sls} -set (resp. γ_{SL}^s -set) of H . Note that every graph without isolated vertices admits a strictly locating stable locating-dominating set and a strictly locating-dominating and stable locating set. Indeed, if H is a graph without isolated vertices, then $V(H)$ is both a strictly locating stable locating-dominating set and a dominating and stable locating set of H . Also, one can easily verify that if $H = K_1 + P_4$, where $V(K_1) = \{a\}$ and $P_4 = [b, c, d, e]$, then $S = \{a, b, d, e\}$ is a strictly locating stable locating-dominating set of H .

Corollary 3. *Let G be a connected non-trivial graph of order m and let H be any graph without isolated vertices.*

- (i) *If $\gamma(H) = 1$, then $\gamma_i^s(G) \leq \gamma(G) + \gamma(G)\gamma_{SL}^s(H) + (m - \gamma(G))\gamma_{sls}^s(H)$.*
- (ii) *If $\gamma(H) \neq 1$, then $\gamma_i^s(G) \leq \min\{\gamma(G) + \gamma(G)\gamma_{SL}^s(H) + (m - \gamma(G))\gamma_{sls}^s(H), m \cdot \gamma_{sl}^s(H)\}$.*

Proof. (i) Suppose $\gamma(H) = 1$. Let A be a γ -set of G . Let D_v be a γ_{SL}^s -set for each $v \in A$ and let it be a γ_i^{sls} -set of H for each $v \in V(G) \setminus A$. By Theorem 5, $S = A \cup [\cup_{v \in A} D_v] \cup [\cup_{v \in V(G) \setminus A} D_v]$ is a stable locating-dominating set of $G \circ H$. It follows that

$$\begin{aligned} \gamma_i^s(G \circ H) &\leq |S| = \gamma(G) + \gamma(G)\gamma_{SL}^s(H) + (m - \gamma(G))\gamma_{sls}^s(H) \\ &= [1 + \gamma_{SL}^s(H) - \gamma_i^{sls}(H)]\gamma(G) + m \cdot \gamma_i^{sls}(H). \end{aligned}$$

(ii) Suppose $\gamma(H) \neq 1$. Let $A = \emptyset$ and let D_v be a γ_{sl}^s -set of H . Then $S' = [\cup_{v \in V(G)} D_v]$ is a stable locating-dominating set of G by Theorem 5. Hence, $\gamma_l^s(G \circ H) \leq |S'| = m \cdot \gamma_{sl}^s(H)$. This and (i) imply that

$$\gamma_l^s(G \circ H) \leq \min\{[1 + \gamma_{SLl}^s(H) - \gamma_l^{sls}(H)]\gamma(G) + m \cdot \gamma_l^{sls}(H), m \cdot \gamma_{sl}^s(H)\}.$$

This establishes the desired results. \square

Consider $G = P_3$ and $H = P_4$. One can easily verify that $m = 3$, $\gamma(G) = 1$, $\gamma_l^{sls}(H) = 4$, $\gamma_{SLl}^s(H) = 3$, and $\gamma_l^s(G \circ H) = 12 = [1 + \gamma_{SLl}^s(H) - \gamma_l^{sls}(H)]\gamma(G) + m \cdot \gamma_l^{sls}(H)$. Thus, the given bound in Corollary 3(i) is tight.

Next, consider $G = P_4$ and $H = P_3$. Then $m = 4$, $\gamma(G) = 2$ and $\gamma_{sl}^s(H) = \gamma_l^{sls}(H) = \gamma_{SLl}^s(H) = 3$. Also,

$$\gamma_l^s(G \circ H) = 12 = m \cdot \gamma_{sl}^s(H) < 14 = [1 + \gamma_{SLl}^s(H) - \gamma_l^{sls}(H)]\gamma(G) + m \cdot \gamma_l^{sls}(H).$$

Hence, the bound in Corollary 3(ii) is also tight.

Conclusion: The concept of stable locating-dominating set, when used to place or install monitoring devices at designated locations in a given system for safeguard by identifying the exact location of an intruder when a problem in a facility occurs, ensures that the remaining devices in a system can still function as expected in an event when exactly one monitor becomes non-functional. Results generated in this study were obtained using other related concepts. It may be interesting and worthwhile to study these concepts and continue this initial investigation on the concept of stable locating-domination.

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