EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 3, 2021, 1044-1056
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Comparison of the Adomian decomposition method and regular perturbation method on non linear equations second kind of Volterra

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Abstract. In this paper, we study convergence of Adomian decomposition method applied to second kind Volterra general integral and show that this method and regular perturbation method converges to the same solution.

2020 Mathematics Subject Classifications: 44Axx, 40C10,45I05

Key Words and Phrases: Adomian Decomposition Method (ADM), Regular Pertubation Method(RPM), Volterra integral equation second kind

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1. Introduction

In the literature, there are few analytical methods or digital successful for solving non linear integral equations. This is due to the strong non linearity of integral equations and the difficulty to find their exact solutions. In this paper, we examine second kind Volterra general integral :

$$\varphi(x) = f(x) + \varepsilon \int_0^x K(x,t)\varphi^p(t)dt; p \geq 2; 0 < \varepsilon \ll 1, a \leq t \leq x \leq T \prec +\infty \qquad (1)$$

Where

φ is the unknown function,

f is continuous function on $\sum_t = [a;T]$,

and $K \in L^2(\Omega)$ is continuous on $\Omega = [a;T] \times [a;T]$.

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.4014

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We used Adomian decomposition method and regular perturbation method permitted us to find the solution of the problem (1).

The structure of the present study consists of an introductory section, Sections 2 and 3, and final conclusions. In Section 2, we prove the convergence of both methods. Section 3 contains numerical examples.

2. Convergence

In this section we show the theorem of convergence.

Theorem 1. *Let us consider the non linear equations second kind of Volterra defined in* (1)*. Then the problem* (P) *converges to a unique solution* $\varphi \in C([a;T])$ *and the Adomian algorithm [1–4, 6–10, 12]*

$$(P_{ADM})\begin{cases} \varphi_0(x) = f(x) \\ \varphi_n(x) = \varepsilon \int_0^x K(x,t)A_{n-1}(t)dt;\ n \geq 1 \end{cases} \tag{2}$$

converges to this solution. In addition, Adomian and regular perturbation algorithms are equivalents.

Proof. The Adomian polynomial are obtained by the following formula :

$$[\frac{\partial^i(\lambda^k A_k)}{\partial\lambda^i}]_{\lambda=0} = [\frac{\partial^i(\lambda^k\varphi_k)^p}{\partial\lambda^i}]_{\lambda=0} \tag{3}$$

By unfolding, we get:

$$\begin{cases} A_0 = \varphi_0^p \\ A_1 = p\varphi_0^{p-1}\varphi_1 \\ A_2 = \frac{1}{2}p[2\varphi_0^{p-1}\varphi_2 + (p-1)\varphi_0^{p-2}\varphi_1^2 \\ A_3 = \frac{1}{6}p[6\varphi_0^{p-1}\varphi_3 - 6p\varphi_0^{p-2}\varphi_1\varphi_2 + (p^2-3p+2)\varphi_0^{p-3}\varphi_1^3] \\ ... \\ A_n = \frac{1}{n!}[\frac{\partial^n(\lambda^k\varphi_k)^p}{\partial\lambda^n}]_{\lambda=0}; \forall n \geq 0 \end{cases} \tag{4}$$

We have following Adomian algorithm:

$$(P_{ADM})\begin{cases} \varphi_0(x) = f(x) \\ \varphi_1(x) = \varepsilon \int_0^x K(x,t)A_0(t)dt \\ \varphi_2(x) = \varepsilon \int_0^x K(x,t)A_1(t)dt \\ ... \\ \varphi_n(x) = \varepsilon \int_0^x K(x,t)A_{n-1}(t)dt;\ n \geq 1 \end{cases} \tag{5}$$

Let us show that the algoritm (P_{ADM}) converges

• f is assumed continuous on $\sum_t = [a;T]$, there exist$m > 0$, such that $\forall x \in \sum_t$, $\forall (x,t) \in \Omega = [0;T] \times [a;T]$, $|f(x)| \leq m$

• K being continuous on $\Omega = [0;T] \times [a;T]$, there exist $M > 0$, such that $\forall (x,t) \in \Omega$, $|K(x,t)| \leq M$.

