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On the Operator $\oplus_m^k$ Related to the Wave Equation and Laplacian

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Abstract. In this article, we study the fundamental solution of the operator $\oplus_m^k$, iterated k-times and is defined by

$$\oplus_m^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + m^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k,$$

where m is a nonnegative real number, $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, k is a nonnegative integer. At first we study the fundamental solution of the operator $\oplus_m^k$ and after that, we apply such the fundamental solution to solve for the solution of the equation $\oplus_m^k u(x)=f(x)$, where $f(x)$ is generalized function and $u(x)$ is unknown function for $x\in\mathbb{R}^n$.

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1. Introduction

We have observed that an operational quantity such as $\delta(x)$ becomes meaningful if it is first multiplied by a sufficiently smooth auxiliary function and then integrated over the entire space. This point of view is also taken as the basis for the definition of an arbitrary generalized function. Accordingly, consider the space D consisting of real-valued function $\phi(x)=\phi(x_1,x_2,\ldots,x_n)$, such that the following hold:

(1) $\phi(x)$ is an infinitely differentiable function defined at every point of $\mathbb{R}^n$. This mean that $D^k\phi$ exists for all multi indices k. Such a function is also call a C^∞ function.

(2) There exists number A such that $\phi(x)$ vanishes for $r>A$. This means that $\phi(x)$ has a compact support. Then $\phi(x)$ is called a *test function*.

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In physical problem, one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instaneously. These forces are described by the Dirac-delta function $\delta(x-\xi)$, which has several significant properties:

$$\delta(x-\xi)=0, \quad x \neq \xi \tag{1}$$

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 0 & \text{for } a,b<\xi \text{ or } \xi<a,b \\ 1 & \text{for } a \leq \xi \leq b \end{cases} \tag{2}$$

and

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1. \tag{3}$$

Equation (3) is a special case of the general formula

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi), \tag{4}$$

where $f(x)$ is a sufficiently smooth function. Relation (4) is called the *sifting property* or the reproducing property of the delta function, and (3) is obtained from it by putting $f(x)=1$. Moreover, Kananthai et al. [1] have studied the fundamental solution of the operator $\oplus^k$ and the weak solution of the equation $\oplus^k u(x) = f(x)$, $f(x)$ is a generalized function where the operator $\oplus^k$ is defined by

$$\begin{aligned}
\oplus^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\
&= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \diamond^k L_1^k L_2^k \\
&= \diamond^k L^k
\end{aligned} \tag{5}$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and k is a nonnegative integer.

Next, Kananthai et al. [2] have studied the relationship between the operator $\oplus^k$ and the wave operator, and the relationship between the operator $\oplus^k$ and the Laplacian. Moreover, equation $\oplus^k K(x) = \delta$ we have $K(x) = [R_{2k}^H(x) * (-1)^k R_{2k}^e(x)] * S_{2k}(x) * T_{2k}(x)$ is the fundamental solution of the operator $\oplus^k$. Later, Kananthai [8] has studied the inversion of the kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$.

In 1988, Trione [11] has studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated k-times such that operator is defined by

$$(\square + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right]^k. \tag{6}$$

Later, Kananthai [7] has studied the fundamental solution for the $(\diamondsuit + m^4)^k$ which related to the Klein-Gordon operator.

From equation (5) the operator $\oplus_m^k$ can be expressed in the form

$$\begin{aligned}
\oplus_m^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+m^2\right)^4-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k\\
&= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+m^2\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+m^2\right)^2+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\\
&= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+m^2\right]^k\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+m^2\right]^k\\
&\quad\times\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+m^2\right]^k\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}-i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+m^2\right]^k, \qquad (7)
\end{aligned}$$

where $i=\sqrt{-1}$, $n=p+q$. And the operator

$$\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$

is introduced by Kananthai [4] and is named the diamond operator which is defined by

$$\diamondsuit^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \qquad (8)$$

Otherwise, the operator $\diamondsuit^k$ can also be expressed in the form $\diamondsuit^k = \square^k\triangle^k = \triangle^k\square^k$, where $\square^k$ is the ultra-hyperbolic operator iterated k-times, is defined by

$$\square^k = \left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\frac{\partial^2}{\partial x_{p+2}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right]^k, \qquad (9)$$

and $\triangle^k$ is the Laplace operator iterated k-times, is defined by

$$\triangle^k = \left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right]^k. \qquad (10)$$

