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Mixed type algorithms for asymptotically nonexpansive mappings in hyperbolic spaces

Tanakit Thianwan

Department of Mathematics, School of Science, University of Phayao, Phayao, 56000, Thailand

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Abstract. In this paper, a new mixed type iteration process for approximating a common fixed point of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings is constructed. We then establish a strong convergence theorem under mild conditions in a uniformly convex hyperbolic space. The results presented here extend and improve some related results in the literature.

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Key Words and Phrases: uniformly convex hyperbolic space, strong convergence, common fixed points, mixed type algorithm, asymptotically nonexpansive mapping

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1. Introduction and preliminaries

Iterative schemes play a prominent role in approximating fixed points of nonlinear mappings. Structural properties of the underlying space, such as strict convexity and uniform convexity, are very much needed for the development of iterative fixed point theory in it. Hyperbolic spaces are general in nature and inherit rich geometrical structure suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory.

Fixed-point iteration processes for nonexpansive self and nonself mappings have been studied extensively by various authors to solve the nonlinear operator equations in Hilbert spaces and Banach spaces (see [10, 11, 17–19, 21, 28, 30, 33, 37] and the references cited therein). Goebel and Kirk [6], in 1972, introduced the class of asymptotically nonexpansive self-mappings, which is an important generalization of the class of nonexpansive self-mappings.

In the last few decades investigations of fixed points by some iterative schemes for asymptotically nonexpansive mappings have attracted many mathematicians.

In 1991, Schu [28] introduced the following modified Mann iteration process

$$u_{n+1} = (1-\vartheta_n)u_n + \vartheta_n \mathcal{T}^n u_n, \quad n \geq 1, \tag{1}$$

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Email address: tanakit.th@up.ac.th (T. Thianwan)

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schu's iteration process (1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces ([4, 18, 22, 24, 28, 29, 34]).

In 2003, Chidume, Ofoedu, and Zegeye [5] introduced the concept of asymptotically nonexpansive nonself-mappings. Also, they studied the following iterative sequence

$$u_{n+1} = \mathcal{P}((1-\vartheta_n)u_n + \vartheta_n \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u_n) \tag{2}$$

to approximate some fixed point of $\mathcal{T}$ under suitable conditions.

If T is a self-mapping, then P becomes the identity mapping so that (2) reduces to (1).

In 2006, Wang [36] considered the following iteration process which is a generalization of (2),

$$\begin{aligned} v_n &= \mathcal{P}((1-\zeta_n)u_n + \zeta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\vartheta_n)u_n + \vartheta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \ n \geq 1, \end{aligned} \tag{3}$$

where $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings and $\{\vartheta_n\}$ and $\{\zeta_n\}$ are real sequences in [0,1). Meanwhile, the results of [36] generalized the results of [5].

The projection type Ishikawa iteration process for approximating common fixed points of two asymptotically nonexpansive nonself-mappings was defined and constructed by Thianwan [35] in a uniformly convex Banach space as follows:

$$\begin{aligned} v_n &= \mathcal{P}((1-\zeta_n)u_n + \zeta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\vartheta_n)v_n + \vartheta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \ n \geq 1, \end{aligned} \tag{4}$$

where $\{\vartheta_n\}$ and $\{\zeta_n\}$ are appropriate real sequences in [0,1). Note that Thianwan process (4) and Wang process (3) are independent neither reduces to the other.

In 2012, Guo, Cho and Guo [9] studied the following iteration scheme:

$$\begin{aligned} v_n &= \mathcal{P}((1-\zeta_n)\mathcal{S}_2^n u_n + \zeta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n), \\ u_{n+1} &= \mathcal{P}((1-\vartheta_n)\mathcal{S}_1^n u_n + \vartheta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n), \ n \geq 1, \end{aligned} \tag{5}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings and $\{\vartheta_n\}$, $\{\zeta_n\}$ are two sequences in [0,1) to approximate common fixed points of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ under proper conditions.

The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. Fixed point theory and hence approximation techniques have been extended to hyperbolic spaces (see [1–3, 25–27] and references therein).

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [13], defined below, which play a significant role in many branches of mathematics.

A hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is a metric space $(\mathcal{X}, d)$ together with a mapping $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ satisfying

$(\mathcal{H}1) : d(z, \mathcal{H}(u, v, \psi)) \leq (1-\psi)\, d(z, u) + \psi d(z, v),$

$(\mathcal{H}2) : d(\mathcal{H}(u, v, \psi)), \mathcal{H}(u, v, \mu) = |\psi - \mu|\, d(u, v),$

$(\mathcal{H}3) : \mathcal{H}(u, v, \psi) = \mathcal{H}(v, u, 1-\psi),$

$(\mathcal{H}4) : d(\mathcal{H}(u, z, \psi), \mathcal{H}(v, w, \psi)) \leq (1-\psi)\, d(u, v) + \psi d(z, w)$

for all $u, v, w, z \in \mathcal{X}$ and $\psi, \mu \in [0,1]$.

A subset $\mathcal{K}$ of a hyperbolic space $\mathcal{X}$ is convex if $\mathcal{H}(u, v, \psi) \in \mathcal{K}$ for all $u, v \in \mathcal{K}$ and $\psi \in [0,1]$. If a space satisfies only $(\mathcal{H}1)$, it coincides with the convex metric space introduced by Takahashi [32]. The concept of hyperbolic spaces in [13] is more restrictive than the hyperbolic type introduced by Goebel et al. [7] since $(\mathcal{H}1) - (\mathcal{H}3)$ together are equivalent to $(\mathcal{X}, d, \mathcal{H})$ being a space of hyperbolic type in [7]. Also it is slightly more general than the hyperbolic space defined by Reich et al. [23].

A hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is said to be

(i) strictly convex [32] if for any $u, v \in \mathcal{X}$ and $\psi \in [0,1]$, there exists a unique element $z \in \mathcal{X}$ such that $d(z, u) = \psi d(u, v)$ and $d(z, v) = (1-\psi) d(u, v)$;

(ii) uniformly convex [31] if for all $u, v, w \in \mathcal{X}$, $r > 0$ and $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that $d(\mathcal{H}(u, v, \frac{1}{2}), u) \leq (1-\delta) r$ whenever $d(u, w) \leq r, d(v, w) \leq r$ and $d(u, v) \geq \epsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [15]).

In the sequel, let $(\mathcal{X}, d)$ be a metric space, and let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. We shall denote the fixed point set of a mapping $\mathcal{T}$ by $\mathcal{F}(\mathcal{T}) = \{u \in \mathcal{K} : \mathcal{T}u = u\}$ and $d(u, \mathcal{F}(\mathcal{T})) = \inf \{d(u, p) : p \in \mathcal{F}(\mathcal{T})\}$.

A self-mapping $\mathcal{T}$ is said to be nonexpansive if $d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v)$ for all $u, v \in \mathcal{K}$. $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d(\mathcal{T}^n u, \mathcal{T}^n v) \leq k_n d(u, v) \tag{6}$$

for all $u, v \in \mathcal{K}$ and $n \geq 1$. $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that $d(\mathcal{T}^n u, \mathcal{T}^n v) \leq \mathcal{L} d(u, v)$ for all $u, v \in \mathcal{K}$ and $n \geq 1$.

It follows that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \forall n \geq 1$. Moreover, each asymptotically nonexpansive mapping is a uniformly $\mathcal{L}$ -Lipschitzian mapping with $\mathcal{L} = \sup_{n \in \aleph} \{k_n\}$. However, the converse of these statements is not true, in general.

Note that, a subset $\mathcal{K}$ of $\mathcal{X}$ is said to be a retract if there exists a continuous mapping $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ such that $\mathcal{P}u = u$ for all $u \in \mathcal{K}$. For more information on nonexpansive retracts and retractions, we refer the reader to ([8, 14]).

For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping (see [5]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(\mathcal{T}(\mathcal{PT})^{n-1}u, \mathcal{T}(\mathcal{PT})^{n-1}v) \leq k_n d(u, v) \tag{7}$$

for all $u, v \in \mathcal{K}$ and $n \geq 1$.

We denote by $(\mathcal{PT})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping, so that (7) reduces to (6).