For $\varphi_n(x)$, we have successively $\forall n \geq 1$

$$\begin{cases} |\varphi_0(x)| \leq m_0 \\ |\varphi_1(x)| \leq m_0^p \dfrac{[\varepsilon M(x-a)]}{1!} \Rightarrow |\varphi_1(x)| \leq m_1 \dfrac{[\varepsilon M(x-a)]}{1!} \\ |\varphi_2(x)| \leq p m_0^p m_1 \dfrac{[\lambda M(x-a)]^2}{2!} \Rightarrow |\varphi_2(x)| \leq m_2 \dfrac{[\varepsilon M(x-a)]^2}{2!} \\ |\varphi_3^k(x)| \leq \frac{1}{2} p[2m_0^{p-1} m_2 + (p-1) m_0^{p-2} m_1^2 \dfrac{[\varepsilon M(x-a)]^3}{3!} \Rightarrow |\varphi_3^k(x)| \leq m_3 \dfrac{[\varepsilon M(x-a)]^3}{3!} \\ \ldots \\ |\varphi_n(x)| \leq m_n \dfrac{[\iota M(x-a)]^n}{n!} \end{cases} \tag{6}$$

Let's put

$$m' = Sup(m_0, m_1, ..., m_n)$$

it follows that :

$$|\varphi_n(x)| \leq m' \frac{[\varepsilon M(x-a)]^n}{n!}$$

Let's put

$$\phi_n(x) = m' \frac{[\varepsilon M(x-a)]^n}{n!}$$

The series:

$$\sum_{n \geq 0} \phi_n(x) = m' \sum_{n \geq 0} \frac{[\varepsilon M(x-a)]^n}{n!}$$

Converging geometrically toward the function:

$$g(x) = m^{'} e^{[\varepsilon M(x-a)]} \quad on\ [0;T]$$

Therefore, the series$\sum_{n \geq 0} \varphi_n^k(x)$ converges normally and thus absolutely to $\varphi(x)$ on $[a;T$ which is the solution to problem (2)

-Let us suppose that the problem (P) admit two distinct solutions φ and ψ

For the function ψ ,we have following Adomian algorithm:

$$\begin{cases} \psi_0(x) = f(x) \\ \psi_n(x) = \varepsilon \int_0^x K(x,t)B_{n-1}(t)dt;\ n \geq 1 \end{cases} \tag{7}$$

Where

$$\begin{cases} B_0 = \psi_0^p \\ B_1 = p\psi_0^{p-1}\ \psi_1 \\ B_2 = \frac{1}{2}p[2\psi_0^{p-1}\varphi_2 + (p-1)\psi_1^{p-2}\psi_1^2 \\ B_3 = \frac{1}{6}p[6\psi_0^{p-1}\psi_3 - 6p\psi_0^{p-2}\psi_1\psi_2 + (p^2 - 3p + 2)\psi_0^{p-3}\psi_1^3] \\ \ldots \\ B_n = \frac{1}{n!}[\frac{\partial^n(\lambda^k\psi_k)^p}{\partial\lambda^n}\ ]_{\lambda=0} \end{cases} \tag{8}$$

are Adomian's polynomial.