By putting $p=1$ and $x_1=t$ (time) in (9), then we obtain the wave operator

$$\square = \frac{\partial^2}{\partial t^2}-\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}. \qquad (11)$$

The operators L_1^k and L_2^k are defined by

$$L_1^k = \left[\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right]^k \tag{12}$$

and

$$L_2^k = \left[\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right]^k, \tag{13}$$

following that

$$L^k = L_1^k L_2^k = L_2^k L_1^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{14}$$

Thus, equation (7) can be written as

$$\begin{aligned} \oplus_m^k &= \left(\Box + m^2\right)^k \left(\triangle + m^2\right)^k \left(L_1 + m^2\right)^k \left(L_2 + m^2\right)^k \\ &= \left(L_2 + m^2\right)^k \left(L_1 + m^2\right)^k \left(\triangle + m^2\right)^k \left(\Box + m^2\right)^k \end{aligned} \tag{15}$$

and from (7) with $q = m = 0$ and $k = 1$, we obtain Laplace operator of p-dimension

$$\oplus_0 = \triangle_p^4,$$

where

$$\triangle_p = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}. \tag{16}$$

In this article, we further study the fundamental solution of the operator $\oplus_m^k$, that is

$$\oplus_m^k H(x, m) = \delta,$$

where $H(x, m)$ is the fundamental solution, δ is the Dirac delta distribution, k is a non-negative integer, m is a nonnegative real number and the operator $\oplus_m^k$ is defined by

$$\oplus_m^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + m^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k. \tag{17}$$

We then also apply such the fundamental solution to solve the solution of the equation $\oplus_m^k u(x) = f(x)$, where $f(x)$ is a given generalized function and $u(x)$ is an unknown function for $x \in \mathbb{R}^n$.

2. Preliminary Notes

In this section, we studied some properties of the ultra-hyperbolic kernel of Marcel Riesz and the fundamental solution of the partial differential operators which will be used as follow.

Definition 1. *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,*

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{18}$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([9], p.72) that

$$R_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{19}$$

$R_\alpha^H(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz. Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{20}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha^H(x) \subset \overline{\Gamma}_+$. Now $R_\alpha^H(x)$ is an ordinary function if $Re\ \alpha \geq n$ and is a distribution of α if $Re\ \alpha < n$.

Now, if $p = 1$ then (19) reduces to the function $M_\alpha(u)$ say, and defined by

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{21}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2})$. The function $M_\alpha(u)$ is called the hyperbolic kernel of Marcel Riesz.

Lemma 1. *Given the equation $\triangle^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\triangle^k$ is the Laplace operator iterated k-times is defined by (10). Then $u(x) = (-1)^k R_{2k}^e(x)$ is the fundamental solution of the operator $\triangle^k$ where*

$$R_{2k}^e(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k} \pi^{\frac{n}{2}} \Gamma(k)} |x|^{2k-n}. \tag{22}$$

Proof. See [4].

Lemma 2. *If $\square^k u(x) = \delta$ for $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ and $u > 0\}$, where $\square^k$is the ultra-hyperbolic operator iterated k-times is defined by (9). Then $u(x) = R^H_{2k}(x)$ is the unique fundamental solution of the operator $\square^k$ where*

$$R^H_{2k}(x) = \frac{u^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} = \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} \tag{23}$$

for

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+2k-n}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p}{2}\right)\Gamma(\frac{p-2k}{2})}. \tag{24}$$

Proof. See [11].

Lemma 3. *Given the equation $\diamondsuit^k u(x) = \delta$ for $x \in \mathbb{R}^n$, then $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the unique fundamental solution of the operator $\diamondsuit^k$, where $\diamondsuit^k$ is the diamond operator iterated k- times is defined by (8), $R^e_{2k}(x)$ and $R^H_{2k}(x)$ are defined by (22) and (23), respectively. Moreover, $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is a tempered distribution.*

Proof. See [4].

