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On general properties of degenerate systems of second order partial differential equations of hypergeometric type

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Abstract. For the first time, the general properties of degenerate related hypergeometric systems such as Horn, Whittaker, Bessel and Laguerre are investigated together. The joint research allowed to reveal their various common properties and to establish a number of new degenerate related systems. They are all private cases of the common system offered by the authors for consideration. For the full study, it is important to classify its regular and irregular special curves and to identify the types of corresponding solutions. In this paper, they are implemented using simple rules. Special attention is paid to the construction of normal and regular solutions, because the solutions of all related degenerate systems such as Horn, Whittaker, Bessel and Laguerre near the irregular singularity on infinity relate to this species. Peculiarities of building normal-regular solutions by the Frobenius-Latysheva method are shown. All constructed normal-regular solutions are expressed through the function of Humbert n variables, which is the solution of degenerate hypergeometric system of Horn type. As an example, the cases $n = 2$ where, along with the application of the Frobenius-Latysheva method, the possibility of outputting new degenerate related systems is demonstrated.

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1. Introduction

M. Laurichella summarized P. Appel's four hypergeometric functions $F_1 - F_4$ of two variables and introduced four hypergeometric functions n of the variables F_A, F_B, F_C, F_D [9]. Then, their various properties and degenerate hypergeometric functions n of the variables were studied [1]. In the case of a single variable function, Lucy J. Slater notes that "The four functions: Kummer's function ${}_1F_1(\alpha;\beta;x)$ and connected to it the second

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solution $U(\alpha;\beta;x)$, two Whittaker's functions $M_{k,m}(x)$ and $W_{k,m}(x)$ are called the degenerated hypergeometric functions". In mathematical physics, most of the functions used, including Weber and Bessel functions, are special cases of these functions. All of them are regular private solutions of some second order differential equations. These equations have the following singularities: regular is in a point $x = 0$ and irregular is in a point $x = \infty$. The Kummer equation

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0 \tag{1}$$

and the Whittaker equation

$$\frac{d^2y}{dx^2} + (\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - \mu^2}{x^2})y = 0 \tag{2}$$

have such properties [10].

In the case of hypergeometric functions of two variables, the range of properties under consideration is expanded. All 20 degenerate hypergeometric functions of two variables are obtained mainly from four hypergeometric functions of P. Appell $F_1 - F_4$ using limit transitions [1, p. 114-120]. Ya.Horn established that they are partial solutions of some systems of partial differential equations of the second order [1, p.226-231]. The systems connected with each other by some common properties are called related. For example, the common property includes the transformation

$$Z(x, y) = Q(x, y) \cdot U(x, y), \tag{3}$$

where $Q(x, y)$ the polynomial of the two variables, with the help of which it is possible to derive one system from the other, as well as to establish a connection between their solutions in the form of degenerate hypergeometric functions of the two variables.

In this paper, instead of the degenerate hypergeometric Kummer function, a comprehensive study of the properties of M.P. Humbert's functions $\Psi_2^{(2)}$ and its generalization for the case n of variables $\Psi_2^{(n)}$ meet our goals [4, p.428]. Let us reveal some basic concepts.

Definition 1. *The degenerate hypergeometric function of M.P. Humbert's $\Psi_2^{(2)}$ two variables $x_j (j = 1, 2)$ is determined using the row*

$$\Psi_2^{(2)}(\lambda;\gamma,\gamma';x_1,x_2) = \sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}}{(\gamma)_m \cdot (\gamma')_n} \cdot \frac{x_1^m}{m!}\frac{x_2^n}{n!}. \tag{4}$$

The row converges absolutely and evenly at $|x_1| < \epsilon, |x_2| < \epsilon$.

Theorem 1. *Humbert's function (4) near a regular singularity (0,0) is a private solution of the Horn system consisting of two differential equations in private derivatives of the second order*

$$\begin{aligned} &x_1 Z_{x_1x_1} + (\gamma_1 - x_1)Z_{x_1} - x_2 Z_{x_2} - \lambda Z = 0, \\ &x_2 Z_{x_2x_2} + (\gamma_2 - x_2)Z_{x_2} - x_1 Z_{x_1} - \lambda Z = 0, \end{aligned} \tag{5}$$

where general unknown.

From (5) using the conversion

$$Z = exp(\frac{x_1}{2} + \frac{x_2}{2}) \cdot x_1^{-\frac{\gamma_1}{2}} \cdot x_2^{-\frac{\gamma_2}{2}} U(x_1, x_2), \tag{6}$$

we get the system

$$\begin{aligned} &x_1 U_{x_1 x_1} - x_1 x_2 \cdot U_{x_2} + -\frac{x_1^2}{4} - \frac{x_1 x_2}{2} + k x_1 + \frac{1}{4} - \mu^2 U = 0, \\ &x_2 U_{x_2 x_2} - x_1 x_2 \cdot U_{x_1} + -\frac{x_2^2}{4} - \frac{x_1 x_2}{2} + k x_2 + \frac{1}{4} - \mu^2 U = 0, \end{aligned} \tag{7}$$

which is analogous to the Whittaker equation (2) [1] out of (5) by conversion (6).

Horn and Whittaker systems are studied well enough, the possibilities of constructing their various solutions associated with degenerate hypergeometric functions of two variables are established. However, these studies have not reached such a level as in the usual case. Especially, it is little known about related systems with Whittaker type system. Studying the relationship between the Horn system (5) and the Whittaker system (7), a number of works [14] and [17] related systems of the Bessel and Laguerre type have been established. Their various properties were revealed and their solutions were built. These ideas need to be expanded, extended to the cases of systems consisting of n equations, and established different properties of their solutions.

Studies of the construction of solutions of the studied systems based on the allocation of a common property based on the transformation (1.3). This allowed the establishment of related systems consisting of equations, the classification of their regular and irregular features using simple rules, the introduction of a new type of normally regular solutions, which are represented by all special hypergeometric functions of one and many variables.

The purpose of this work is to study a number of related systems consisting of n differential equations in private second order derivatives, the solutions of which are degenerated hypergeometric functions of n variables, to establish their common properties. To achieve this goal, paragraph 2 establishes the most common system from which all related systems of the Horn, Whittaker, Bessel and Laguerre types are derived in paragraph 3. The connection between systems of the Horn and Whittaker types of different orders with the solution of the species $\Psi_2^{(n)}$ proved by M.P. Humbert is used [1, p.135]. Peculiarities of application of Frobenius-Latysheva method [13] generalized by Zh.Tasmambetov in the case of the systems under study to the construction of their solution are shown.

2. Main results

2.1. About special curves of the general degenerate hypergeometric system

Setting the task. Let's introduce the system consisting of differential equations in private derivatives of the second order of type

$$x_j^2 [r_{20}^{(j)} - \alpha_{20}^{(j)} x_j] F_{x_j x_j} + x_j [r_{10}^{(j)} - \alpha_{10}^{(j)} x_j] F_{x_j} + [r_{01}^{(j)} - \alpha_{01}^{(j)} x_j] \sum_{k \neq j} x_k F_{x_k}$$

$$+[r_{00}^{(j)} - \alpha_{00}^{(j)} x_j]F = 0, \tag{8}$$

where $r_{20}^{(j)}, r_{10}^{(j)}, r_{01}^{(j)}, r_{00}^{(j)}, \alpha_{20}^{(j)}, \alpha_{10}^{(j)}, \alpha_{01}^{(j)}, \alpha_{00}^{(j)} (j = \overline{1,n})$ some constants.