Let's put $\omega(x) = \varphi(x) - \psi(x)$, $w(x)$ checks the following Adomian algorithm:

$$\begin{cases} \omega_0(x) = \varphi_0(x) - \psi_0(x) \\ \omega_n(x) = \varepsilon \int_0^x K(x,t)[A_{n-1}(t) - B_{n-1}(t)]dt;\ n \geq 1 \end{cases} \tag{9}$$

By unfolding the algorithm (9) for $k \geq 1$,we get:

$$\begin{cases} \omega_0(x) = f(x) - f(x) = 0 \\ \omega_1(x) = \varepsilon \int_0^x K(x,t)[\varphi_0^p(t) - \psi_0^p(t)]dt = 0 \\ \omega_2(x) = \varepsilon \int_0^x K(x,t)[A_1(t) - B_1(t)]dt = \varepsilon \int_0^x K(x,t)[p\varphi_0^p(t)\varphi_1(t) - p\psi_0^p(t)\psi_1(t)]dt = 0 \\ \ldots \\ \omega_n(x) = \varepsilon \int_0^x K(x,t)[A_{n-1}(t) - B_{n-1}(t)]dt = 0; \forall n \geq 0 \end{cases}$$

we have

$$\omega(x) = \sum_{n\geq 0} \omega_n(x) = 0$$

since

$$\forall n \geq 0, \omega_n(x) = 0$$

And thus

$$\omega(x) = \varphi(x) - \psi(x) = 0 \Leftrightarrow \varphi(x) = \psi(x)$$

and

$$\varphi = \psi \forall x \in \Sigma_t = [a; T].$$

This proves the uniqueness of the solution of the equation (1) and the convergence of the Adomian algorithm.

Let us show that the Adomian algorithm and the regular perturbation converges to the same solution

Let us consider the problem (P)
• Applying the Adomian decomposition algorithm to (1) , we get:

$$(P_{ADM}) \left\{ \begin{array}{c} \varphi_0(x) = f(x) \\ \varphi_n(x) = \varepsilon \int_0^x K(x,t) A_{n-1}(t)dt; \; n \geq 1 \end{array} \right. \tag{10}$$

• Regular perturbation method [5, 11]

Let us suppose that ψ is another solution of the problem (P)
This method consist to search the approximate solution by an asymptotic expression :

$$\psi(x) = \sum_{n\geq 0} \varepsilon^n \psi_n(x) \tag{11}$$

where ε is a small parameter of the problem.

Let's introduce (11) to (1), we get:

$$\sum_{n\geq 0} \varepsilon^n \psi_n(x) = f(x) + \varepsilon \int_0^x K(x,t) (\sum_{n\geq 0} \varepsilon^n \psi_n(t))^p(t)dt; \tag{12}$$

Using Binomial Newton formula, we get

$$\begin{aligned} [\psi_0 + (\varepsilon\psi_1 + \varepsilon^2\psi_2)]^p \quad = \quad & \psi_0^p + p\psi_0^{p-1}(\varepsilon\psi_1 + \varepsilon^2\psi_2) + \\ & \frac{p(p-1)}{2}\psi_0^{p-2}(\varepsilon^2\psi_1^2 + 2\varepsilon^3\psi_1\psi_2 + \varepsilon^4\psi_2^2) + ... + (\varepsilon\psi_1 + \varepsilon^2\psi_2)^p \end{aligned}$$

By identification according to the growth power of ε, we get:

$$\left\{ \begin{array}{l} \varepsilon^0 : \psi_0(x) = f(x) \\ \varepsilon^1 : \psi_1(x) = \int_0^x K(x,t)\psi_0^p(t)dt \\ \varepsilon^2 : \psi_2(x) = \int_0^x K(x,t)p\psi_0^{p-1}(t)\psi_1(t)dt \\ \varepsilon^3 : \psi_3(x) = \int_0^x K(x,t)\frac{1}{2}p[2\psi_0^{p-1}\psi_2(t) + (p-1)\psi_0^{p-2}(t)\psi_1^2(t)]dt \\ ... \\ \varepsilon^n : \psi_n(x) = ... \end{array} \right.$$

• **Comparision of the solution of the both methods**

Let's put $\phi_n(x) = \varphi_n(x) - \varepsilon^n \psi_n(x)$
By unfolding, we get:

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$$\begin{cases} \phi_0(x) = \varphi_0(x) - \psi_0(x) f(x) = f(x) - f(x) = 0 \\ \quad \phi_1(x) = \varepsilon \int_0^x K(x,t)\varphi_0^p(t)dt - \varepsilon \int_0^x K(x,t)\psi_0^p(t)dt = \varepsilon \int_0^x K(x,t)[\varphi_0^p(t) - \psi_0^p(t)]dt = 0 \\ \quad \phi_2(x) = \varepsilon \int_0^x K(x,t)p\varphi_0^{p-1}(t)\varphi_1(t)dt - \varepsilon^2 \int_0^x K(x,t)p\psi_0^{p-1}(t)\psi_1(t)]dt \\ \qquad\qquad = \varepsilon \int_0^x K(x,t)[pf_0^{p-1}(t)[\varphi_1(t) - \varepsilon\psi_1(t)]dt = 0 \\ \ ... \\ \phi_n(x) = \varphi_n(x) - \varepsilon^n \psi_n(x) = 0, \forall n \geq 0 \end{cases}$$

we have

$$\sum_{n\geq 0} \phi_n(x) = 0$$

$$\Longleftrightarrow \sum_{n\geq 0}[\varphi_n(x) - \varepsilon^n \psi_n(x)] \ = 0 \Longleftrightarrow \sum_{n\geq 0}[\varphi_n(x) = \sum_{n\geq 0} \varepsilon^n \psi_n(x)] \Longleftrightarrow \ \varphi(x) = \psi(x)$$

and

$$\varphi = \psi \forall x \in \Sigma_t = [a;T].$$

And thus

$$\omega(x) = \varphi(x) - \psi(x) = 0 \Leftrightarrow \varphi(x) = \psi(x)$$

and

$$\varphi = \psi, \forall x \in \Sigma_t = [a;T].$$

This proves that t of the Adomian decomposition algorithm and regular perturbation method converges to the same solution .

3. Numerical examples

3.1. Example 1

Let's consider the following non linear integral equation of second kind of Volterra :

$$\varphi(x) = \sqrt{x} - \frac{16}{15}\varepsilon x^2 \sqrt{x} + \varepsilon \int_0^x \frac{\varphi^4(t)}{\sqrt{x-t}} dt; 0 < \varepsilon \ll 1 \tag{13}$$

• **Solving by the Adomian decomposition method**
- Applying the **Adomian** algorithm, it follows that (13)

$$\begin{cases} \quad \varphi_0(x) = \sqrt{x} \\ \quad \varphi_1(x) = -\frac{16}{15}\varepsilon x^2\sqrt{x} + \varepsilon \int_0^x \dfrac{A_0(t)}{\sqrt{x-t}}dt \\ ... \\ \varphi_n(x) = \varepsilon \int_0^x \dfrac{A_{n-1}(t)}{\sqrt{x-t}}dt, \qquad\qquad n \geq 2 \end{cases} \tag{14}$$

where

$$\begin{cases} A_0 = \varphi_0^4 \\ A_1 = 4\varphi_0^3\ \varphi_1 \\ A_2 = 4\varphi_0^3\varphi_2 + 6\varphi_0^2\varphi_1^2 \\ ... \\ A_n = \dfrac{1}{n!}[\dfrac{\partial^n(\lambda^k\varphi_k)^p}{\partial\lambda^n}\ ]_{\lambda=0} \end{cases} \tag{15}$$

- Computation of

$$\varepsilon \int_0^x \frac{A_0(t)}{\sqrt{x-t}}dt$$

Let's take $T = x - t$ and integrating, we get:

$$\begin{aligned} \varepsilon \int_0^x \frac{A_0(t)}{\sqrt{x-t}}dt &= \varepsilon \int_0^x \frac{\varphi_0^4(t)}{\sqrt{x-t}}dt = \varepsilon \int_0^x \frac{t^2}{\sqrt{x-t}}dt \\ &= \varepsilon \int_0^x (x-T)^2 T^{\frac{-1}{2}}\,dT = \frac{16}{15}\varepsilon x^2\sqrt{x} \end{aligned}$$