Lemma 4. *Given the equation $L_1^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where L_1^k is the operator defined by (12), then $u(x) = (-1)^k(-i)^{\frac{q}{2}} S_{2k}(x)$ is the fundamental solution of the operator L_1^k, where*

$$S_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}[x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + \cdots + x_{p+q}^2)]^{\left(\frac{2k-n}{2}\right)}. \tag{25}$$

Lemma 5. *Given the equation $L_2^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where L_2^k is the operator defined by (13), then $u(x) = (-1)^k(i)^{\frac{q}{2}} T_{2k}(x)$ is the fundamental solution of the operator L_2^k, where*

$$T_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}[x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + \cdots + x_{p+q}^2)]^{\left(\frac{2k-n}{2}\right)}. \tag{26}$$

Lemma 6. *Given the equation $L^k u(x) = \delta$ for $x \in \mathbb{R}^n$, then $u(x) = S_{2k}(x) * T_{2k}(x)$ is the fundamental solution of the operator L^k, which is defined by (14), $S_{2k}(x)$ and $T_{2k}(x)$ are defined by (25) and (26), respectively.*

Proof. The proof of the Lemma 4, Lemma 5 and Lemma 6 are given in [2].

Lemma 7. *The function $R^H_{-2k}(x)$ and $(-1)^k R^e_{-2k}(x)$ are the inverse in the convolution algebra of $R^H_{2k}(x)$ and $(-1)^k R^e_{2k}(x)$, respectively.*

Lemma 8. (1) *The function $S_{2k}(x)$ and $T_{2k}(x)$ are the fundamental solution of the operator L_1^k and L_2^k , respectively, where $S_{2k}(x)$ and $T_{2k}(x)$ are defined by (25) and (26) , respectively.*

(2) *The function $S_{-2k}(x)$ and $T_{-2k}(x)$ are the inverse in the convolution algebra of $S_{2k}(x)$ and $T_{2k}(x)$, respectively.*

Proof. The proof of the Lemma 7 and Lemma 8 are given in [2].

Lemma 9. *Given the equation $(\Box + m^2)^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\Box$ is the ultra-hyperbolic operator defined by (9). Then $u(x) = W_{2k}(x,m)$ is the fundamental solution of the operator $(\Box + m^2)^k$. In particular, for $m = 0$ we have $W_{2k}(x, m = 0) = R^H_{2k}(x)$, where*

$$W_{2k}(x,m) = \sum_{r=0}^{+\infty} \binom{-k}{r} m^{2r} R^H_{2k+2r}(x), \tag{27}$$

$R^H_{2k+2r}(x)$ is defined by (23).

Proof. Since the operator $\Box$ defined in equation (9) is a linearly continuous and have $1-1$ mapping, then it has inverse. From Lemma 2 and equation (27) we obtain

$$\begin{aligned} W_{2k}(x,m) &= \sum_{r=0}^{+\infty} \binom{-k}{r} (m^2)^r \Box^{-k-r} \delta \\ &= (\Box + m^2)^{-k} \delta. \end{aligned} \tag{28}$$

By applying the operator $(\Box + m^2)^k$ to both sides of equation (28), we obtain

$$(\Box + m^2)^k W_{2k}(x,m) = (\Box + m^2)^k \cdot (\Box + m^2)^{-k} \delta.$$

Therefore,

$$(\Box + m^2)^k W_{2k}(x,m) = \delta.$$

Since

$$W_{2k}(x,m) = \binom{-k}{0} m^{2(0)} R^H_{2k+2(0)}(x) + \sum_{r=1}^{+\infty} \binom{-k}{r} m^{2r} R^H_{2k+2r}(x). \tag{29}$$

The second summand of the right-hand member of (29) vanishes for $m = 0$ and then, we have $W_{2k}(x, m = 0) = R^H_{2k}(x)$ which is the fundamental solution of the ultra hyperbolic operator $\Box^k$.