It is necessary to establish regular and irregular singularities of the given system (8) and define the type of the corresponding solutions. The special curves are set by equating the coefficients to zero for the older derivatives $F_{x_j x_j} : x_j^2[r_{20}^{(j)} - \alpha_{20}^{(j)} x_j] = 0 \Rightarrow x_j = 0,\, x_j = r_{20}^{(j)}/\alpha_{20}^{(j)}$ $(j = \overline{1,n})$. Hence, it is not difficult to make sure that the final singularities are: $\underbrace{(0,0,\cdots,0)}_{n}$, $\underbrace{(0,0,\cdots,0,r_{20}^{(1)}/\alpha_{20}^{(1)})}_{n}$, $\cdots$, $\underbrace{(r_{20}^{(1)}, r_{20}^{(2)},\cdots,r_{20}^{(n)})}_{n}$. The singularities on infinity $(\infty,\infty,\cdots,\infty)$, $(\infty,\infty,\cdots,\infty,0)$, $\cdots$ are added to them. In an ordinary case, usually two features $\underbrace{(0,0,\cdots,0)}_{n}$ and $\underbrace{(\infty,\infty,\cdots,\infty,0)}_{n}$, are distinguished from them. Therefore, we will also limit ourselves to building regular and irregular solutions near these singularities.

2.2. Simple rules for classifying singularities and determining the type of solution near singular curves

The establishment of regular and irregular singular curves plays an important role in the analytical theory of the systems under study. Further, based on our established simple rules for classification of singular curves of systems consisting of two second order equations [13], we formulate a rule for classification of singular curves, in the case of a system consisting of n second order equations.

Rule 1. If the system (8) has coefficients $r_{20}^{(j)} \neq 0$, then system (8) is particularly regular and the corresponding solution is a generalized stepwise row of variables

$$F(x_1,x_2,\cdots,x_n) = x_1^{\rho_1}\cdot x_2^{\rho_2}\cdots x_n^{\rho_n}$$
$$\times \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} A_{m_1,m_2,\cdots,m_n} x_1^{m_1}\cdot x_2^{m_2}\cdots x_n^{m_n},\ A_{0,0,\cdots,0}\neq 0, \tag{9}$$

with unknown indicators of the row ρ_j $(j = \overline{1,n})$ and $A_{m_1,m_2,\cdots,m_n}$ $()m_1,m_2,\cdots,m_n = 0,1,2,\cdots)$ with unknown coefficients of the row.

Definition 2. *Function defined with the row*

$$F(x_1,x_2,\cdots,x_n) = \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} A_{m_1,m_2,\cdots,m_n} x_1^{m_1}\cdot x_2^{m_2}\cdots x_n^{m_n},$$

is called a generalized hypergeometric function if the relationship

$$\frac{A_{m_1+1,\cdots,m_n}}{A_{m_1,m_2,\cdots,m_n}},\cdots,\frac{A_{m_1,\cdots,m_n+1}}{A_{m_1,m_2,\cdots,m_n}}$$

are rational functions of indices $m_1,m_2,\cdots,m_n$ [3].

Further, we will mainly deal with the construction of degenerated hypergeometric functions of many variables in the form of generalized hypergeometric functions as solutions to the systems under study.

Rule 2. If coefficients $r_{20}^{(j)} = 0$ are used in (8), the system $(0, 0, \cdots, 0)$ is particularly irregular. The solution of the system (8) with an irregular singularity shall be presented in the form of

$$F(x_1, x_2, \cdots, x_n) = expQ(x_1, x_2, \cdots, x_n) x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n}$$
$$\times \sum_{m_1, m_2, \cdots, m_n = 0}^{\infty} A_{m_1, m_2, \cdots, m_n} x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}, \ A_{0,0,\cdots,0} \neq 0, \tag{10}$$

where ρ_j $(j = \overline{1, n})$, $A_{m_1, m_2, \cdots, m_n}$ $()m_1, m_2, \cdots, m_n = 0, 1, 2, \cdots)$ are unknown constants.

Degree of polynomial to variables

$$Q(x_1, x_2, \cdots, x_n) = \frac{\alpha_{p0\cdots 0}}{p} x_1^p + \cdots + \frac{\alpha_{0\cdots 0p}}{p} x_n^p + \cdots$$
$$+ \alpha_{10\cdots 0} x_1 + \cdots + \alpha_{0\cdots 01} x_n, \tag{11}$$

where $\alpha_{0\cdots 0p}, \cdots, \alpha_{0\cdots 01}$ are unknown coefficients are determined by the notion of rank.

Definition 3. *The rank p is determined by the highest degrees of the system's coefficients by equality*

$$p = 1 + k, \ k = \max \frac{\tau_s - \tau_0}{s}, \ (j = \overline{1, n}), \tag{12}$$

is called the row order (10) and can be an integer and a fraction (positive or negative) number.

K.Ya.Latysheva, along with the notion of rank, used the notion of anti-rank m to classify special points [14].

Definition 4. *Antitrank m, defined by the smallest degree of system factors, is the number of*

$$m = -1 - \chi = -\min \frac{\pi_j - \pi_0}{j}, \ (j = \overline{1, n}). \tag{13}$$

Usually, the concept of anti-range m links to a singularity $(0, 0, ..., 0)$, and the concept of rank to a singularity $(\infty, \infty, ..., \infty)$. In the case of the system of species (8), these concepts are generalized by Zh. Tasmambetov [13].

Rule 3. If the system has (8) coefficients $\alpha_{20}^{(j)} = 0$, the singularity $(\infty, \infty, ..., \infty)$ is particularly irregular. In the case where $\alpha_{20}^{(j)} \neq 0$, the singularity is particularly regular. In this case, the system (8) will have a regular singularity $(\infty, \infty, ..., \infty)$ near to the corresponding solution:

$$F(x_1, x_2, \cdots, x_n) = x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n}$$

$$\times \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} B_{m_1,m_2,\cdots,m_n} x_1^{-m_1} \cdot x_n^{-m_n},\ B_{0,0,\cdots,0} \neq 0, \tag{14}$$

where ρ_j $(j = \overline{1,n})$, $B_{m_1,m_2,\cdots,m_n}$ $()m_1, m_2, \cdots, m_n = 0, 1, 2, \cdots)$ are unknown constants.

The solution of the system (8) in the vicinity of an irregular singularity $(\infty, ..., \infty)$ is presented as follows

$$F(x_1, x_2, \cdots, x_n) = expQ(x_1, x_2, \cdots, x_n) x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n}$$
$$\times \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} B_{m_1,m_2,\cdots,m_n} x_1^{-m_1} \cdot x_2^{-m_2} \cdots x_n^{-m_n},\ B_{0,0,\cdots,0} \neq 0, \tag{15}$$

where ρ_j $(j = \overline{1,n})$, $B_{m_1,m_2,\cdots,m_n}$ $()m_1, m_2, \cdots, m_n = 0, 1, 2, \cdots)$ are unknown constants; $Q(x_1, ..., x_n)$ polynomial from n variables of type (11) is common to solutions (10) and (15). Let us call the expression (15) the normal row of Tome of variables and (15) is the formal solution of the type system (8).

The advantage of using the concept of rank and anti-rank is that by the type of the given system it is possible to establish regularity and irregularity of special curves, and the type of the proposed solution. Then, using the Frobenius-Latysheva method, we will build a definite solution depending on the regularity and irregularity of the special curves.

If at the same time $p \leq 0$ and $m \leq 0$, therefore, singularities $(0, 0, ..., 0)$ and $(\infty, \infty, ..., \infty)$ and are regular, then the polynomial $Q(x_1, ..., x_n)$ in (10) and (15) will be identically equal to zero and there are solutions of the species (9) and (14) at the same time.