By induction on n , we get:

$$\begin{cases} \varphi_0(x) = \sqrt{x} \\ \varphi_1(x) = -\dfrac{16}{15}\varepsilon x^2\sqrt{x} + \varepsilon \int_0^x \dfrac{A_0(t)}{\sqrt{x-t}}dt = -\dfrac{16}{15}\varepsilon x^2\sqrt{x} + \dfrac{16}{15}\varepsilon x^2\sqrt{x} = 0 \\ \varphi_2(x) = \varepsilon \int_0^x \dfrac{A_1(t)}{\sqrt{x-t}}dt = \varepsilon \int_0^x \dfrac{4\varphi_0^3\ (t)\varphi_1(t)}{\sqrt{x-t}}dt = 0 \\ \varphi_3(x) = \varepsilon \int_0^x \dfrac{A_2(t)}{\sqrt{x-t}}dt = \varepsilon \int_0^x \dfrac{[4\varphi_0^3(t)\varphi_2(t) + 6\varphi_0^2(t)\varphi_1^2(t)]}{\sqrt{x-t}}dt = 0 \\ ... \\ \varphi_p(x) = 0, \quad \forall\ n \geq 1 \end{cases} \tag{16}$$

Let's put

$$\varphi(x) \quad = \quad \sum_{n\geq 0} \varphi_n(x)$$

$$= \quad \varphi_0(x)$$

Then the **exact solution** of the problem (13) is:

$$\varphi(x) = \sqrt{x}, \tag{17}$$

• Solving by the regular perturbation method

Let us search the solution of (13) by an asymptotic expression :

φ

$$\varphi(x) = \sum_{n \geq 0} \varepsilon^n \varphi_n(x) \tag{18}$$

where ε is a small parameter of the problem.

Let's introduce (18) to (13), we get:

$$\sum_{n \geq 0} \varepsilon^n \varphi_n(x) = \sqrt{x} - \frac{16}{15}\varepsilon x^2 \sqrt{x} + \varepsilon \int_0^x \frac{(\sum_{n \geq 0} \varepsilon^n \varphi_n(t))^4}{\sqrt{x-t}} dt; \tag{19}$$

Using Binomial Newton formula, we get

$$[\varphi_0 + (\varepsilon\varphi_1 + \varepsilon^2\varphi_2)]^4 = \varphi^4 + 4\varphi_0^3(\varepsilon\varphi_1 + \varepsilon^2\varphi_2) + 6\varphi_0^2(\varepsilon^2\varphi_1 + 2\varepsilon^3\varphi_1\varphi_2 + \varepsilon^4\varphi_2^2) + 4\psi_0(\varepsilon\varphi_1 + \varepsilon^2\varphi_2)^3 + (\varepsilon\varphi_1 + \varepsilon^2\varphi_2)^4..$$

By identification and by unfolging according to the growth power of ε, we get:

$$\begin{cases} \varepsilon^0 : \varphi_0(x) = \sqrt{x} \\ \quad \varepsilon^1 : \varphi_1(x) = -\frac{16}{15}\varepsilon x^2 \sqrt{x} + \int_0^x \frac{\varphi_0^4(t)}{\sqrt{x-t}} dt = -\frac{16}{15} x^2 \sqrt{x} + \frac{16}{15} \\ \quad x^2\sqrt{x} = 0 \\ \varepsilon^2 : \varphi_2(x) = \int_0^x \frac{4\varphi_0^3(t)\varphi_1(t)}{\sqrt{x-t}} dt = 0 \\ \varepsilon^3 : \varphi_3(x) = \int_0^x \frac{[4\varphi_0^3(t)\varphi_2(t) + 6\varphi_0^2(t)\varphi_1^2(t)]}{\sqrt{x-t}} dt = 0 \\ ... \\ \varepsilon^n : \varphi_n(x) = 0; \forall\ n \geq 1 \end{cases}$$