In addition, if $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is asymtotically nonexpansive in light of (7) and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is a nonexpansive retraction, then $\mathcal{PT} : \mathcal{K} \to \mathcal{K}$ is asymtotically nonexpansive in light of (6) (see also (8)). Indeed, for all $u, v \in \mathcal{K}$ and $n \geq 1$, by (7), it follows that

$$\begin{aligned} d((\mathcal{PT})^n u, (\mathcal{PT})^n v) &= d(\mathcal{PT}(\mathcal{PT})^{n-1}u, \mathcal{PT}(\mathcal{PT})^{n-1}v) \\ &\leq d(\mathcal{T}(\mathcal{PT})^{n-1}u, \mathcal{T}(\mathcal{PT})^{n-1}v) \\ &\leq k_n d(u, v). \end{aligned}$$

Therefore, we now introduce the following definition.

Definition 1. *For any nonempty subset $\mathcal{K}$ of a metric space $(\mathcal{X}, d)$. Let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. A nonself-mapping $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is called asymptotically nonexpansive with respect to $\mathcal{P}$ if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$d((\mathcal{PT})^n u, (\mathcal{PT})^n v) \leq k_n d(u, v) \tag{8}$$

for all $u, v \in \mathcal{K}$ and $n \geq 1$.

In the sequel, we shall need the following lemmas.

Lemma 1. *([20]) Let $\{\eta_n\}, \{\vartheta_n\}$ and $\{\zeta_n\}$ be sequences of non-negative real numbers such that*

$$\eta_{n+1} \leq (1 + \vartheta_n)\eta_n + \zeta_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \vartheta_n < \infty$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$, then $\lim_{n\to\infty} \eta_n$ exists.

Lemma 2. *([12]) Let $\{u_n\}$ and $\{v_n\}$ be two sequences of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ such that, for $r \in [0, \infty)$, $\lim_{n\to\infty} sup\ d(u_n, a) \leq r$, $\lim_{n\to\infty} sup\ d(v_n, a) \leq r$, and*

$$\lim_{n\to\infty} d(\mathcal{H}(u_n, v_n, \varsigma_n), a) = r,$$

where $\varsigma_n \in [a, b]$ with $0 < a \leq b < 1$, then we have, $\lim_{n\to\infty} d(u_n, v_n) = 0$.

2. Main results

In this section, we suggest a new iterative algorithm for mixed type asypmtotically nonexpansive mappings and establish the strong convergence theorem in a uniformly convex hyperbolic space.

Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpasive self-mappings and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ be two asymptotically nonexpasive nonself-mappings. We will denote the set of common fixed points of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ by $\mathcal{F}$, that is, $\mathcal{F} := \mathcal{F}(\mathcal{S}_1) \cap \mathcal{F}(\mathcal{S}_2) \cap \mathcal{F}(\mathcal{T}_1) \cap \mathcal{F}(\mathcal{T}_2)$. The algorithm is defined as follows: $u_1 \in K$,

$$\begin{aligned} v_n &= \mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n, \zeta_n), \\ u_{n+1} &= \mathcal{H}(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n, \vartheta_n), \end{aligned} \tag{9}$$

where $\{\vartheta_n\}$ and $\{\zeta_n\}$ are two sequences in $[0, 1)$.

The following lemmas are needed.

Lemma 3. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpasive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ be two asymptotically nonexpasive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively and $\mathcal{F} \neq \emptyset$. Suppose that $\{\vartheta_n\}$ and $\{\zeta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ using algorithm (9). Then $\lim_{n\to\infty} d(u_n, q)$ exists for any $q \in \mathcal{F}$.*