If rank $p > 0$ and antirank $m \leq 0$, i.e. when the singularity $(\infty, \infty, ..., \infty)$ is irregular, and $(0, 0, ..., 0)$ - regular, there is a solution of the species (10). For such solutions, K.Y. Latysheva introduced the term normal-regular [6]. That is why we make sure that the transformation of the species

$$F(x_1, x_2, \cdots, x_n) = expQ(x_1, x_2, \cdots, x_n) \cdot U(x_1, x_2, \cdots, x_n), \tag{16}$$

where

$$Q(x_1, x_2, \cdots, x_n) = \frac{\alpha_{p0\cdots0}}{p} x_1^p + \frac{\alpha_{0p\cdots0}}{p} x_2^p + \cdots + \alpha_{0\cdots01} x_n, \tag{17}$$

multi-degree polynomial p with undefined coefficients $\alpha_{p0\cdots0}, \alpha_{0p\cdots0}, \cdots, \alpha_{0\cdots01}$, $U(x_1, x_2, \cdots, x_n)$ a row of species (9), is used in the construction of normal and regular solutions. Normal-regular solutions are interesting to us because the solutions of all related degenerate systems like Horn, Whittaker, Bessel and Laguerre belong to this species.

Their common properties include:

1. For all the above mentioned related degraded systems, a special curve $(0, 0, ..., 0)$ is regular and an irregular curve $(\infty, \infty, ..., \infty)$ is irregular.

2. The system obtained from the initial system (8) by means of the transformation (16) is called auxiliary. Out of here, unknown polynomial coefficients $\alpha_{p0\cdots0}, \alpha_{0p\cdots0}, \cdots, \alpha_{0\cdots01}$, $Q(x_1, x_2, \cdots, x_n)$ (17) are determined.

3. The degree of polynomial $Q(x_1, x_2, \cdots, x_n)$ is determined by the notion of rank p.

2.3. Application of Frobenius-Latysheva method to the construction of normal and regular solutions.

In this paper, the main method for constructing solutions of the system under study is the Frobenius-Latysheva method [13], which allows to make a classification of regular and irregular special curves, to build solutions near these singularities, as was stated in the previous paragraph 2.1. Obtaining related degraded systems of the Horn, Whittaker, Bessel, and Laguerre types by means of transformation (16).

Application of the Frobenius-Latysheva method implies first determination of special curves, establishment of compatibility and integrability conditions as well as drawing up a system of Frobenius characteristic functions. In general, establishing the conditions of compatibility is difficult. Usually, when studying specific systems of the species (8) the conditions of compatibility are provided by the method of construction of the proposed Campe de Ferrier [1]. The integration condition is always fulfilled for it.

Definition 5. *A system of Frobenius characterization functions is a system obtained by substitution in (8) instead of a function* $F = x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n}$.

Hence, for the construction of a normal-regular solution (8) a system of constitutive equations with respect to the singularity of $(0, 0, ..., 0)$ is defined.

Definition 6. *A system of constitutive equations in relation to a singularity* $(0, 0, ..., 0)$ *is called a system of*

$$f^{(j)}_{0,...,0}(\rho_1, ..., \rho_n) = r^{(j)}_{20}\rho_j(\rho_j - 1) + r^{(j)}_{10}\rho_j + r^{(j)}_{01}\sum_{(k\neq j)}\rho_k + r^{(j)}_{0,0} = 0, \tag{18}$$

of which the indicators of series (9) and (10) are defined as $(\rho^{(j)}_1, \rho^{(j)}_2, \cdots, \rho^{(j)}_n)$, $(j = \overline{1, n})$.

The numbers $(\rho^{(j)}_1, \rho^{(j)}_2, \cdots, \rho^{(j)}_n)$, $(j = \overline{1, n})$ allows us to determine the number of linear-independent private solutions of the system (8) near the singularities described in paragraph 2.1 above.

If in the system (8) there are constant $r^{(j)}_{01} = 0$ and $r^{(j)}_{0,0} = 0$ then we obtain a hypergeometric system of the following type

$$x_j[r^{(j)}_{20} - \alpha^{(j)}_{20}x_j]F_{x_jx_j} + x_j[r^{(j)}_{10} - \alpha^{(j)}_{10}x_j]F_{x_j} - \alpha^{(j)}_{01}\sum_{k\neq j}x_kF_{x_k} - \alpha^{(j)}_{0,0}x_jF = 0. \tag{19}$$

Then the system of constitutive equations (18) with respect to the singularity $(0, 0, ..., 0)$ is recorded as

$$f^{(j)}_{0,...,0}(\rho_1, ..., \rho_n) = r^{(j)}_{20}\rho_j(\rho_j - 1) + r^{(j)}_{10}\rho_j + r^{(j)}_{01} = 0, \ (j = \overline{1, n}), \tag{20}$$

The Horn type system belongs to the hypergeometric type system.

Let us proceed to the construction of a normal and regular solution of the hypergeometric type system. As in the case of functions of two variables [12], we will consider the right part of the normal-regular solution (16) as a product of two co-multipliers:

A) $expQ(x_1, x_2, ..., x_n)$-determining multiplier with undefined polynomial coefficients (11).

B)

$$x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n} \times \sum_{m_1, m_2, \cdots, m_n = 0}^{\infty} A_{m_1, m_2, \cdots, m_n} x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}, \; A_{0,0,\cdots,0} \neq 0- \tag{21}$$

a generalized stepwise series of variables representing a solution near the particular where $\rho_j (j = \overline{1,n})$, $A_{m_1,\cdots,m_n}$ are the unknown constants.

In the case of A), the statement is true.

Theorem 2. *In order for an auxiliary system to have at least one solution of the kind (2.9), the equality must be fulfilled*

$$b_{n00\ldots0}^{(j)} = 0, b_{0n0\ldots0}^{(j)} = 0, \cdots, b_{00\ldots0}^{(j)} = 0, (j = \overline{1,n}) \tag{22}$$

obtained from the auxiliary system by equating the coefficients to zero at the highest degrees of independent variables $x_1, x_2, \cdots, x_n$ *from which unknown polynomial coefficients* $\alpha_{n0\cdots0}, \alpha_{0n\cdots0}, \cdots, \alpha_{0\cdots0n}$, $Q(x_1, x_2, \cdots, x_n)$ *are determined (11).*

This is the first necessary condition for the existence of a normal-regular solution.

If the system (22) has multiple roots, there are so-called subnormal solutions, i.e. solutions for fractional degrees of independent variables.

Theorem 3. *In order for a system of Horn type with an irregular feature (19) to have a normal-regular solution of the species (16), it is necessary* $(\rho_1^t, \rho_2^t, \cdots, \rho_n^t,)$ $t = \overline{1,n}$ *to be the root of the system of defining equations for the species feature* $(0, 0, \cdots, 0)$ *(20) obtained from the auxiliary system by substitution instead of the unknown* $F(x_1, x_2, \cdots, x_n) = x_1^{\rho_1} \cdot x_2^{\rho_2} \cdots x_n^{\rho_n}$.

As it can be seen from this, the second necessary condition is related to the definition of unknown constants $\rho_j (j = \overline{1,n})$, $A_{m_1,\cdots,m_n}$ of the generalized steppe series of n variables (10), i.e., the series (21). This series represents the solution of the auxiliary system in the vicinity of $(0, 0, \cdots, 0)$. The coefficient of this series $A_{m_1,\cdots,m_n})$ is determined from the sequences of the recurrent systems.

The proofs of two theorems 2.1 and 2.2 are similar to the proofs of such theorems as in the case of functions of two variables [11]. The specific application of the two theorems 2.1 and 2.2 will be shown in paragraph 3. The specific application of the two theorems (2) and (3 will be shown in paragraph 3.