Lemma 10. *Given the equation $(\triangle + m^2)^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\triangle$ is Laplace operator defined by (10). Then $u(x) = Y_{2k}(x,m)$ is the fundamental solution of the operator $(\triangle + m^2)^k$. In particular, for $m = 0$ we have $Y_{2k}(x, m = 0) = (-1)^k R^e_{2k}(x)$, where*

$$Y_{2k}(x,m) = \sum_{r=0}^{+\infty} \binom{-k}{r} m^{2r} (-1)^{k+r} R^e_{2k+2r}(x), \tag{30}$$

$R^e_{2k+2r}(x)$ is defined by (22).

Proof. Since the operator $\triangle$ defined by equation (10) is a linearly continuous and have $1-1$ mapping, then it has inverse. From Lemma 1 and equation (30), we obtain

$$\begin{aligned} Y_{2k}(x,m) &= \sum_{r=0}^{+\infty}\binom{-k}{r}(m^2)^r \triangle^{-k-r}\,\delta \\ &= (\triangle+m^2)^{-k}\delta. \end{aligned} \tag{31}$$

By applying the operator $(\triangle+m^2)^k$ to both sides of equation (31), we obtain

$$(\triangle+m^2)^k Y_{2k}(x,m) = (\triangle+m^2)^k\cdot(\triangle+m^2)^{-k}\delta.$$

Therefore,

$$(\triangle+m^2)^k Y_{2k}(x,m) = \delta.$$

Since

$$Y_{2k}(x,m) = \binom{-k}{0}m^{2(0)}(-1)^k R^e_{2k+2(0)}(x) + \sum_{r=1}^{+\infty}\binom{-k}{r}m^{2r}(-1)^k R^e_{2k+2r}(x). \tag{32}$$

The second summand of the right-hand member of (32) vanishes for $m=0$ and then, we have $Y_{2k}(x,m=0) = (-1)^k R^e_{2k}(x)$ which is the fundamental solution of the Laplace operator $\triangle^k$.

Lemma 11. *Given the equation $(L_1+m^2)^k u(x) = \delta$ for $x\in\mathbb{R}^n$, where L_1 is the operator defined by* (12). *Then $u(x) = M_{2k}(x,m)$ is the fundamental solution of the operator $(L_1+m^2)^k$. In particular, for $m=0$ we have $M_{2k}(x,m=0) = (-1)^k(-i)^{\frac{q}{2}}S_{2k}(x)$ where*

$$M_{2k}(x,m) = \sum_{r=0}^{+\infty}\binom{-k}{r}m^{2r}(-1)^{k+r}(-i)^{\frac{q}{2}}S_{2k+2r}(x), \tag{33}$$

$S_{2k+2r}(x)$ is defined by (25).

Proof. Since the operator L_1 defined in equation (12) is a linearly continuous and have $1-1$ mapping ,then it has inverse. From Lemma 4 and equation (33), we obtain

$$\begin{aligned} M_{2k}(x,m) &= \sum_{r=0}^{+\infty}\binom{-k}{r}(m^2)^r L_1^{-k-r}\delta \\ &= (L_1+m^2)^{-k}\delta. \end{aligned} \tag{34}$$

By applying the operator $(L_1+m^2)^k$ to both sides of equation (34), we obtain

$$(L_1+m^2)^k M_{2k}(x,m) = (L_1+m^2)^k\cdot(L_1+m^2)^{-k}\delta.$$

Therefore,

$$(L_1+m^2)^k M_{2k}(x,m) = \delta.$$

Since

$$\begin{aligned} M_{2k}(x,m) &= \binom{-k}{0} m^{2(0)}(-1)^{k+0}(-i)^{\frac{q}{2}} S_{2k+2(0)}(x) \\ &\quad + \sum_{r=1}^{+\infty} \binom{-k}{r} m^{2r}(-1)^{k+r}(-i)^{\frac{q}{2}} S_{2k+2r}(x). \end{aligned} \tag{35}$$

The second summand of the right-hand member of (35) vanishes for $m = 0$ and then, we have $M_{2k}(x, m = 0) = (-1)^k(-i)^{\frac{q}{2}} S_{2k}(x)$ which is the fundamental solution of the operator L_1^k .