Let's put

$$\varphi(x) \quad = \quad \sum_{n \geq 0} \varepsilon^n \varphi_n(x)$$

$$= \varphi_0(x)$$

Then the **exact solution** of the problem (13) is:

$$\varphi(x) = \sqrt{x}, \tag{20}$$

3.2. Example 2

Let's consider the following a system of non linear integral equations of second kind of Volterra :

$$\begin{cases} u(x) = e^x + \dfrac{\varepsilon(e^{-mx} - e^x)}{m+1} + \varepsilon \int_0^x e^{x-t} v^m(t)dt;\ 0 < \varepsilon \ll 1; m \geq 2 \\ v(x) = e^{-x} - \dfrac{\varepsilon(e^{2mx} - e^x)}{2m-1} + \varepsilon \int_0^x e^{x-t} w^m(t)dt;\ 0 < \varepsilon \ll 1; m \geq 2 \\ w(x) = e^{2x} - \dfrac{\varepsilon(e^{mx} - e^x)}{m-1} + \varepsilon \int_0^x e^{x-t} u^m(t)dt;\ 0 < \varepsilon \ll 1; m \geq 2 \end{cases} \tag{21}$$

- **Solving by the Adomian decomposition method**
  - Applying the **Adomian** algorithm, it follows that (13) :

$$\begin{cases} u_0(x) = e^x \\ u_1(x) = \dfrac{\varepsilon(e^{-mx} - e^x)}{m+1} + \varepsilon \int_0^x e^{x-t} A_0(t)dt \\ \dots \\ u_n(x) = \varepsilon \int_0^x e^{x-t} A_{n-1}(t)dt \qquad n \geq 2 \end{cases} \quad \begin{cases} v_0(x) = e^{-x} \\ v_1(x) = -\dfrac{\varepsilon(e^{2mx} - e^x)}{2m-1} + \varepsilon \int_0^x e^{x-t} B_0(t)dt \\ \dots \\ v_n(x) = \varepsilon \int_0^x e^{x-t} B_{n-1}(t)dt \qquad n \geq 2 \end{cases} \tag{22}$$

$$\begin{cases} w_0(x) = e^{2x} \\ w_1(x) = -\dfrac{\varepsilon(e^{mx} - e^x)}{m-1} + \varepsilon \int_0^x e^{x-t} C_0(t)dt \\ \dots \\ w_n(x) = \varepsilon \int_0^x e^{x-t} C_{n-1}(t)dt \qquad n \geq 2 \end{cases} \tag{23}$$

where the Adomian polynomial's are given by:

$$\begin{cases} A_0 = v_0^m \\ A_1 = m v_0^{m-1}\ v_1 \\ A_2 = \frac{1}{2} m [2 v_0^{m-1} v_2 + (m-1) v_1^{m-2} v_1^2 \\ \dots \\ A_n = \frac{1}{n!} [\frac{\partial^n (\lambda^k v_k)^m}{\partial \lambda^n}\ ]_{\lambda=0} \end{cases} \quad \begin{cases} B_0 = w_0^m \\ B_1 = m w_0^{m-1}\ w_1 \\ B_2 = \frac{1}{2} m [2 w_0^{m-1} w_2 + (m-1) w_1^{m-2} w_1^2 \\ \dots \\ B_n = \frac{1}{n!} [\frac{\partial^n (\lambda^k w_k)^m}{\partial \lambda^n}\ ]_{\lambda=0} \end{cases} \tag{24}$$

$$\begin{cases} C_0 = u_0^m \\ C_1 = m u_0^{m-1}\ u_1 \\ C_2 = \dfrac{1}{2} m[2u_0^{m-1}u_2 + (m-1)u_1^{m-2}u_1^2 \\ ... \\ C_n = \dfrac{1}{n!}[\dfrac{\partial^n(\lambda^k u_k)^m}{\partial \lambda^n}\ ]_{\lambda=0} \end{cases} \tag{25}$$