3. Peculiarities of building the solution of Horn type system consisting of equations.

Setting the task. Peculiarities of construction of the solution of the Horn type system consisting of the equations of the kind

$$x_j \frac{\partial^2 F}{\partial x_j^2} + [\gamma_j - x_j] \frac{\partial F}{\partial x_j} - \sum_{(k \neq j} x_k \frac{\partial F}{\partial x_k} - \lambda F = 0, (j = \overline{1,n}), \tag{23}$$

where $F = F(x_1, x_2, \cdots, x_n)$ is a common unknown for all system equations (23).

It is necessary to establish related systems with the Horn type system (23) and disclose properties of solutions of such systems. Further, to show the features of Frobenius-Latysheva method application to building solutions of various related systems.

We will highlight the main properties of Humbert's functions $\Psi_2^{(n)}$, using their relationship between the systems of the Horn type and Whittaker and $\Psi_2^{(n)}$ solutions [2].

Definition 7. *The generated hypergeometric function of M.P.Humbert* $\Psi_2^{(n)}(n = 2, 3, ...)$ *n variables* $x_j(j = \overline{1,n})$ *is determined using the series*

$$\Psi_2^{(n)}(\lambda; \gamma_1, \gamma_2, \cdots, \gamma_n; x_1, x_2, \cdots, x_n)$$
$$= \sum_{m_1, m_2, \cdots, m_n = 0}^{\infty} \frac{(\lambda)_{m_1+m_2+\cdots+m_n}}{(\gamma_1)_{m_1} \cdot (\gamma_2)_{m_2} \cdot \cdots \cdot (\gamma_n)_{m_n}} \cdot \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \cdots \frac{x_n^{m_n}}{m_n!}. \tag{24}$$

The row converges absolutely and evenly at $|x_1| < \epsilon, |x_2| < \epsilon, \cdots, |x_n| < \epsilon$.

The property about private solutions in the vicinity of $(0, 0, \cdots, 0)$.

Theorem 4. *The Humbert function (24) is a private solution of the Horn type system (23) near the peculiarity* $(0, 0, \cdots, 0)$.

When $n = 2$ a similar theorem 1 has been proved, where it is established that Humbert's function is a private solution of the Horn system (5) near a regular singularity $(0, 0)$. It is also established that for the system (5) the singularity $(0, 0)$ is a regular feature and $(\infty, \infty, \cdots \infty)$ - an irregular singularity. Theorem 2 is a generalization of theorem 1, where for the Horn system (23) the singularity $(0, 0, \cdots, 0)$ is regular and $(\infty, \infty, \cdots \infty)$ irregular.

The property in relation to the number of solutions of the system (23) is established by the following theorem.

Theorem 5. *The system (23) in singularity* $(0, 0, \cdots, 0)$ *has regular solutions in the form of generalized steppe series of two variables*

$$F(x_1, \cdots, x_n) = x_1^{\rho_1} \cdots x_n^{\rho_n} \sum_{m_1, \cdots, m_n = 0}^{\infty} A_{m_1, m_2, \cdots, m_n} x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}, \tag{25}$$

where $\rho_j(j = \overline{1,n})$, $A_{m_1,m_2,\cdots,m_n}(m_n = 0, 1, 2, \cdots)$ *unknown coefficients, which are expressed through the Humbert function* $\Psi_2^{(n)}$.

To determine the number of solutions, it is first necessary to determine the number of roots of the system of defining equations with respect to the singularity $(0, 0, \cdots, 0)$:

$$f^{(j)}_{0,\cdots,0}(\rho_1, \cdots, \rho_n) \equiv \rho_j(\rho_j - 1) + \gamma_j\rho_j = 0, \ (j = \overline{1, n}) \tag{26}$$

If the roots are simple, then it is possible to determine the roots 2^n, that is, the indexes of the series as $(\rho^t_1, \rho^t_2, \cdots, \rho^t_n)(t = \overline{1, n})$. These indicators correspond to 2^n regular solutions in the form of generalized step series n of variables (25). Taking into account the indicators of the series, they are presented in the form of [4]:

$$\begin{aligned}
&1\{\Psi_2((\lambda; \gamma_1, \gamma_2, \cdots, \gamma_n; x_1, x_2, \cdots, x_n),\\
&n\{x_1^{1-\gamma_1}\Psi_2(\lambda + 1 - \gamma_1, 2 - \gamma_1, \gamma_2, \cdots, \gamma_n; x_1, \cdots, x_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&1\{x_1^{1-\gamma_1} \cdots x_n^{1-\gamma_n}\Psi_2(\lambda + n - \gamma_1 - \cdots - \gamma_n; 2 - \gamma_1, \cdots, 2 - \gamma_n; x_1, \cdots, x_n).
\end{aligned} \tag{27}$$

The solution corresponding to the indicator is a particular solution (3.2) of a Horn-type system (3.1) near the singularity .

In a well-studied private case $n = 2$, the corresponding Horn system (5) consisting of two differential equations in second order private derivatives has $2^2(n = 2)$ linear-independent private solutions, according to the general theory of such systems, when the system is joint and the condition of integrability is fulfilled [15]. In this case, the system of defining equations (26) is written in the following form

$$f^{(j)}_{0,0}(\rho_1, \rho_2) \equiv \rho_j(\rho_j - 1) + \gamma_j\rho_j = 0, \ (j = 1, 2) \tag{28}$$

and has four pairs of indicators (ρ^t_1, ρ^t_2) $(t = \overline{1, 4})$, so the statement is fair.

Theorem 6. *The Horn system (5) has four linear-independent private solutions close to the singularity* $(0, 0)$ *and the total solution is presented as the following sum.*

$$\begin{aligned}
&Z(x_1, x_2) = A \cdot Z(x_1, x_2) + B \cdot Z(x_1, x_2) + C \cdot Z(x_1, x_2) + D \cdot Z(x_1, x_2) =\\
&= A \cdot \Psi_2(\lambda; \gamma, \gamma'; x_1, x_2) + B \cdot x_1^{1-\gamma}\Psi_2(\lambda + 1 - \gamma, 2 - \gamma, \gamma'; x_1, x_2)\\
&+C \cdot x_2^{1-\gamma'}\Psi_2(\lambda + 1 - \gamma', \gamma, 2 - \gamma'; x_1, x_2)\\
&+D \cdot x_1^{1-\gamma}x_2^{1-\gamma'}\Psi_2(\lambda + 2 - \gamma - \gamma', 2 - \gamma, 2 - \gamma'; x_1, x_2)
\end{aligned}$$

where all particular decisions $Z_t(x_1, x_2)(t = \overline{1, 4})$ are expressed through Humbert functions Ψ_2.

A similar theorem can be formulated for a general case.

Theorem 7. *The Horn type system (23) consisting of* n *differential equations in second-order private derivatives has* 2^n *linearly independent private solutions close to the feature* $(0, 0, \cdots, 0)$ *and its general solution is presented as a sum of* $F(x_1, x_2, \cdots, x_n) = \sum_{l=0}^{2^n} C_l \cdot F_l(x_1, x_2, \cdots, x_n)$, *where* $F(x_1, x_2, \cdots, x_n)$ *private system solutions (8);* C_i $(i = 1, 2, \cdots, 2^n)$ *- arbitrary constants.*

Based on theorem 5, the auxiliary system has 2^n linearly independent private solutions (27), because the system of defining equations in relation to the singularity $(0, 0, \cdots, 0)$ (26) has 2^n simple roots. This shows that the second necessary condition has been fulfilled. The existence of normal and regular solutions is ensured by the following statement.