Lemma 12. *Given the equation $(L_2 + m^2)^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where L_2 is the operator defined by* (13)*. Then $u(x) = N_{2k}(x, m)$ is the fundamental solution of the operator $(L_2 + m^2)^k$. In particular, for $m = 0$ we have $N_{2k}(x, m = 0) = (-1)^k(i)^{\frac{q}{2}} T_{2k}(x)$, where*

$$N_{2k}(x,m) = \sum_{r=0}^{+\infty} \binom{-k}{r} m^{2r}(-1)^{k+r}(i)^{\frac{q}{2}} T_{2k+2r}(x), \tag{36}$$

$T_{2k+2r}(x)$ is defined by (26)*.*

Proof. Since the operator L_2 defined in equation (13) is a linearly continuous and have $1 - 1$ mapping, then it has inverse. From Lemma 5 and equation (36), we obtain

$$\begin{aligned} N_{2k}(x,m) &= \sum_{r=0}^{+\infty} \binom{-k}{r} (m^2)^r L_2^{-k-r} \delta \\ &= (L_2 + m^2)^{-k} \delta. \end{aligned} \tag{37}$$

By applying the operator $(L_2 + m^2)^k$ to both sides of equation (37), we obtain

$$(L_2 + m^2)^k N_{2k}(x,m) = (L_2 + m^2)^k \cdot (L_2 + m^2)^{-k} \delta.$$

Therefore,

$$(L_2 + m^2)^k N_{2k}(x,m) = \delta.$$

Since

$$\begin{aligned} N_{2k}(x,m) &= \binom{-k}{0} m^{2(0)}(-1)^{k+0}(i)^{\frac{q}{2}} T_{2k+2(0)}(x) \\ &\quad + \sum_{r=1}^{+\infty} \binom{-k}{r} m^{2r}(-1)^{k+r}(i)^{\frac{q}{2}} T_{2k+2r}(x). \end{aligned} \tag{38}$$

The second summand of the right-hand member of (38) vanishes for $m = 0$ and then, we have $N_{2k}(x, m = 0) = (-1)^k(i)^{\frac{q}{2}} T_{2k}(x)$ which is the fundamental solution of the operator L_2^k .

Lemma 13. *The convolution $W_{2k}(x,m) * Y_{2k}(x,m)$ exists and is a tempered distribution where $W_{2k}(x,m)$ and $Y_{2k}(x,m)$ are defined by* (27) *and* (30)*, respectively.*

Proof. See [7].

Lemma 14. *The convolution $M_{2k}(x,m) * N_{2k}(x,m)$ exists and is a tempered distribution where $M_{2k}(x,m)$ and $N_{2k}(x,m)$ are defined by* (33) *and* (36)*, respectively.*

Proof. From (33) and (36), we have

$$\begin{aligned} M_{2k}(x,m) * N_{2k}(x,m) &= \left(\sum_{r=0}^{+\infty} \binom{-k}{r} m^{2r}(-1)^{k+r}(-i)^{\frac{q}{2}} S_{2k+2r}(x)\right) \\ &\quad * \left(\sum_{r=0}^{+\infty} \binom{-k}{r} m^{2r}(-1)^{k+r}(i)^{\frac{q}{2}} T_{2k+2r}(x)\right) \\ &= \sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty} \binom{-k}{r}\binom{-k}{s} m^{2r+2s} S_{2k+2r}(x) * T_{2k+2r}(x). \end{aligned}$$

Since the function $S_{2k+2r}(x)$ and $T_{2k+2r}(x)$ are tempered distributions, see([5],p.34, [2], p.302 and [6], p.97) and the convolution of functions

$$S_{2k+2r}(x) * T_{2k+2r}(x)$$

exists and is also a tempered distribution, see ([3], p.152). Thus, $M_{2k}(x,m) * N_{2k}(x,m)$ exists and also is a tempered distribution.

Lemma 15. *(The convolution $W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$). The function $W_{2k}(x,m) * Y_{2k}(x,m)$ and $M_{2k}(x,m) * N_{2k}(x,m)$ are tempered distributions. The convolution $W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$ exists and also a tempered distribution.*

Proof. See [10].