By unfoldind on n , we get:

$$\begin{cases} u_0(x) = e^x \\ u_1(x) = 0 \\ u_2(x) = 0 \\ ... \\ u_n(x) = 0; \forall\ n \geq 1 \end{cases} \quad \begin{cases} v_0 = e^{-x} \\ v_1(x) = 0 \\ v_2(x) = 0 \\ ... \\ v_n(x) = 0; \forall\ n \geq 1 \end{cases} \quad \begin{cases} w_0 = e^{2x} \\ w_1(x) = 0 \\ w_2(x) = 0 \\ ... \\ w_n(x) = 0; \forall\ n \geq 1 \end{cases} \tag{26}$$

Let's put

$$\begin{aligned} (u(x), v(x), w(x)) &= (\sum_{n\geq 0} u_n(x), \sum_{n\geq 0} v_n(x), \sum_{n\geq 0} w_n(x)) \\ &= (u_0(x), v_0(x), w_0(x)) \end{aligned}$$

Then the **exact solution** of the problem (21) is:

$$(u(x), v(x), w(x)) = (\ e^x,\ e^{-x}, e^{2x}) \tag{27}$$

• **Solving by the regular perturbation method**

Let us search the solution of (13) by an asymptotic expression :

$$(u(x), v(x), w(x)) = (\sum_{n\geq 0} \varepsilon^n u_n(x), \sum_{n\geq 0} \varepsilon^n v_n(x), \sum_{n\geq 0} \varepsilon^n w_n(x)) \tag{28}$$

where ε is a small parameter of the problem.

Let's introduce (18) to (13) , we get:

$$\begin{cases} \sum_{n\geq 0} \varepsilon^n u_n(x) = e^x + \dfrac{\varepsilon(e^{-mx} - e^x)}{m+1} + \varepsilon \int_0^x e^{x-t}(\sum_{n\geq 0} \varepsilon^n v_n(t))^m dt \\ \sum_{n\geq 0} \varepsilon^n v_n(x) = e^{-x} - \dfrac{\varepsilon(e^{mx} - e^x)}{2m-1} + \varepsilon \int_0^x e^{x-t}(\sum_{n\geq 0} \varepsilon^n w_n(t))^m dt \\ \sum_{n\geq 0} \varepsilon^n w_n(x) = e^{2x} - \dfrac{\varepsilon(e^{-mx} - e^x)}{m-1} + \varepsilon \int_0^x e^{x-t}(\sum_{n\geq 0} \varepsilon^n u_n(t))^m dt \end{cases} \tag{29}$$

Using Binomial Newton formula, we get

$$
\begin{aligned}
[u_0 + (\varepsilon u_1 + \varepsilon^2 u_2)]^m &= u_0^m + p u_0^{m-1}(\varepsilon u_1 + \varepsilon^2 u_2) + \frac{m(m-1)}{2} u_0^{m-2}(\varepsilon^2 u_1^2 + 2\varepsilon^3 u_1 u_2 + \varepsilon^4 u_2^2) + ... \\
[v_0 + (\varepsilon u_1 + \varepsilon^2 v_2)]^m &= v_0^m + p v_0^{m-1}(\varepsilon v_1 + \varepsilon^2 v_2) + \frac{m(m-1)}{2} v_0^{m-2}(\varepsilon^2 v_1^2 + 2\varepsilon^3 v_1 v_2 + \varepsilon^4 v_2^2) + ... \\
[w_0 + (\varepsilon w_1 + \varepsilon^2 w_2)]^m &= w_0^m + p w_0^{m-1}(\varepsilon w_1 + \varepsilon^2 w_2) + \frac{m(m-1)}{2} w_0^{m-2}(\varepsilon^2 w_1^2 + 2\varepsilon^3 w_1 w_2 + \varepsilon^4 w_2^2) + ...
\end{aligned}
$$

By identification according to the growth power of ε, we get:

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
\varepsilon^0 : u_0(x) = e^x \\
\varepsilon^1 : u_1(x) = \dfrac{(e^{-mx} - e^x)}{m+1} + \int_0^x e^{x-t} v_0^m(t)dt \\
\varepsilon^2 : u_2(x) = \int_0^x e^{x-t} m v_0^{m-1}(t) v_1(t)dt \\
\varepsilon^2 : u_3(x) = \int_0^x e^{x-t} \dfrac{1}{2} m[2v_0^{m-1}(t)v_2(t) + (m-1)v_1^{m-2}(t)v_1^2(t)]dt \\
... \\
\varepsilon^{n:2} u_n(x) = ...
\end{array}
\right. \\
\left\{
\begin{array}{l}
\varepsilon^0 : v_0(x) = e^{-x} \\
\varepsilon^1 : v_1(x) = -\dfrac{(e^{mx} - e^x)}{2m-1} + \int_0^x e^{x-t} w_0^m(t)dt \\
\varepsilon^2 : v_2(x) = \int_0^x e^{x-t} m w_0^{m-1}(t) w_1(t)dt \\
\varepsilon^2 : v_3(x) = \int_0^x e^{x-t} \dfrac{1}{2} m[2w_0^{m-1}(t)w_2(t) + (m-1)w_1^{m-2}(t)w_1^2(t)]dt \\
... \\
\varepsilon^{n:2} v_n(x) = ...
\end{array}
\right. \\
\left\{
\begin{array}{l}
\varepsilon^0 : w_0(x) = e^{2x} \\
\varepsilon^1 : w_1(x) = \frac{(e^{-mx} - e^x)}{m+1} + \int_0^x e^{x-t} u_0^m(t)dt \\
\varepsilon^2 : w_2(x) = \int_0^x e^{x-t} m u_0^{m-1}(t) u_1(t)dt \\
\varepsilon^2 : w_3(x) = \int_0^x e^{x-t} \frac{1}{2} m[2u_0^{m-1}(t)u_2(t) + (m-1)u_1^{m-2}(t)u_1^2(t)]dt \\
... \\
\varepsilon^{n:2} w_n(x) = ...
\end{array}
\right.
\end{array}
\right. \tag{30}
$$

By unfoldind on n , we get :

$$
\left\{
\begin{array}{l}
u_0(x) = e^x \\
u_1(x) = 0 \\
u_2(x) = 0 \\
... \\
u_n(x) = 0; \forall\ n \geq 1
\end{array}
\right.
\qquad
\left\{
\begin{array}{l}
v_0 = e^{-x} \\
v_1(x) = 0 \\
v_2(x) = 0 \\
... \\
v_n(x) = 0; \forall\ n \geq 1
\end{array}
\right.
\qquad
\left\{
\begin{array}{l}
w_0 = e^{2x} \\
w_1(x) = 0 \\
w_2(x) = 0 \\
... \\
w_n(x) = 0; \forall\ n \geq 1
\end{array}
\right. \tag{31}
$$

Let's put

$$
\begin{aligned}
(u(x), v(x), w(x)) &= (\sum_{n\geq 0} \varepsilon^n u_n(x), \sum_{n\geq 0} \varepsilon^n v_n(x), \sum_{n\geq 0} \varepsilon^n w_n(x)) \\
&= (u_0(x), v_0(x), w_0(x))
\end{aligned}
$$

Then the **exact solution** of the problem (21) is:

$$
(u(x), v(x), w(x)) = (\ e^x,\ e^{-x}, e^{2x}) \tag{32}
$$

4. Conclusion

In this paper, we first showed that the Adomian decomposition method converges when applied to Volterra general integral equations of second kind.

Then we showed that the Adomian decomposition method and regular perturbation method converges to the same solution when applied to Volterra general integral equations of second kind.

Lastly, we used these both method to solve a non linear integral equation of second kind and a system of non linear integral equations of second kind of Volterra .

We showed that using the both method, we get the same solution. There are then the very powerful numerical tools for the resolution of non linear equations and systems of nonlinear equations of Volterra .

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