Theorem 8. *If systems of characteristic equations (22) have only simple pairs of roots, a system of the Horn type with an irregular singularity* $(\infty, \infty, \cdots, \infty)$ *of positive rank* $p > 0$ *and antirank* $m \leq 0$ *allows* 2^n *for normal-regular solutions of the species (16).*

However, here the reasoning is for the most general case, where rank $p = 1 + k$ is any number. In fact, the rank of the Horn type system is $p = 1 > 0$, therefore, the polynomial (11) will be the polynomial of the first degree.

$$Q(x_1, x_2, \cdots, x_n) = \alpha_{10\cdots0}x_1 + \alpha_{01\cdots0}x_2 + \cdots + \alpha_{0\cdots01}x, \tag{29}$$

with undetermined coefficients $\alpha_{10\cdots0}, \alpha_{01\cdots0}, \cdots, \alpha_{0\cdots01} \cdots$ They are determined from a system of characteristic equations (22). These reasonings lead us to the basic property of the transformation (16).

All major related systems of the Gorn and Whittaker type are derived from them by means of special cases of transformation (16).

In the following we will show the derivation of the related systems of Whittaker, Bessel and Campers by means of a species transform (16).

3.1. Normal-regular solutions for Whittaker-type systems

The Kummer equation (1) is a special case of the Horn system (5). The Whittaker functions $M_{k,m}(x)$ and $W_{k,m}(x)$ being solutions of the Whittaker equation (2, obtained from Kummer equations (1, by means of transformation have found wide application in problems of science and technology. This stimulated the study of Whittaker function of many variables, mainly due to the works of M.P.Humbert [4, 5].

Theorem 9. *The system*

$$x_j^2 \frac{\partial^2 U}{\partial x_j^2} - x_j \sum_{r \neq j} x_r \frac{\partial U}{\partial x_r} + \left[-\frac{x_j^2}{4} - \frac{x_j}{2} \sum_{r \neq j} x_r + kx_j + \frac{1}{4} - \mu_j^2 \right] U = 0, \tag{30}$$

obtained from a Horn type system (23) by means of conversion

$$F(x_1, x_2, \cdots, x_n) = exp\left(\frac{x_1}{2} + \cdots + \frac{x_n}{2}\right) \cdot x_1^{-\frac{\gamma_1}{2}} \cdots x_n^{-\frac{\gamma_n}{2}} U(x_1, x_2, \cdots, x_n), \tag{31}$$

is a Whittaker-type system [1, p.135] and has a normally regular solution of the form

$$W_{\lambda,\mu_1,\cdots,\mu_n}(x_1, x_2, \cdots, x_n) = \sum \frac{\Gamma(-2\omega_1\mu_1) \cdots \Gamma(-2\omega_n\mu_n)}{\Gamma\left(1 - \frac{n}{2} - \omega_1\mu_1 - \cdots - \omega_n\mu_n - k\right)}$$
$$\times M_{k,\omega_1,\mu_1,\cdots,\omega_n,\mu_n}, \tag{32}$$

where presentation proved [1, p.135] by M.P. Humbert is fair

$$M_{k,\mu_1,\cdots,\mu_n}(x_1,\cdots,x_n)=x_1^{\mu_1+\frac{1}{2}}\cdots x_n^{\mu_n+\frac{1}{2}}\cdot exp\big(-\frac{x_1+\cdots+x_n}{2}\big)$$
$$\times\Psi_2\big(\mu_1+\cdots+\mu_n-k+\frac{n}{2},2\mu_1+1,\cdots,2\mu_n+1,x_1,\cdots,x_n\big). \tag{33}$$

The peculiarity of transformation (31) is an unknown function $U(x_1,\cdots,x_n)$ from x_j $(j=\overline{1,n})$ n variables. For a newly obtained system $U(x_1,\cdots,x_n)$ it is a common unknown function. In its turn, in order to build normal and regular solutions of the system (30), the following transformation should be performed

$$U(x_1,\cdots,x_n)=exp(\alpha_{10\cdots0}x_1+\cdots+\alpha_{0\cdots01}x_n)\cdot x_1^{\rho_1}\cdot x_2^{\rho_2}\cdot\cdots\cdot x_n^{\rho_n}$$
$$\times\Phi(x_1,\cdots,x_n), \tag{34}$$

with unknown constants $\alpha_{10\cdots0},\cdots,\alpha_{0\cdots01}$, ρ_j $(j=\overline{1,n})$ and an unknown function $\Phi(x_1,\cdots,x_n)$. This is because the statement is true.

Theorem 10. *The system (30) obtained by conversion (31) from the system (23) has the same rank as the original Horn system (23).*

The fairness of this statement can be verified by direct verification. Since the rank $p=1$, the first-degree polynomial in the determinant multiplier
$exp(\alpha_{10\cdots0}x_1+\cdots+\alpha_{0\cdots01}x_n)$ remains a polynomial $Q(x_1,\cdots,x_n)$ with unknown coefficients$\alpha_{10\cdots0},\cdots,\alpha_{0\cdots01}$.

As an example, let us give a special case of the system (30) obtained from the system (5) [17].

After the conversion

$$U(x_1,x_2)=exp\big(\alpha_{10}x_1+\alpha_{01}x_2\big)\cdot\Phi(x_1,x_2) \tag{35}$$

it leads to an auxiliary system with a relatively unknown function $\Phi(x_1,x_2)$, which has four normally regular solutions

$$U_1(x_1,x_2)=exp\big(-\frac{x_1}{2}-\frac{x_2}{2}\big)\cdot\Phi(x_1,x_2)=M_{k,\mu,\nu}(x_1,x_2),$$
$$U_2(x_1,x_2)=exp\big(-\frac{x_1}{2}-\frac{x_2}{2}\big)\cdot\Phi(x_1,x_2)=M_{k,\mu,-\nu}(x_1,x_2),$$
$$U_3(x_1,x_2)=exp\big(-\frac{x_1}{2}-\frac{x_2}{2}\big)\cdot\Phi(x_1,x_2)=M_{k,-\mu,\nu}(x_1,x_2),$$
$$U_4(x_1,x_2)=exp\big(-\frac{x_1}{2}-\frac{x_2}{2}\big)\cdot\Phi(x_1,x_2)=M_{k,-\mu,-\nu}(x_1,x_2) \tag{36}$$

where unknown functions $\Phi_i(x_1,x_2)$, $(i=\overline{1,4})$ are expressed through the Humbert function Ψ_2 :

$$\Phi_1(x_1,x_2)=x_1^{\frac{1}{2}+\mu}x_2^{\frac{1}{2}+\nu}\Psi_2(\mu+\nu+1-k,2\mu+1,2\nu+1;x_1,x_2),$$
$$\Phi_2(x_1,x_2)=x_1^{\frac{1}{2}+\mu}x_2^{\frac{1}{2}-\nu}\Psi_2(\mu-\nu+1-k,2\mu+1,2\nu-1;x_1,x_2),$$

$$\Phi_3(x_1,x_2)=x_1^{\frac{1}{2}-\mu}x_2^{\frac{1}{2}+\nu}\Psi_2(-\mu+\nu+1-k,2\mu-1,2\nu+1;x_1,x_2),$$
$$\Phi_4(x_1,x_2)=x_1^{\frac{1}{2}-\mu}x_2^{\frac{1}{2}-\nu}\Psi_2(\mu-\nu-1-k,2\mu-1,2\nu-1;x_1,x_2). \quad (37)$$

Here, at first two necessary conditions are checked (Theorems 2 and 3). Indeed, if $n=2$ then the transform (34) is presented in the form of (20) and the system of characteristic equations (22) is presented:

$$f_{10}^{(1)}(\alpha_{10},\alpha_{01})=\alpha_{10}^2-\frac{1}{4}=0, f_{01}^{(2)}(\alpha_{10},\alpha_{01})=\alpha_{01}^2-\frac{1}{4}=0, \quad (38)$$

and also has four pairs of roots:

$$\big(\alpha_{10}^{(1)},\alpha_{01}^{(1)}\big)=\big(\frac{1}{2},\frac{1}{2}\big),\ \big(\alpha_{10}^{(1)},\alpha_{01}^{(2)}\big)=\big(\frac{1}{2},-\frac{1}{2}\big)$$
$$\big(\alpha_{10}^{(2)},\alpha_{01}^{(1)}\big)=\big(-\frac{1}{2},\frac{1}{2}\big),\ \big(\alpha_{10}^{(2)},\alpha_{01}^{(2)}\big)=\big(-\frac{1}{2},-\frac{1}{2}\big) \quad (39)$$

defining the four polynomials of the first degree (11): $Q_i(x_1,x_2)=\alpha_{10}^{(i)}x_1+\alpha_{01}^{(i)}x_2, (i=\overline{1,4})$.

