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New generalized Hermite−Hadamard type inequalities for $p-$convex functions in the mixed kind

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Abstract. In this article, we would like to state generalized results related to well-known Hermite − Hadamard dual inequality for $p-$convex functions using the newly introduced notion of $(s, r)-$convexity ($s-$convex function in mixed kind) with different techniques. Hence various established and new results would be captured as special case.

2020 Mathematics Subject Classifications: 26A46, 26A51, 26D07, 26D99

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1. Introduction and Preliminaries

The field of Mathematical Inequality is very conspicuous and lucid for researchers. The basic theory of convex function holds a very powerful solution to the problems faced by researchers during a detailed analysis. This field of mathematical research provides an important contrivance in growth of various branches of research and is given considerable attention in literature. Convexity has its applications in various fields of professional and daily life like management sciences, architecture, arts, industrial and pharmaceutical research and many more. The Hermite−Hadamard dual inequalities have a number of different applications, due to which it is a general need that one should study them, specially those involving $p-$convex functions. For further study related to the topic we refer the reader following articles [2], [3] − [6], [17] and [21] − [22].

Before we proceed further it is worth mentioning here, we introduce some notation which we would use in this article: I is a real interval, I° is interior of interval I, $M_p = \frac{b^p - a^p}{p}$ and $\beta_r(a, b) = \int_0^r t^{a-1}(1-t)^{b-1}dt, \quad a, b > 0$ is incomplete Beta function. It is worth mentioning that throughout this article we used the convention that $0^0 = 1$.

We shall start with some useful definitions and results:

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Theorem 1. *[10] Let $f : I \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(y)dy \le \frac{f(a)+f(b)}{2} \tag{1}$$

This result is known as Hermite−Hadamard dual inequality for convex function. For concave function f, both inequalities would be in reverse order. It is to be noted that Hadamard's inequality may be regarded as a refinement and it follows easily from Jensen's inequality. Hadamard's inequality for convex function has been given an illustrious attention in recent years and a considerable variety of refinements (see [1] – [5], [9] and [11] – [16]).

We recall here definition of $p-$convex function from [13]:

Definition 1. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be $p-$convex, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \le tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. *If we choose $p = 1$ and $p = -1$ in Definition 1, we get the ordinary convex function [4] and harmonically convex function [11].*

Here, we are going to introduce some new types of $p-$convex function, which we call as quasi $p-$convex function and $P-p-$convex function respectively.

Definition 2. *Let $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \subset (0,\infty) \to [0,\infty)$ is known as quasi $p-$convex, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \le \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 2. *If we choose $p = 1$ in Definition 2, we get the quasi convex function [14].*

Definition 3. *Let $p \in \mathbb{R} \setminus \{0\}$. We say that $f : I \subset (0,\infty) \to [0,\infty)$ is a $P-p-$convex function, if f is a non-negative and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \le f(x) + f(y).$$

Remark 3. *If we choose $p = 1$ in Definition 3, we get the $P-$convex function [8].*

Now, we are going to present the definitions of $s-p-$ convex functions of first and second kind extracted from [1], which can be used to generalize the results for Hermite−Hadamard type inequality given in [16].

Definition 4. *[1] Let $s \in [0,1], p \in \mathbb{R} \setminus \{0\}$. A function $f : I \subset (0,\infty) \to [0,\infty)$ is said to be the $s-p-$convex function in 1^{st} kind, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq t^s f(x) + (1-t^s)f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 4. *Following result will be obtained by replacing different values of s and p:*

(i) Note that in the above definition we also include $s = 0$. Further, if we put $s = 0$, we easily get the refinement of Definition 2, i.e.,

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq f(x) \leq max\{f(x), f(y)\}$$

(ii) If we choose $p = 1$ in Definition 4, we get the $s-$convex function in 1^{st} kind [19].

Definition 5. *[1] Let $s \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \subset (0,\infty) \to [0,\infty)$ is said to be the $s-p-$convex function in 2^{nd} kind, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 5. *Following result will be obtained by replacing different values of s and p:*

(i) In the similar manner, we have slightly improved the above definition by including $s = 0$. Further, if we put $s = 0$, we easily get the Definition 3.

(ii) If we choose $p = 1$ in Definition 5, we get the $s-$convex function in 2^{nd} kind [7].

Now, we are going to give the definition of $s-p-$convex function in mixed kind (or $(s,\ r)-p-$convex function) by further generalizing the Definitions 4 and 5 such that we can easily obtained both the definitions by imposing certain restrictions on r and s.