Proof. Let $q \in \mathcal{F}$. Setting $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. Using algorithm (9), we have

$$\begin{aligned} d(v_n, q) &= d\left(\mathcal{H}\left(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n, \zeta_n\right), q\right) \\ &\leq (1 - \zeta_n)\, d(\mathcal{S}_2^n u_n, q) + \zeta_n d\left((\mathcal{P}\mathcal{T}_2)^n u_n, q\right) \\ &\leq (1 - \zeta_n)\, h_n d(u_n, q) + \zeta_n h_n d(u_n, q) \\ &= h_n d(u_n, q), \end{aligned} \tag{10}$$

and so

$$\begin{aligned} d(u_{n+1}, q) &= d\left(\mathcal{H}\left(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n, \vartheta_n\right), q\right) \\ &\leq (1 - \vartheta_n)\, d(\mathcal{S}_1^n v_n, q) + \vartheta_n d\left((\mathcal{P}\mathcal{T}_1)^n v_n, q\right) \\ &\leq (1 - \vartheta_n)\, h_n d(v_n, q) + \vartheta_n h_n d(v_n, q) \\ &= h_n d(v_n, q) \\ &\leq h_n^2 d(u_n, q) \end{aligned}$$

$$= \left(1 + (h_n^2 - 1)\right) d\left(u_n, q\right). \tag{11}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, we have $\sum_{n=1}^{\infty}(h_n^2 - 1) < \infty$. It follows from Lemma 1 that $\lim_{n\to\infty} d\left(u_n, q\right)$ exists. □

Lemma 4. *Let $(\mathcal{X}, d, \mathcal{W})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ be two asymptotically non-expansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively, and $\mathcal{F} \neq \emptyset$. From an arbitrary $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ using algorithm (9) and the following conditions hold:*

(i) $\{\vartheta_n\}$ and $\{\zeta_n\}$ are real sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$;

(ii) $d(u, \mathcal{T}_i v) \leq d(\mathcal{S}_i u, \mathcal{T}_i v)$ for all $u,\ v \in \mathcal{K}$ and $i = 1,\ 2$.

Then, $\lim_{n\to\infty} d(u_n, \mathcal{S}_i u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_i)u_n) = 0$ for $i = 1,\ 2$.

Proof. Let $q \in \mathcal{F}$. Set $h_n = \max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. By Lemma 3, we have $\lim_{n\to\infty} d(u_n, q)$ exists. Assume that $\lim_{n\to\infty} d(u_n, q) = c$, letting $n \to \infty$ in the inequality (11), we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n, \vartheta_n), q) = c. \tag{12}$$

In addition, using (10), we have $d(\mathcal{S}_1^n v_n, q) \leq h_n^2 d(u_n, q)$. Taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} d(\mathcal{S}_1^n v_n, q) \leq c. \tag{13}$$

Taking the lim sup on both sides in the inequality (10), we obtain $\limsup_{n\to\infty} d(v_n, q) \leq c$, and so

$$\limsup_{n\to\infty} d((\mathcal{P}\mathcal{T}_1)^n v_n, q) \leq \limsup_{n\to\infty} h_n d(v_n, q) = c. \tag{14}$$

Using (12), (13), (14), and Lemma 2, we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) = 0. \tag{15}$$

By the condition (ii), we have

$$d(v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) \le d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n). \tag{16}$$

Letting $n \to \infty$ in the inequality (16), by (15), we have

$$\lim_{n\to\infty} d(v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) = 0. \tag{17}$$

Using (11), we have

$$\begin{aligned} d(u_{n+1}, q) &\le (1-\vartheta_n) d(\mathcal{S}_1^n v_n, q) + \vartheta_n d((\mathcal{P}\mathcal{T}_1)^n v_n, q) \\ &\le (1-\vartheta_n) d(\mathcal{S}_1^n v_n, q) + \vartheta_n d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) + \vartheta_n d(\mathcal{S}_1^n v_n, q) \\ &= d(\mathcal{S}_1^n v_n, q) + \vartheta_n d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) \\ &\le h_n d(v_n, q) + \vartheta_n d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n). \end{aligned} \tag{18}$$

Taking the lim inf on both sides in the inequality (18), using (15), $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $\lim_{n\to\infty} d(u_{n+1}, q) = c$, we have

$$\liminf_{n\to\infty} d(v_n, q) \ge c. \tag{19}$$

Since $\limsup_{n\to\infty} d(v_n, q) \le c$, by (19), we have $\lim_{n\to\infty} d(v_n, q) = c$. This implies that

$$\begin{aligned} c = \lim_{n\to\infty} d(v_n, q) &\le \lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n, \zeta_n), q) \\ &\le \lim_{n\to\infty} d(u_n, q) = c, \end{aligned}$$

and so

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n, \zeta_n), q) = c. \tag{20}$$