3. Main Results

In this main results, we obtained two theorems and such a solution $H(x,m)$ related to the partial differential operator depends on the condition of p, q, k and m.

Theorem 1. *Given the equation*

$$\oplus_m^k H(x,m) = \delta, \tag{39}$$

where $\oplus_m^k$ is the operator iterated k- times defined by (7)*, δ is the Dirac delta distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer and m is a nonnegative real number. Then we obtain*

$$H(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m) \tag{40}$$

is the fundamental solution for the operator $\oplus_m^k$ *iterated k-times, where* $\oplus_m^k$ *is defined by* (7). *In particular, for* $q = m = 0$ *then* (39) *becomes*

$$\triangle_p^{4k} H(x,0) = \delta, \tag{41}$$

we obtain

$$H(x,0) = Y_{8k}(x,0) = R_{8k}^e(x) \tag{42}$$

is the fundamental solution of (41)*, where* $\triangle_p^{4k}$ *is the Laplace operator of p-dimension, iterated* $4k$*-times which is defined by* (16). *Moreover, from* (40)

$$H(x,0) = \left[R_{2k}^H(x) * (-1)^k R_{2k}^e(x)\right] * S_{2k}(x) * T_{2k}(x) \tag{43}$$

is the fundamental solution of O-plus operator $\oplus^k$ *and from* (43) *we obtain*

$$[(-1)^k R_{-2k}^e(x) * S_{-2k}(x) * T_{-2k}(x)] * H(x,0) = R_{2k}^H(x) \tag{44}$$

is the fundamental solution of the ultra-hyperbolic operator $\Box^k$ *iterated k-times defined by* (9)*, where* $R_{-2k}^e(x)$*,* $S_{-2k}(x)$ *and* $T_{-2k}(x)$ *are inverse of* $R_{2k}^e(x)$*,* $S_{2k}(x)$ *and* $T_{2k}(x)$ *respectively. From* (43) *and* (44) *with* $p = 1, q = n-1, k = 1$ *and* $x_1 = t$*, we obtain*

$$[(-1)^k R_{-2}^e(x) * S_{-2}(x) * T_{-2}(x)] * H(x,0) = M_2(u) \tag{45}$$

is the fundamental solution of the wave operator is defined by (11) *where* $M_2(u)$ *is defined by* (21) *with* $\alpha = 2$.

Proof. From (15) and (39) we have

$$\oplus_m^k H(x,m) = \left(\left(\Box + m^2\right)^k \left(\triangle + m^2\right)^k \left(L_1 + m^2\right)^k \left(L_2 + m^2\right)^k\right) H(x,m) = \delta.$$

Convolving both sides of the above equation by the convolution

$$W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$$

and the properties of convolution with derivatives, we obtain

$$\begin{aligned} &\left(\Box + m^2\right)^k W_{2k}(x,m) * \left(\triangle + m^2\right)^k Y_{2k}(x,m) \\ &\quad * \left(L_1 + m^2\right)^k M_{2k}(x,m) * \left(L_2 + m^2\right)^k N_{2k}(x,m) * H(x,m) \\ &\qquad = Y_{2k}(x,m) * W_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m) * \delta. \end{aligned} \tag{46}$$

Thus

$$H(x,m) = \delta * \delta * \delta * \delta * H(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m) \tag{47}$$

by Lemma 9, Lemma 10, Lemma 11 and Lemma 12. Thus we obtain (40) as required. In particular, for $q = m = 0$ then (39) becomes

$$\triangle_p^{4k} H(x,0) = \delta$$

where $\triangle_p^{4k}$ is the Laplace operator of p-dimension iterated $4k$-times. By Lemma 10 , we have

$$H(x,0) = Y_{8k}(x,0) = R^e_{8k}(x) \tag{48}$$

is the fundamental solution of (41). From Lemma 9, Lemma 10, Lemma 11 and Lemma 12 , we have

$$\begin{aligned} H(x,m=0) &= W_{2k}(x,0) * Y_{2k}(x,0) * M_{2k}(x,0) * N_{2k}(x,0) \\ &= \left[R^H_{2k}(x) * (-1)^k R^e_{2k}(x)\right] * S_{2k}(x) * T_{2k}(x) \end{aligned} \tag{49}$$