Definition 6. *Let $(s,r) \in [0,1]^2, p \in \mathbb{R} \setminus \{0\}$. A function $f : I \subset (0,\infty) \to [0,\infty)$ is said to be the $(s,r)-p-$convex function (or $s-p-$convex function in mixed kind), if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq t^{rs} f(x) + (1-t^r)^s f(y), \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 6. *Following well known results will be obtained by taking the different combinations of values of r, s and p.*

(i) If we choose $s = 1$ in (2), we get $s-p-$convex function in 1^{st} kind.

(ii) If we choose $r = 1$ in (2), we get $s-p-$convex function in 2^{st} kind.

(iii) If we choose $r = s = 1$ *in* (2), *we get* $p-$*convex function.*

(iv) If we choose $r = 0$ *in* (2), *we get refinement of quasi* $p-$*convex function.*

(v) If we choose $r = 1$ *and* $s = 0$ *in* (2), *we get* $P - p-$*convex function.*

(vi) If we choose $p = 1$ *in* (2), *we get* $(s, r)-$*convex function in mixed kind [14].*

(vii) If we choose $p = s = 1$ *in* (2), *we get* $s-$*convex function in* 1^{st} *kind.*

(viii) If we choose $p = r = 1$ *in* (2), *we get* $s-$*convex function in* 2^{st} *kind.*

(ix) If we choose $p = r = s = 1$ *in* (2), *we get ordinary convex function.*

(x) If we choose $p = 1$ *and* $r = 0$ *in* (2), *we get refinement of quasi convex function.*

(xi) If we choose $p = r = 1$ *and* $s = 0$ *in* (2), *we get* $P-$*convex function.*

Renowned Hölder's inequality in its general integral form is given as follows [18]:

Theorem 2. *Let* $1 \le p, q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L_p$ *and* $\phi \in L_q$, *then* $f\phi \in L_1$ *and*

$$\int |f(u)\phi(u)|du \le \|f\|_p \|\phi\|_q \tag{3}$$

where $f \in L_p$ *if* $\|f\|_p = \left(\int |f(u)|^p du\right)^{\frac{1}{p}} < \infty$.

Note that if we put $p = q = 2$, the above inequality becomes Cauchy – Schwarz inequality. Also, if we put $q = 1$ and let $p \to \infty$, then we get,

$$\int |f(u)\phi(u)|du \le ||f||_\infty ||\phi||_1$$

where $||f||_\infty$ stands for the essential supremum of $|f|$, i.e.,

$$||f||_\infty = ess \sup_{\forall u} |f(u)|.$$

Definition 7. *Let* f, ϕ *are real valued functions defined on* $[a, b]$ *and if* $|f|$ *and* $|f||\phi|^q$ *are integrable on* $[a, b]$, *then for* $q \ge 1$ *we have:*

$$\int_a^b |f(u)||\phi(u)|du \le \left(\int_a^b |f(u)|du\right)^{1-\frac{1}{q}} \left(\int_a^b |f(u)||\phi(u)|^q du\right)^{\frac{1}{q}}.$$

The above inequality is known as power mean inequality (see [21]).

In [12], İ. Işcan stated and proved a result related to Hermite−Hadamard dual inequality for $p-$convex functions which we recall here:

Theorem 3. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a $p-$convex function, $p \in \mathbb{R} \setminus \{0\}$ and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequalities holds:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{1}{M_p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

Remark 7. *By taking different values of p, we easily obtain following results:*

(i) It can be verified that Theorem 1 is obtained by taking $p = 1$ in the above result.

(ii) It can be verified that Theorem 3 of [11] is obtained by taking $p = -1$ in the above result.

Now we state the following identity which will be used to derive the main results of this article.

Lemma 1. *[20] Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$, $p \in \mathbb{R} \setminus \{0\}$. If $f' \in L[a, b]$ then the following identity holds:*

$$\begin{aligned} &\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\ = \quad & M_p^2 \int_0^1 \frac{k(t)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt. \end{aligned}$$

where

$$k(t) = \begin{cases} t, & t \in [0, \frac{1}{2}), \\ t-1, & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

This article is organized as: In the next section, we are going to estimate the bounds of one of the Hermite$-$Hadamard inequalities (by taking absolute difference of first term and middle term of (4)) by using first differentiable $p-$convex functions in mixed kind. These results would capture various results stated in [15], [16] and [20] as special cases and the last section gives us conclusion with some remarks and future ideas.