In addition, we have

$$\limsup_{n\to\infty} d(\mathcal{S}_2^n u_n, q) \le \limsup_{n\to\infty} h_n d(u_n, q) = c \tag{21}$$

and

$$\limsup_{n\to\infty} d((\mathcal{P}\mathcal{T}_2)^n u_n, q) \le \limsup_{n\to\infty} h_n d(u_n, q) = c. \tag{22}$$

It follows from (20), (21), (22), and Lemma 2 that

$$\lim_{n\to\infty} d(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n) = 0. \tag{23}$$

Now, we prove that $\lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_1)u_n) = 0 = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_2)u_n)$. Indeed, by the condition (ii), we have

$$d(u_n, (\mathcal{P}\mathcal{T}_2)^n u_n) \leq d(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n). \tag{24}$$

Using (23) and (24), we have

$$\lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_2)^n u_n) = 0. \tag{25}$$

Using algorithm (9), we have

$$\begin{aligned} d(v_n, \mathcal{S}_2^n u_n) &\leq (1-\zeta_n) d(\mathcal{S}_2^n u_n, \mathcal{S}_2^n u_n) + \zeta_n d(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n) \\ &= \zeta_n d(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n). \end{aligned}$$

It follows from (23) that

$$\lim_{n\to\infty} d(v_n, \mathcal{S}_2^n u_n) = 0. \tag{26}$$

Furthermore, we have

$$\begin{aligned} d(v_n, u_n) \leq\ & d(v_n, \mathcal{S}_2^n u_n) + d(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n) \\ & + d((\mathcal{P}\mathcal{T}_2)^n u_n, u_n). \end{aligned} \tag{27}$$

It follows from (23), (25), (26), and (27) that

$$\lim_{n\to\infty} d(u_n, v_n) = 0. \tag{28}$$

By the condition (ii), we have

$$d(u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) \leq d(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n).$$

Since

$$\begin{aligned} d(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) &\leq d(\mathcal{S}_1^n u_n, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_1)^n v_n, (\mathcal{P}\mathcal{T}_1)^n u_n) \\ &\leq h_n d(u_n, v_n) + d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n) \\ &\quad + h_n d(v_n, u_n), \end{aligned} \tag{29}$$

using (15), (28) and (29), we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) = 0, \tag{30}$$

and so

$$\lim_{n\to\infty} d(u_n, (\mathcal{PT}_1)^n u_n) = 0. \tag{31}$$

In addition,

$$\begin{aligned} d(u_{n+1}, \mathcal{S}_1^n v_n) &= d((\mathcal{H}(\mathcal{S}_1^n v_n, (\mathcal{PT}_1)^n v_n, \alpha_n)), \mathcal{S}_1^n v_n) \\ &\leq (1-\vartheta_n) d(\mathcal{S}_1^n v_n, \mathcal{S}_1^n v_n) + \vartheta_n d((\mathcal{PT}_1)^n v_n, \mathcal{S}_1^n v_n) \\ &= \vartheta_n d((\mathcal{PT}_1)^n v_n, \mathcal{S}_1^n v_n). \end{aligned}$$

Thus, it follows from (15) that

$$\lim_{n\to\infty} d(u_{n+1}, \mathcal{S}_1^n v_n) = 0. \tag{32}$$

In addition

$$d(u_{n+1}, (\mathcal{PT}_1)^n v_n) \leq d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, (\mathcal{PT}_1)^n v_n).$$

Using (15) and (32), we have

$$\lim_{n\to\infty} d(u_{n+1}, (\mathcal{PT}_1)^n v_n) = 0. \tag{33}$$

It follows from (30) and (31) that

$$\begin{aligned} d(\mathcal{S}_1^n u_n, u_n) &\leq d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n) + d((\mathcal{PT}_1)^n u_n, u_n) \\ &\to 0 \text{ (as } n \to \infty). \end{aligned} \tag{34}$$