Similarly, we make sure that the second necessary condition is met (Theorem 3).

A system of constitutive equations for species singularities $(0,0)$ (18):

$$f_{00}^{(1)}(\rho_1,\rho_2)=\rho_j^{(t)}(\rho_j^{(t)}-1)+\frac{1}{4}-\mu_j^2=0,\ (j=1,2),(t=1,2) \quad (40)$$

has four pairs of roots:

$$(\rho_1^{(1)},\rho_2^{(1)})=\big(\frac{1}{2}+\mu_1,\frac{1}{2}+\mu_2\big),\ (\rho_1^{(1)},\rho_2^{(2)})=\big(\frac{1}{2}+\mu_1,-\frac{1}{2}+\mu_2\big),$$
$$(\rho_1^{(2)},\rho_2^{(1)})=\big(-\frac{1}{2}+\mu_1,\frac{1}{2}+\mu_2\big),\ (\rho_1^{(2)},\rho_2^{(2)})=\big(-\frac{1}{2}+\mu_1,-\frac{1}{2}+\mu_2\big), \quad (41)$$

Implementation of the second necessary condition (40) ensures the existence of four linear-independent private solutions (37). The fulfillment of two necessary conditions ensures the existence of four normal-regular solutions if $n=2, 2^n=2^2$ (36).

On the basis of theorem 8, in general, there are 2^n normal and regular solutions of the species (32). On the previous example, we made sure that this requires two necessary conditions (Theorems 2 and 3). In fact, in this case, the system of characteristic equations obtained by transformation (31) looks like

$$f_{10\cdots 0}^{(1)}(\alpha_{10\cdots 0},\cdots,\alpha_{0\cdots 01})=\alpha_{10\cdots 0}^2-\frac{1}{4}=0,\cdots$$

$$f_{0\cdots 01}^{(1)}(\alpha_{10\cdots 0},\cdots,\alpha_{0\cdots 01})=\alpha_{0\cdots 01}^2-\frac{1}{4}=0,$$

It has 2^n species roots (39) that define the 2^n polynomial of the first degree of the species (29):

$$Q(x_1,x_2,\cdots,x_n)=\alpha_{10\cdots 0}^{(i)}x_1+\cdots+\alpha_{0\cdots 01}^{(i)}x_n,\ i=2^2,\cdots,2^n.$$

In the same way, we make sure the second necessary condition is fulfilled. Indeed, a system of constitutive equations concerning a singularity is presented in the form of $(0, 0, \cdots, 0)$

$$f^{(1)}_{0,\cdots,0}(\rho_1, \cdots, \rho_n) = \rho_j(\rho_j - 1) + \frac{1}{4} - \mu_j^2 = 0, \ (j = \overline{1, n})$$

The fulfillment of the second necessary condition shows that the Whittaker system (30) also has regular linear-independent private solutions of the species (25) as the related Horn system (23). Further, the simultaneous fulfillment of two conditions shows the existence of normal-regular solutions of the species (32).

The technique applied by us is used, further, at constructions of the solutions of Bessel and Laguerre type systems related to Whittaker's system.

3.2. Systems with solutions in the form of degenerate hypergeometric functions of two variables reduced to Bessel functions

The degenerated hypergeometric function of a single variable, which is reduced to Bessel functions, can be obtained by limiting the transition from Gauss and Kummer functions. Indeed, the limit transition is fair

$$\lim_{\epsilon\to 0} F\big(\frac{1}{\epsilon}, \frac{1}{\epsilon}; \gamma; \epsilon^2 x\big) = \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} \cdot \frac{x^m}{m!} = J(\gamma; x), \tag{42}$$

where the function $J(\gamma; x)$ is called a function reducible to the Bessel function, because the equality is fair.

$$J_k(x) = \frac{\big(\frac{x}{2}\big)^k}{\Gamma(k+1)} \cdot J\big(k+1, -\frac{x^2}{2}\big), \tag{43}$$

and

$$x^2 \frac{d^2 J_k}{dx^2} + x \frac{dJ_k}{dx} + (x^2 - k^2) J_k = 0, \tag{44}$$

Equation (44) is a basic Bessel equation, and $J_k(x)$ a certain equation (43) is called the first order Bessel or cylindrical function or Bessel order k function from the argument x [8, p.15].

Definition 8. *The degenerated hypergeometric Kummer function $G(\alpha, \gamma; x)$ of a single variable using the limit transition is given as follows*

$$\begin{aligned} F_1(;\gamma; x) &= 1 + \frac{1}{\gamma} x + \frac{1}{2!\gamma(\gamma+1)} x^2 + \cdots \\ &= \lim_{\alpha\to\infty} \big[1 + \frac{\alpha}{1!\gamma}\frac{x}{\alpha} + \frac{\alpha(\alpha+1)}{2!\gamma(\gamma+1)}\frac{x^2}{\alpha^2} + \cdots\big] = J(\gamma; x). \end{aligned} \tag{45}$$

Here is a generalization of this definition in case of functions of two variables.

Definition 9. *The degenerated hypergeometric function of the Humbert* $\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)$ *of two variables* $x_j (j=1,2)$ *by means of a limit transition is given as follows*

$$\lim_{\lambda\to\infty}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=J(\gamma_1;x_1)J(\gamma_2;x_2)=j(\gamma_1,\gamma_2;x_1,x_2), \tag{46}$$

where $J(\gamma_j;x_j)(j=1,2)$*-functions reduced to Bessel functions (42) or (45).*

For the following system we have introduced, the statement is fair.

Theorem 11. *Bessel type system*

$$\begin{aligned} &x_1^2\cdot U_{x_1x_1}-x_1x_2\cdot U_{x_2}+\{-\frac{1}{4}x_1^2-\frac{1}{2}x_1x_2+kx_1+\alpha(1-\alpha)\}U=0,\\ &x_2^2\cdot U_{x_2x_2}-x_1x_2\cdot U_{x_1}+\{-\frac{1}{4}x_2^2-\frac{1}{2}x_1x_2+kx_2+\beta(1-\beta)\}U=0, \end{aligned} \tag{47}$$

where $k=\alpha+\beta-\lambda$ *and* α,β,λ *- some parameters, and* $U=U(x_1,x_2)$ *- the total unknown, is obtained from the Horn system (4) by means of conversion*

$$Z=exp(\frac{x_1}{2}+\frac{x_2}{2})\cdot x_1^{-\frac{\gamma_1}{2}}\cdot x_2^{-\frac{\gamma_2}{2}}U(x_1,x_2), \tag{48}$$

and, under the conditions of compatibility and integrability, has four linear-independent private normal and regular solutions