2. Estimations of bound of Hermite$-$Hadamard (Left) Inequality for mixed kind $s-$Convex Function

Now we are going to state and prove three generalized results related to Hermite $-$Hadamard type inequalities for $p-$ convex function in mixed kind using Definition 6, Definition 7 and Theorem 2.

Theorem 4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ and $a<b$. If $|f'|$ is $s-p-$convex in the mixed kind on I for some fixed $r,s \in [0,1]$ on $[a,b]$ for $p \in \mathbb{R}\setminus\{0\}$, then following inequality holds:*

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \le M_p^2[A(p)|f'(a)|+B(p)|f'(b)|].$$

where

$$A(p) = \left[\int_0^{1/2} \frac{t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{t^{rs}-t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right]$$

$$B(p) = \left[\int_0^{1/2} \frac{t(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{(1-t)(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right]$$

Proof. By using Lemma 1 and then by applying the definition of *mixed* kind $s-p-$convexity of $|f'|$ on I, we have,

$$\begin{aligned}
&\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\
\le\ & M_p^2\left[\int_0^{1/2} \frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\right. \\
&\left.+\int_{1/2}^1 \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\right] \\
\le\ & M_p^2\left[\int_0^{1/2} \frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left\{t^{rs}|f'(a)|+(1-t^r)^s|f'(b)|\right\}dt\right. \\
&\left.+\int_{1/2}^1 \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left\{t^{rs}|f'(a)|+(1-t^r)^s|f'(b)|\right\}dt\right] \\
=\ & M_p^2\left[\left\{\int_0^{1/2} \frac{t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{t^{rs}-t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(a)|\right. \\
&\left.+\left\{\int_0^{1/2} \frac{t(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{(1-t)(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(b)|\right]
\end{aligned}$$

which completes the proof.

Remark 8. *In Theorem* 4, *we can get the following results:*

(i) If one takes $r = s = 1$, *then one has Theorem* 3.3 *of [20].*

(ii) If one takes $p = r = s = 1$, *then one has the Theorem* 2.2 *of [15].*

Corollary 1. *In Theorem* 4, *one can see the following:*

(i) If one takes $s = 1$ *then one has the following Hermite–Hadamard type inequality for* $s-p-$*convex functions in* 1^{st} *kind:*

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2\left[\left\{\int_0^{1/2} \frac{t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{t^s - t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(a)|\right.$$
$$\left.+\left\{\int_0^{1/2} \frac{t-t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{1-t-t^s+t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(b)|\right].$$

(ii) If one takes $r = 1$, *then one has the following Hermite–Hadamard type inequality for* $s-p-$*convex functions in* 2^{nd} *kind:*

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2\left[\left\{\int_0^{1/2} \frac{t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{t^s - t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(a)|\right.$$
$$\left.+\left\{\int_0^{1/2} \frac{t(1-t)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{(1-t)^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right\}|f'(b)|\right].$$

(iii) If one takes $p = 1$, *then one has the following Hermite–Hadamard type inequality for* $(s,r)-$*convex functions in mixed kind:*

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right|$$

$$\leq M_1^2 \left[\left\{ \beta_{1/2^r}\left(\frac{2}{r}, s+1\right) + \beta_{1-1/2^r}\left(s+1, \frac{1}{r}\right) - \beta_{1-1/2^r}\left(s+1, \frac{2}{r}\right) \right\} \frac{|f'(b)|}{r} + \frac{(2^{rs+1}-1)}{2^{rs+1}(rs+1)(rs+2)}|f'(a)| \right].$$

(iv) If one takes $p = s = 1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 1^{st} kind:

$$\left| \int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right) \right|$$
$$\leq M_1^2 \left[\frac{(2^{s+1}-1)}{2^{s+1}(s+1)(s+2)}|f'(a)| + \left(\frac{1}{4} - \frac{(2^{s+1}-1)}{2^{s+1}(s+1)(s+2)}\right)|f'(b)| \right].$$

(v) If one takes $p = r = 1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 2^{nd} kind:

$$\left| \int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right) \right|$$
$$\leq M_1^2 \frac{(2^{s+1}-1)}{2^{s+1}(s+1)(s+2)}(|f'(a)| + |f'(b)|).$$