In addition,

$$d(\mathcal{S}_1^n u_n, (\mathcal{PT}_2)^n u_n) \leq d(\mathcal{S}_1^n u_n, u_n) + d(u_n, (\mathcal{PT}_2)^n u_n).$$

Thus, it follows from (25) and (34) that

$$\lim_{n\to\infty} d(\mathcal{S}_1^n u_n, (\mathcal{PT}_2)^n u_n) = 0. \tag{35}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_1^n v_n, (\mathcal{PT}_2)^n u_n) &\leq d(\mathcal{S}_1^n v_n, \mathcal{S}_1^n u_n) + d(\mathcal{S}_1^n u_n, (\mathcal{PT}_2)^n u_n) \\ &\leq h_n d(v_n, u_n) + d(\mathcal{S}_1^n u_n, (\mathcal{PT}_2)^n u_n). \end{aligned}$$

Using (28) and (35), we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_2)^n u_n) = 0. \tag{36}$$

It follows from (28), (32) and (36) that

$$\begin{aligned} d(u_{n+1}, (\mathcal{P}\mathcal{T}_2)^n v_n) &\leq d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_2)^n u_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_2)^n u_n, (\mathcal{P}\mathcal{T}_2)^n v_n) \\ &\leq d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_2)^n u_n) + h_n d(u_n, v_n) \\ &\quad \to 0 \text{ (as } n \to \infty). \end{aligned} \tag{37}$$

Again, since $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} v_{n-1}, u_n \in \mathcal{K}$ for $i = 1, 2$ and T_1, T_2 are two asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned} d((\mathcal{P}\mathcal{T}_i)^n v_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) &= d(((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} v_{n-1}), (\mathcal{P}\mathcal{T}_i) u_n) \\ &\leq max\{l_1^{(1)}, l_1^{(2)}\} d((\mathcal{P}\mathcal{T}_i)^{n-1} v_{n-1}, u_n). \end{aligned} \tag{38}$$

Using (33), (37), and (38), for $i = 1, 2$, we have

$$\lim_{n\to\infty} d((\mathcal{P}\mathcal{T}_i)^n v_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) = 0. \tag{39}$$

Moreover, we have

$$d(u_{n+1}, v_n) \leq d(u_{n+1}, (\mathcal{P}\mathcal{T}_1)^n v_n) + d((\mathcal{P}\mathcal{T}_1)^n v_n, v_n).$$

Using (17) and (33), we have

$$\lim_{n\to\infty} d(u_{n+1}, v_n) = 0. \tag{40}$$

In addition, for $i = 1, 2$, we have

$$\begin{aligned} d(u_n, (\mathcal{P}\mathcal{T}_i) u_n) &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d((\mathcal{P}\mathcal{T}_i)^n u_n, (\mathcal{P}\mathcal{T}_i)^n v_{n-1}) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n v_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) \\ &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + max\{sup_{n\geq 1} l_n^{(1)}, sup_{n\geq 1} l_n^{(2)}\} d(u_n, v_{n-1}) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n v_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n). \end{aligned}$$

Thus, it follows from (25), (31), (39), and (40) that

$$\lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_1) u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_2) u_n) = 0.$$

Finally, we prove that

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_1 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_2 u_n) = 0.$$

In fact, for $i = 1, 2$, we have

$$\begin{aligned} d(u_n, \mathcal{S}_i u_n) &\le d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d(\mathcal{S}_i u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) \\ &\le d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d(\mathcal{S}_i^n u_n, (\mathcal{P}\mathcal{T}_i)^n u_n). \end{aligned}$$

Thus, it follows from (23), (25), (30), and (31) that

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_1 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_2 u_n) = 0.$$

The proof is completed. □

The following example presents the condition (ii) in Lemma 4 which is satisfied by the mappings $\mathcal{S}_i$ and $\mathcal{T}_i$, $i = 1, 2$, when $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ and $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are given in the next example.