is the fundamental solution of the O-plus operator $\oplus^k$ in [2]. Now we will relate the fundamental solution $H(x,m=0)$ given by (43) to the fundamental solution of the wave equation is defined by (11). Now from (43) and by Lemma 7 and Lemma 8(2) and the properties of inverses in convolution algebra, we obtain

$$\begin{aligned} &[(-1)^k R^e_{-2k}(x) * S_{-2k}(x) * T_{-2k}(x)] * H(x,m=0) \\ &\qquad = \delta * \delta * \delta * R^H_{2k}(x) = R^H_{2k}(x). \end{aligned}$$

Actually, by Lemma 2 $R^H_{2k}(x)$ is the fundamental solution of the ultra-hyperbolic operator $\Box^k$ iterated k-times is defined by (9). In particular, by putting $p = 1, q = n-1, k = 1$ and $x_1 = t$ in (43) and (44) then $R^H_2(x)$ reduce to $M_2(u)$ is defined by (21) with $\alpha = 2$. Thus we obtain

$$[(-1)^k R^e_{-2}(x) * S_{-2}(x) * T_{-2}(x)] * H(x,m=0) = M_2(u)$$

is the fundamental solution of the wave operator is defined by (11) where $u = t^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2$.

Theorem 2. *Given the equation*

$$\oplus^k_m u(x) = f(x), \tag{50}$$

where $f(x)$ is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = H(x,m) * f(x) \tag{51}$$

is a solution of the equation (50)*, where $H(x,m)$ is the fundamental solution of equation* (39)*.*

Proof. Convolving both sides of (50) by $H(x,m)$, where $H(x,m)$ is the fundamental solution for $\oplus^k_m$ in Theorem 1, we obtain

$$H(x,m) * \oplus^k_m u(x) = H(x,m) * f(x)$$

or,

$$\oplus^k_m H(x,m) * u(x) = H(x,m) * f(x)$$

applying the Theorem 1, we have

$$\delta * u(x) = H(x,m) * f(x).$$

Therefore,

$$u(x) = H(x,m) * f(x).$$

Example 1. *Consider the equation*

$$(m^4 + \triangle^2)^k (m^4 - \triangle^2)^k u(x) = f(x), \tag{52}$$

where $\triangle^2$ *is the biharmonic operator defined by*

$$\triangle^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^2, \tag{53}$$

$x \in \mathbb{R}^n$, $f(x)$ *is a given generalized function and* $u(x)$ *is an unknown function. For solving the product of biharmonic operators, we can rewrite the equation (52) as*

$$(m^8 - \triangle^4)^k u(x) = f(x) \tag{54}$$

and we know that the operator in the equation (54) is the operator $\oplus_m^k$ *with* $p = 0$ *and* $n = q$, *we obtain the function* $H(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$, *where* $W_{2k}(x,m), Y_{2k}(x,m), M_{2k}(x,m)$ *and* $N_{2k}(x,m)$ *are defined by (27), (30), (33) and (36), respectively , and*

$$R_{2k}^e(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} \left(x_1^2 + \cdots + x_n^2\right)^{\left(\frac{2k-n}{2}\right)}, \quad n = p+q \tag{55}$$

$$R_{2k}^H(x) = \frac{u^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} = \frac{(-x_1^2 - x_2^2 - \cdots - x_n^2)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} \tag{56}$$

for

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+2k-n}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k}{2}\right)\Gamma(\frac{-2k}{2})}, \tag{57}$$

$$S_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} \left(-i\left(x_1^2 + \cdots + x_n^2\right)\right)^{\left(\frac{2k-n}{2}\right)} \tag{58}$$

and

$$T_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)} \left(i\left(x_1^2 + \cdots + x_n^2\right)\right)^{\left(\frac{2k-n}{2}\right)}. \tag{59}$$

Convolving both sides of (54) by the new fundamental solution

$$H(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$$

, *we obtain that* $u(x) = f(x) * H(x,m)$ *is the solution of (54).*

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