Theorem 5. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ and $a < b$. If $|f'|^q$, $q \geq 1$ is $s - p-$convex in the mixed kind on I for some fixed $r, s \in [0, 1]$ and for $p \in \mathbb{R} \setminus \{0\}$, then following inequality holds:*

$$\left| \int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \right|$$
$$\leq M_p^2 \left\{ (Z_3(p))^{1-\frac{1}{q}} \left[(Z_4(p))|f'(a)|^q + Z_5(p)|f'(b)|^q\right]^{\frac{1}{q}} + (Z_6(p))^{1-\frac{1}{q}} \left[(Z_7(p))|f'(a)|^q + Z_8(p)|f'(b)|^q\right]^{\frac{1}{q}} \right\}.$$

where

$$Z_3(p) = \int_0^{1/2} \frac{t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}}dt, \quad Z_4(p) = \int_0^{1/2} \frac{t^{rs+1}}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}}dt$$

$$Z_5(p) = \int_0^{1/2} \frac{t(1-t^r)^s}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}}dt, \quad Z_6(p) = \int_{1/2}^{1} \frac{(1-t)}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}}dt$$

$$Z_7(p)=\int\limits_{1/2}^{1}\frac{t^{rs}(1-t)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt, \quad Z_8(p)=\int\limits_{1/2}^{1}\frac{(1-t)(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt$$

Proof. By using Lemma 1, Power mean inequality and then by applying the definition of *mixed* kind $s-p-$convexity of $|f|^q$ on I, we have,

$$\begin{aligned}
&\left|\int\limits_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\
\leq\ & M_p^2\left[\int\limits_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\right. \\
&\left.+\int\limits_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\right] \\
\leq\ & M_p^2\left[\left(\int\limits_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\right. \\
&\left(\int\limits_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int\limits_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}} \\
&\left.\left(\int\limits_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & M_p^2\left[\left(\int\limits_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\right. \\
&\left(\int\limits_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left\{t^{rs}|f'(a)|^q+(1-t^r)^s|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\left.\left(\int_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left\{t^{rs}|f'(a)|^q+(1-t^r)^s|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}\right]$$

$$= M_p^2\left[\left(\int_{0}^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\right.$$

$$\left(|f'(a)|^q\int_{0}^{1/2}\frac{t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt+|f'(b)|^q\int_{0}^{1/2}\frac{t(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1}\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\left.\left(|f'(a)|^q\int_{1/2}^{1}\frac{t^{rs}-t^{rs+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt+|f'(b)|^q\int_{1/2}^{1}\frac{(1-t)(1-t^r)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}\right]$$

which completes the proof.

Remark 9. *In Theorem* 5, *we can get the following results:*

(i) If one takes $p=r=s=1$, *then one has first result of Corollary* 2 *of [16].*

(ii) If one takes $r=s=1$, *then one has second result of Corollary* 2 *of [16].*

(iii) If one takes $p=-1$ *and* $r=s=1$, *then one has fifth result of Corollary* 2 *of [16].*

Corollary 2. *In Theorem* 5, *one can see the following:*

(i) If one takes $s=1$ *then one has the following Hermite–Hadamard type inequality for* $s-p-$*convex functions in* 1^{st} *kind:*

$$\left|\int_a^b\frac{f(x)}{x^{1-p}}dx-M_pf\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|\le M_p^2\left[\left(\int_0^{1/2}\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\right.$$

$$\left(|f'(a)|^q \int_0^{1/2} \frac{t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + |f'(b)|^q \int_0^{1/2} \frac{t-t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1} \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\left.\left(|f'(a)|^q \int_{1/2}^{1} \frac{t^s-t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + |f'(b)|^q \int_{1/2}^{1} \frac{1-t-t^s+t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}\right].$$

(ii) If one takes $r = 1$, then one has the following Hermite–Hadamard type inequality for $s-p-$convex functions in 2^{nd} kind:

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \le M_p^2\left[\left(\int_0^{1/2} \frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\right.$$

$$\left(|f'(a)|^q \int_0^{1/2} \frac{t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + |f'(b)|^q \int_0^{1/2} \frac{t(1-t)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1} \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\left.\left(|f'(a)|^q \int_{1/2}^{1} \frac{t^s-t^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + |f'(b)|^q \int_{1/2}^{1} \frac{(1-t)^{s+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{\frac{1}{q}}\right].$$

(iii) If one takes $p = 1$, then one has the following Hermite–Hadamard type inequality for $(s, r)-$convex functions in mixed kind:

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \le M_1^2\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$

$$\left[\left\{\frac{|f'(a)|^q}{2^{rs+2}(rs+2)} + \frac{|f'(b)|^q}{r}\beta_{1/2^r}\left(\frac{2}{r}, s+1\right)\right\}^{\frac{1}{q}} + \left\{\frac{(2^{rs+1}-1)|f'(a)|^q}{2^{rs+1}(rs+1)(rs+2)}\right.\right.$$

$$\left.\left.+\frac{|f'(b)|^q}{r}\left(\beta_{1-1/2^r}\left(s+1, \frac{1}{r}\right) + \beta_{1-1/2^r}\left(s+1, \frac{2}{r}\right)\right)\right\}^{\frac{1}{q}}\right].$$

(iv) *If one takes $p = s = 1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 1^{st} kind:*

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \leq M_1^2\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$
$$\left[\left\{\frac{|f'(a)|^q}{2^{s+2}(s+2)} + \left(\frac{1}{8} - \frac{1}{2^{s+2}(s+2)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{(2^{s+2}-(s+3))|f'(a)|^q}{2^{s+2}(s+1)(s+2)} + \left(\frac{1}{4} - \frac{(2^{s+1}-1)}{2^{s+1}(s+1)(s+2)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}\right].$$

(v) *If one takes $p = r = 1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 2^{nd} kind:*

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \leq M_1^2\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$
$$\left[\left\{\frac{|f'(a)|^q}{2^{s+2}(s+2)} + \frac{(2^{s+2}-(s+3))|f'(b)|^q}{2^{s+2}(s+1)(s+2)}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{(2^{s+2}-(s+3))|f'(a)|^q}{2^{s+2}(s+1)(s+2)} + \frac{|f'(b)|^q}{2^{s+2}(s+2)}\right\}^{\frac{1}{q}}\right].$$

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ and $a < b$. If $|f'|^{q_2}$, $q_2 \geq 1$ is $s-p-$convex in the mixed kind on I for some fixed $r, s \in [0,1]$ and for $p \in \mathbb{R}\setminus\{0\}$, then following inequality holds:*

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2\left[Z_9(p)\left(\frac{|f'(a)|^{q_2}}{2^{rs+1}(rs+1)} + \frac{\beta_{1/2^r}\left(\frac{1}{r}, s+1\right)|f'(b)|^{q_2}}{r}\right)^{\frac{1}{q_2}}\right.$$
$$\left.+Z_{10}(p)\left(\frac{(2^{rs+1}-1)|f'(a)|^{q_2}}{2^{rs+1}(rs+1)} + \frac{\beta_{1-1/2^r}\left(s+1, \frac{1}{r}\right)|f'(b)|^{q_2}}{r}\right)^{\frac{1}{q_2}}\right].$$

where

$$Z_9(p) = \left(\int_0^{1/2}\left(\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{q_1}dt\right)^{\frac{1}{q_1}},$$

$$Z_{10}(p) = \left(\int_{1/2}^{1} \left(\frac{1-t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}}$$

with $\frac{1}{q_1} + \frac{1}{q_2} = 1$.

Proof. By using Lemma 1, Hölder's inequality and then by applying the definition of *mixed* kind $s-p-$convexity of $|f|^{q_2}$ on I, we have,

$$\begin{aligned}
&\left| \int_a^b \frac{f(x)}{x^{1-p}} dx - M_p f\left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\
\leq\ & M_p^2 \left[\int_0^{1/2} \frac{t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \left| f'\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| dt \right. \\
&\left. + \int_{1/2}^{1} \frac{1-t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \left| f'\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| dt \right]. \\
\leq\ & M_p^2 \left[\left(\int_0^{1/2} \left(\frac{t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_0^{1/2} \left| f'\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right|^{q_2} dt \right)^{\frac{1}{q_2}} \right. \\
&\left. + \left(\int_{1/2}^{1} \left(\frac{1-t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_{1/2}^{1} \left| f'\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right|^{q_2} dt \right)^{\frac{1}{q_2}} \right] \\
\leq\ & M_p^2 \left[\left(\int_0^{1/2} \left(\frac{t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}} \right. \\
&\left(\int_0^{1/2} \left(t^{rs} |f'(a)|^{q_2} + (1-t^r)^s |f'(b)|^{q_2} \right) dt \right)^{\frac{1}{q_2}} \\
&+ \left(\int_{1/2}^{1} \left(\frac{1-t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}}
\end{aligned}$$

$$\left.\left(\int_{1/2}^{1}\left(t^{rs}|f'(a)|^{q_2}+(1-t^r)^s|f'(b)|^{q_2}\right)dt\right)^{\frac{1}{q_2}}\right]$$