Example 1. *([16]) Let $\mathcal{X}$ be the real line with metric $d(u, v) = |u - v|$ and $\mathcal{K} = [-1, 1]$. Define $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ by $\mathcal{H}(u, v, \psi) := \psi u + (1 - \psi)v$ for all $u, v \in \mathcal{X}$ and $\psi \in [0, 1]$. Then $(\mathcal{X}, d, \mathcal{H})$ is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\mathcal{K}$ is a nonempty closed convex subset of $\mathcal{X}$. Define two mappings $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ by*

$$\mathcal{T}u = \begin{cases} -2\sin\frac{u}{2}, & \text{if } u \in [0, 1], \\ 2\sin\frac{u}{2}, & \text{if } u \in [-1, 0) \end{cases}$$

and

$$\mathcal{S}u = \begin{cases} u, & \text{if } u \in [0, 1], \\ -u, & \text{if } u \in [-1, 0). \end{cases}$$

Clearly, $\mathcal{F}(\mathcal{T}) = \{0\}$ and $\mathcal{F}(\mathcal{S}) = \{u \in \mathcal{K}; 0 \le u \le 1\}$. Now, we show that $\mathcal{T}$ is nonexpansive. In fact, if $u, v \in [0, 1]$ or $u, v \in [-1, 0)$, then

$$d(\mathcal{T}u, \mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v| = 2|\sin\frac{u}{2} - \sin\frac{v}{2}| \le |u - v| = d(u, v).$$

If $u \in [0, 1]$ and $v \in [-1, 0)$ or $u \in [-1, 0)$ and $v \in [0, 1]$, then

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}v) &= |\mathcal{T}u - \mathcal{T}v| \\ &= 2|\sin\frac{u}{2} + \sin\frac{v}{2}| \\ &= 4|\sin\frac{u+v}{4}\cos\frac{u-v}{4}| \\ &\le |u + v| \\ &\le |u - v| \end{aligned}$$

$$= d(u, v).$$

That is, $\mathcal{T}$ is nonexpansive. It follows that $\mathcal{T}$ is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that $\mathcal{S}$ is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$. Next, to show that $\mathcal{S}$ and $\mathcal{T}$ satisfy the condition (ii) in Lemma 4, we have to consider the following cases:

Case 1. Let $u, v \in [0, 1]$. It follows that

$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u + 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$

Case 2. Let $u, v \in [-1, 0)$. It follows that

$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u - 2\sin\frac{v}{2}| \leq | - u - 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$

Case 3. Let $u \in [-1, 0)$ and $v \in [0, 1]$. It follows that

$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u + 2\sin\frac{v}{2}| \leq | - u + 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$

Case 4. Let $u \in [0, 1]$ and $v \in [-1, 0]$. It follows that

$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u - 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$

Hence the condition (ii) in Lemma 4 is satisfied. □

Now, we can prove a strong convergence theorem.

Theorem 1. *Let $\mathcal{K}, \mathcal{X}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the hypotheses of Lemma 4. Suppose that $\{\vartheta_n\}$, $\{\zeta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$ and $\mathcal{S}_i, \mathcal{T}_i$ for all $i = 1, 2$ satisfy the condition (ii) in Lemma 4. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$f(d(u, \mathcal{F})) \leq d(u, \mathcal{S}_1 u) + d(u, \mathcal{S}_2 u) + d(u, (\mathcal{P}\mathcal{T}_1)u) + d(u, (\mathcal{P}\mathcal{T}_2)u)$$

for all $u \in \mathcal{K}$, where $d(u, \mathcal{F}) = \inf\{d(u, q) : q \in \mathcal{F}\}$. Then the sequence $\{u_n\}$ defined by algorithm (9) converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$.

Proof. By Lemma 4, we have $\lim_{n\to\infty} d(u_n, \mathcal{S}_i u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_i)u_n) = 0$ for $i = 1, 2$. It follows from hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(u_n, \mathcal{F})) \leq \lim_{n\to\infty} (d(u_n, \mathcal{S}_1 u_n) + d(u_n, \mathcal{S}_2 u_n) \\ + d(u_n, (\mathcal{P}\mathcal{T}_1)u_n) + d(u_n, (\mathcal{P}\mathcal{T}_2)u_n)) = 0.\end{aligned}$$