$$\begin{aligned} &U_1(x_1,x_2)=exp\Big(-\frac{x_1}{2}-\frac{x_2}{2}\Big)x_1^{\alpha}x_2^{\beta}\Psi_2(\lambda,2\alpha,2\beta;x_1,x_2),\\ &U_2(x_1,x_2)=exp\Big(-\frac{x_1}{2}-\frac{x_2}{2}\Big)x_1^{\alpha}x_2^{1-\beta}\Psi_2(\lambda-2\beta+1,2\alpha,2\beta-2;x_1,x_2),\\ &U_3(x_1,x_2)=exp\Big(-\frac{x_1}{2}-\frac{x_2}{2}\Big)x_1^{1-\alpha}x_2^{\beta}\Psi_2(\lambda-2\alpha+1,2\alpha-2,2\beta;x_1,x_2),\\ &U_4(x_1,x_2)=exp\Big(-\frac{x_1}{2}-\frac{x_2}{2}\Big)x_1^{1-\alpha}x_2^{1-\beta}\Psi_2(\lambda-2\alpha-2\beta+1,2\alpha-2,2\beta-2;x_1,x_2). \end{aligned}$$

which are expressed through the degenerate hypergeometric Humbert function Ψ_2*, which is reduced at the values of the parameters* $\gamma_j=2\alpha_j$ $(j=1,2)$ *to the Bessel function of the two variables by a limit transition*

$$\lim_{\lambda\to 0}\Psi_2(\lambda,2\alpha,2\beta;\lambda x_1,\lambda x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(2\alpha_1)_{m_1}(2\alpha_2)_{m_2}}\frac{x_1^m}{m_1!}\frac{x_2^n}{m_2!}.$$

It should be noted that (47) is related to the Whittaker system (5), therefore, to build its solution you can use the method given in the previous paragraph 3.1. The transformation (48) is a special case of the general transformation (16). Here, it shows the application of the view transform (16) to output systems related to the Whittaker system. This system can be generalized in case of: systems of n equations, as well as theorem 11.

Theorem 12.

$$x_j^2 U_{x_j x_j} - x_j \sum_{(r \neq j)} x_r U_r + \big[- \frac{x_j^2}{4} - \frac{x_j}{2} \sum_{(r \neq j)} x_r + k x_j + \alpha_j (1 - \alpha_j) \big] U = 0, (j = \overline{1, n}), \quad (49)$$

where $k = \alpha + \beta - \lambda$ and $\alpha_i (i = 1, 2, \cdots, n), \lambda$- some parameters, and $U(x_1, \cdots, x_n)$ - total unknown, from a system of Horn type (23) is obtained by means of conversion

$$F(x_1, \cdots, x_n) = exp\big(\frac{x_1}{2} + \cdots + \frac{x_n}{2}\big) \cdot x_1^{-\frac{\gamma_1}{2}} \cdots x_n^{-\frac{\gamma_n}{2}} U(x_1, \cdots, x_n),$$

and under the conditions of compatibility and integrability [17] has 2^n linear-independent private normal-regular solutions of the species (31), which are expressed through the degenerate hypergeometric Humbert function $\Psi_2^{(n)}$, which is reduced at the values of the parameters $\gamma_j = 2\alpha_j (j = 1, 2, \cdots, n)$ to the Bessel function of two variables by a limit transition

$$\begin{aligned} &\lim_{\lambda \to 0} \Psi_2^{(n)} (\lambda, 2\alpha_1, 2\alpha_2, \cdots, 2\alpha_n; \lambda x_1, \lambda x_2, \cdots, \lambda x_n) \\ &= \sum_{m_1, m_2, \cdots, m_n = 0}^{\infty} \frac{1}{(2\alpha_1)_{m_1} (2\alpha_2)_{m_2} \cdots (2\alpha_n)_{m_n}} \frac{x_1^m}{m_1!} \frac{x_2^n}{m_2!} \cdots \frac{x_n^m}{m_n!}. \end{aligned}$$

3.3. About the output of related systems such as Laguerrs one.

In some works [14],[16] the possibilities of constructing the solution of the Laguerre type systems and their connection with the systems of Horn and Whittaker types were studied. It has been established that out of all 20 systems the solutions of which are degenerated hypergeometric functions of two variables, the closest to the Laguerre type system is the Horn type system (4).

Theorem 13. *Laguerre type system*

$$\begin{aligned} &x_1 Z_{x_1 x_1} + (1 + \alpha_1 - x_1) Z_{x_1} - x_2 Z_{x_2} - \lambda Z = 0, \\ &x_2 Z_{x_2 x_2} + (1 + \alpha_2 - x_2) Z_{x_2} - x_1 Z_{x_1} - \lambda Z = 0, \end{aligned} \quad (50)$$

out of the Horn system it is obtained by means of replacement, and one of the particular solutions is a polynomial

$$\begin{aligned} &L_{n,n}^{(\alpha_1, \alpha_2)} (x_1, x_2) = |Psi_2[-n, \alpha_1 + 1, \alpha_2 + 1; x_1, x_2] \\ &\sum_{m_1, m_2 = 0}^{\infty} \frac{(-n)_{m_1 + m_2}}{(\alpha_1 + 1)_{m_1} (\alpha_2 + 1)_{m_2}} \cdot \frac{x_1^{m_1}}{m_1!} \cdot \frac{x_2^{m_2}}{m_2!}. \end{aligned} \quad (51)$$

The system (50) is called the main Laguerre system, and (51) is a generalized polynomial of two variables. The connection between the Humbert and the Laguerre polynomial functions of two variables (51) has been studied in the work [14].

To obtain another related system of the Laguerre type, we again apply the species transformation (16) [16].

Theorem 14. *The system related to the Laguerre system*

$$x_1^2 \cdot U_{x_1x_1} - x_1x_2 \cdot U_{x_2} + \{-\frac{1}{4}x_1^2 - \frac{x_1x_2}{2} + kx_1 + \frac{1-\alpha^2}{4}\}U = 0,$$
$$x_2^2 \cdot U_{x_2x_2} - x_1x_2 \cdot U_{x_1} + \{-\frac{1}{4}x_2^2 - \frac{x_1x_2}{2} + kx_2 + \frac{1-\beta^2}{4}\}U = 0, \tag{52}$$

where $k = (\alpha+\beta+2-2\lambda)/2$, *obtained from the Horn system (1.4) by means of a conversion*

$$Z = exp(\frac{x_1}{2} + \frac{x_2}{2}) \cdot x_1^{-\frac{\alpha+1}{2}} x_2^{-\frac{\beta+1}{2}} U(x,y) \tag{53}$$

and has four linear-independent private solutions in the form of normal-regular series.

$$U(x,y) = exp\big(-\frac{x_1}{2} - \frac{x_2}{2}\big) \cdot x_1^{-\frac{\alpha+1}{2}} x_2^{-\frac{\beta+1}{2}} \Psi_2[-n, \alpha+1, \beta+1; x_1, x_2]$$
$$= exp\big(-\frac{x_1}{2} - \frac{x_2}{2}\big) \cdot L_{n,n}^{(\alpha,\beta)}(x_1, x_2)$$

dependent on the Laguerre polynomials of two variables (51).