$$= M_p^2\left[\left(\int_0^{1/2}\left(\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{q_1}dt\right)^{\frac{1}{q_1}}\right.$$

$$\left(\frac{|f'(a)|^{q_2}}{2^{rs+1}(rs+1)}+\frac{\beta_{1/2^r}\left(\frac{1}{r},s+1\right)|f'(b)|^{q_2}}{r}\right)^{\frac{1}{q_2}}$$

$$+\left(\int_{1/2}^{1}\left(\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{q_1}dt\right)^{\frac{1}{q_1}}$$

$$\left.\left(\frac{(2^{rs+1}-1)|f'(a)|^{q_2}}{2^{rs+1}(rs+1)}+\frac{\beta_{1-1/2^r}\left(s+1,\frac{1}{r}\right)|f'(b)|^{q_2}}{r}\right)^{\frac{1}{q_2}}\right]$$

which completes the proof.

Remark 10. *In Theorem* 6, *we can get the following results:*

(i) If one takes $p = r = s = 1$, *then one has Theorem* 2.3 *of [15].*

(ii) If one takes $r = s = 1$, *then one has first result of Corollary* 3 *of [16].*

(iii) If one takes $p = -1$ *and* $r = s = 1$, *then one has fourth result of Corollary* 3 *of [16].*

Corollary 3. *In Theorem* 6, *one can see the following:*

(i) If one takes $s = 1$ *then one has the following Hermite–Hadamard type inequality for* $s-p-$*convex functions in* 1^{st} *kind:*

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \leq M_p^2 \times$$

$$\left[\left(\int_0^{1/2}\left(\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{p}dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+(2^s-1)|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{1/2}\left(\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{p}dt\right)^{\frac{1}{p}}$$

$$\left(\frac{(2^s-1)\,|f'(a)|^q+\left(2^s-2^{s+1}+1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\Bigg].$$

(ii) If one takes $r=1$, then one has the following Hermite–Hadamard type inequality for $s-p-$convex functions in 2^{nd} kind:

$$\left|\int_a^b \frac{f(x)}{x^{1-p}}dx - M_p f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \leq M_p^2 \times$$

$$\left[\left(\int_0^{1/2}\left(\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{1/2}\left(\frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right].$$

(iii) If one takes $p=1$, then one has the following Hermite–Hadamard type inequality for $(s,r)-$convex functions in mixed kind:

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \leq M_1^2 \times$$

$$\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q}{2^{rs+1}(rs+1)}+\frac{\beta_{1/2^r}\left(\frac{1}{r},s+1\right)|f'(b)|^q}{r}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(2^{rs+1}-1)|f'(a)|^q}{2^{rs+1}(rs+1)}+\frac{\beta_{1-1/2^r}\left(s+1,\frac{1}{r}\right)|f'(b)|^q}{r}\right)^{\frac{1}{q}}\right].$$

(iv) If one takes $p=s=1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 1^{st} kind:

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \leq M_1^2 \times$$

$$\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+(2^s-1)\,|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(2^s-1)\,|f'(a)|^q+\left(2^s-2^{s+1}+1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right].$$

(v) *If one takes $p = r = 1$, then one has the following Hermite–Hadamard type inequality for $s-$convex functions in 2^{nd} kind:*

$$\left|\int_a^b f(x)dx - M_1 f\left(\frac{a+b}{2}\right)\right| \leq M_1^2 \times$$

$$\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right].$$

3. Conclusion and Remarks

3.1. Conclusion

Hermite−Hadamard dual inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in literature. We have given its generalization by introducing new generalized notion of $(s, r)-$convex functions in the mixed kind. This new class of functions contains many important classes including class of $s-$convex in the first and in the second kind (and hence contains class of convex functions). It also contains class of $P-$convex functions and class of quasi−convex functions. In section 2, we have stated three different results related to estimation of bound of difference of left and middle term of Hermite−Hadamard dual inequality in absolute sense. Here we used different techniques including power mean inequality and Hölder's inequality. These results capture various results stated in articles [15], [16] and [20].

Now we are going to give some remarks and future ideas for readers.

3.2. Remarks and Future Ideas

(i) We can also state all the inequalities given in this article in reverse direction for concave function by using simple relation f is concave iff $-f$ is convex.

(ii) One may also work on Fejer inequality by introducing weights in Hermite−Hadamard dual inequality.

(iii) One may do similar work by using various different classes of functions.

(iv) One may try to state all results stated in this article in discrete case.

(v) One may also state all results stated in this article in higher dimensions.

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