Thus $\lim_{n\to\infty} f(d(u_n, \mathcal{F})) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$. Using Lemma 11, we have $\lim_{n\to\infty} d(u_n, \mathcal{F})$ exists. It implies that $\lim_{n\to\infty} d(u_n, \mathcal{F}) = 0$. Now, we show that $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}$. In fact, from (11), we have

$$d(u_{n+1}, q) \leq (1 + (h_n^2 - 1))d(u_n, q)$$

for each $n \geq 1$, where $h_n = \max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$ and $q \in F$. For any $m, n, m > n \geq 1$, we have

$$d(u_m, q) \leq (1 + (h_{m-1}^2 - 1))d(u_{m-1}, q)$$

$$
\begin{aligned}
&\leq e^{h^2_{m-1}-1} d(u_{m-1}, q)\\
&\leq e^{h^2_{m-1}-1} e^{h^2_{m-2}-1} d(u_{m-2}, q)\\
&\vdots\\
&\leq e^{\sum_{i=n}^{m-1}(h_i^2-1)} d(u_n, q)\\
&\leq \mathcal{M} d(u_n, q),
\end{aligned}
$$

where $\mathcal{M} = e^{\Sigma_{i=1}^{\infty}(h_i^2-1)}$. Thus, for any $q \in \mathcal{F}$, we have

$$
\begin{aligned}
d(u_n, u_m) &\leq d(u_n, q) + d(u_m, q)\\
&\leq (1+\mathcal{M}) d(u_n, q).
\end{aligned}
$$

Taking the infimum over all $q \in \mathcal{F}$, we have

$$d(u_n, u_m) \leq (1+\mathcal{M}) d(u_n, \mathcal{F}).$$

Thus it follows from $\lim_{n\to\infty} d(u_n, \mathcal{F}) = 0$ that $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space $\mathcal{X}$, the sequence $\{u_n\}$ converges strongly to some $q^* \in \mathcal{K}$. It is easy to prove that $\mathcal{F}(\mathcal{S}_1), \mathcal{F}(\mathcal{S}_2), \mathcal{F}(\mathcal{T}_1)$ and $\mathcal{F}(\mathcal{T}_2)$ are all closed and so $\mathcal{F}$ is a closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty} d(u_n, \mathcal{F}) = 0$ gives that $d(q^*, \mathcal{F}) = 0$. Therefore $q^* \in \mathcal{F}$. This completes the proof. □

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are self-mappings, then $\mathcal{P}$ becomes the identity mapping. By using to the same ideas and techniques as in Lemma 3, Lemma 4 and Theorem 1, we can also obtain a strong convergence theorem for asymptotically nonexpansive mappings in a uniformly convex hyperbolic space. Therefore we can state the following result without proofs.

Theorem 2. *Let $(\mathcal{X}, d, \mathcal{W})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpansive mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpansive mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively, and $\mathcal{F} \neq \emptyset$. Suppose that $\{\vartheta_n\}$, $\{\zeta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$ and $\mathcal{S}_i, \mathcal{T}_i$ for all $i = 1, 2$ satisfy the condition (ii) in Lemma 4. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$f(d(u, \mathcal{F})) \leq d(u, \mathcal{S}_1 u) + d(u, \mathcal{S}_2 u) + d(u, \mathcal{T}_1 u) + d(u, \mathcal{T}_2 u)$$

for all $u \in \mathcal{K}$, where $d(u, \mathcal{F}) = \inf\{d(u, q) : q \in \mathcal{F}\}$. Then the sequence $\{u_n\}$ defined by

$$
\begin{aligned}
v_n &= \mathcal{H}(\mathcal{S}_2^n u_n, \mathcal{T}_2^n u_n, \zeta_n),\\
u_{n+1} &= \mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1^n v_n, \vartheta_n)
\end{aligned}
$$

converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$.

3. Conclusions

Author constructed a new mixed type iterative method to approximate a common fixed point for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in a uniformly convex hyperbolic space. An asymptotically nonexpansive nonself-mapping with respect to a nonexpansive retraction is defined in Definition 1. An illustrative example is also provided as Example 1. Author proved strong convergence result which is stronger than that of delta and weak convergence results.

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