These results can be generalized for a system (3.1) consisting of n equations

Theorem 15. *Provided system of Horn type (23) consisting of n equations of parameters* $\gamma_1 = \alpha_1 + 1,\ \gamma_2 = \alpha_2 + 1, \cdots, \gamma_n = \alpha_n + 1$ $(\alpha_1 > -1, \alpha_2 > -1, \cdots, \alpha_n > -1, \alpha_1 \neq 0, \alpha_2 \neq 0, \cdots, \alpha_n \neq 0)$ *then the system of differential equations in second order particular derivatives (23) has* 2^n *linear-independent particular solutions.*

$$F_1(x_1, \cdots, x_n) = \Psi_2^{(n)}(\lambda; 1+\alpha_1, \cdots, 1+\alpha_n; x_1, \cdots, x_n,), \tag{54}$$
$$n\{F_2(x_1, \cdots, x_n) = x^{-\alpha_1}\Psi_2^{(n)}(\lambda-\alpha_1, 1-\alpha_1, \gamma_2, \cdots, \gamma_n; x_1, \cdots, x_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$1\{F_{2n}(x_1, \cdots, x_n) = x^{-\alpha_1} \cdots x^{-\alpha_n}$$
$$\times\Psi_2^{(n)}(\lambda+n-\alpha_1-\alpha_2-\cdots-\alpha_n; 1-\alpha_1, \cdots, 1-\alpha_n; x_1, \cdots, x_n). \tag{55}$$

expressed through Humbert functions $\Psi_2^{(n)}$ *from n variables. Of them, the solution*

$$F_1(x_1, \cdots, x_n) = \Psi_2^{(n)}(-n, 1+\alpha_1, 1+\alpha_2, \cdots 1+\alpha_n; x_1, \cdots, x_n), \tag{56}$$

at $\lambda = -n$ *(n > 0 - integer) defines a generalized polynomial of Laguerre n variables of the species*

$$L_{n,n,\cdots,n}^{(\alpha_1,\alpha_2,\cdots,\alpha_n)}(x_1, x_2, \cdots, x_n)$$
$$= \sum_{m_1,m_2,\cdots,m_n=0}^{n} \frac{(-n)_{m_1+m_2+\cdots+m_n}}{(1+\alpha_1)_{m_1}(1+\alpha_2)_{m_2}\cdots(1+\alpha_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots\frac{x_n^{m_n}}{m_n!}. \tag{57}$$

If $\alpha_j = 0 (j = \overline{1,n})$, the polynomial Laguerre (54) is represented as

$$F_1(x_1, \cdots, x_n) = \Psi_2^{(n)}(\lambda, 1, 2, \cdots, n; x_1, \cdots, x_n)$$

and at $\lambda = -n$ ($n > 0$- an integer) defines a simple Laguerre polynomial of n variables of the species

$$L_{n,n,\cdots,n}^{(0,0,\cdots,o)}(x_1, x_2, \cdots, x_n) = \sum_{m_1,m_2,\cdots,m_n=0}^{n} \frac{(-n)_{m_1+m_2+\cdots+m_n}}{(1)_{m_1}(1)_{m_2}\cdots(1)_{m_n}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \cdots \frac{x_n^{m_n}}{m_n!}.$$

One can find many different interesting properties of Laguerre polynomial. For example, for a product of n Laguerre polynomials [11, p.49], equality is performed.

$$L_{m_1}^{(\alpha_1)}(z_1 x) \cdots L_{m_n}^{(\alpha_k)}(z_n x) = \sum_{k=0}^{m_1+\cdots+m_n} \gamma_k L_k^{(\alpha)}(x),$$

where the Laurichella row coefficient: $\gamma_k (k \geq 0)$

$$\gamma_k = \binom{m_1 + \alpha_1}{m_1} \ldots \binom{m_n + \alpha_n}{m_n}$$
$$\times F_A^{(n+1)}[\alpha + 1, -m_1, \cdots, -m_n, -k; \alpha_1 + 1, \cdots, \alpha_n + 1, \alpha + 1; z_1, \cdots, z_n, 1].$$

Conclusion

Thus, in this paper, the general properties of the eliminated systems consisting of n differential equations in private derivatives of the second order of hypergeometric type have been studied. Ya.Horn has established that in a particular case if $n = 2$, the solutions of such systems are 14 complete and 20 degenerate hypergeometric functions of two variables. M.P.Humbert established a connection [4, 5] between systems of Horn and Whittaker type, with solutions of $\Psi_2^{(n)}$ different orders. In a number of works by Zh.Tasmambetov, systems of the Bessel and Laguerre types were revealed and their different properties [13, 14, 17] were studied. However, connection of these systems with systems of Horn and Whittaker types is not studied enough. It is mostly limited to the case $n = 2$. Their comprehensive study in ordinary differential equations, in particular, the studied links between the equations of Kummer, Whittaker, Bessel and others, is not enough.

Conclusions. The present work is devoted to research the general properties of the degenerate related systems of type Horn, Whittaker, Bessel and Laguerre consisting of the n equations in private derivatives of the second order. All of them are partial cases of the common system (8) introduced by us, so it is important to highlight its main properties. These properties are considered in point 2, where regular and irregular special curves are established. Classification of special curves is determined with the help of simple rules 1, 2, 3. These rules also make it possible to define solutions (9) and (14) in the vicinity of regular singularities $(0,0,\cdots,0)$ and $(\infty,\infty,\cdots,\infty)$ respectively. Normal-regular (10) and normal (15) solutions exist near regular singularities. They have a common multiplier $expQ(x_1,x_2,\cdots,x_n)$ where the degree of polynomial is determined by the notion of rank $p = 1 + k$ equality (12). Normal-regular solutions are given special attention in this paragraph because the solutions of all related degenerate systems such as Horn, Whittaker, Bessel and Lagerr belong to this species. In constructing normal-regular solutions, the main point is to apply the transformation (16), the unknown coefficients of which are determined by the Frobenius-Latysheva method (point 2.2) to the system (8). The paper shows how special cases of this (3), (6), (23), (34), (35), (48), and (53) transformation are applied to the derivation of related systems such as Horn, Whittaker, Bessel and Laguerre, as well as to the study their basic properties. The above related systems belong to a hypergeometric system (19) derived from a common system (8) while $r_{0,1}^{(j)} = 0$ and $r_{0,0}^{(j)} = 0(j = \overline{1,n})$.

Paragraph 3 sets out the peculiarities of building solutions for degenerate hypergeometric systems. The main degenerate hypergeometric system is the Horn type system. The rest systems such as Whittaker, Bessel and Laguerre systems are derived from it through various replacements and transformations. An important private solution of the Horn type system (23) is the degenerate hypergeometric function of the Humbert n variables
$\Psi_2^{(n)}(\lambda,\gamma_1,\gamma_2,\cdots,\gamma_n;x_1,x_2,\cdots,x_n)$. It is a generalization of the degenerate hypergeometric Kummer function ${}_1F_1(\alpha;\beta;x)$, and the solutions of all the above related systems, both near a regular feature and an irregular feature, are expressed through various special cases of the Humbert function $\Psi_2^{(n)}$. For example, a Horn-type system has 2^n regular solutions near a particular $(0,0,\cdots,0)$ feature, while a Laguerre-type system derived from a Horn-type system (23), after being replaced $\gamma_j = 1+\alpha_j(\alpha_j > -1, \alpha_j \neq 0, j = \overline{1,n})$, has 2^n linearly independent private solutions (3.32)-(3.33) expressed through Humbert functions $\Psi_2^{(n)}$ from n variables. Of these, the solution (56), at $\lambda = -n$ ($n > 0$ - integer) defines a generalized Lagerr polyne from n variables. At the same time, all related systems near an irregular feature $(\infty,\infty,\cdots,\infty)$ have normal-regular solutions obtained by means of the species transformation (16), where the polynomial $Q(x_1,x_2,\cdots,x_n)$ has a species (29). As an example, cases $n = 2$ where the Frobenius-Latysheva method has been demonstrated are considered.

The well-known reference book by E. Kamke [7] provides an extensive list of related equations with the Bessel equations [7, p.400] , Whittaker [7, p.430], Gauss hypergeometric equation [7, p.425] and others, where it is emphasized that a large number of other equations found in various applications of mathematics can be reduced to the above equations. Unfortunately, studies of special functions of many variables and their applications in different tasks of science and technology have not received such development. In the present work, we have decided to fill this gap by proposing some related systems for consideration and building their solutions. We think that research in this direction is waiting for its continuation.

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