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Definite Integral of Power and Algebraic Functions in terms of the Lerch Function

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Abstract. Bierens de haan (1867) evaluated a definite integral involving the cotangent function and this result was also listed in Gradshteyn and Ryzhik (2007). The objective of this present note is to use this integral along with Cauchy's integral formula to derive a definite logarithmic integral in terms of the Lerch function. We will use this integral formula to produce a table of known and new results in terms of special functions and thereby expanding the list of definite integrals in both text books.

2020 Mathematics Subject Classifications: 30E20,33-01, 33-03, 33-04, 33-33B, 33E20,33E33

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1. Introduction

A thorough review of the Bierens de haan (1867) and Gradshteyn and Rhyzik's (2007) books of integral tables showcases a vast number of difficult and unknown integral formulas. We shall derive and evaluate the integral

$$\int_0^\infty \left(\frac{\left(\frac{bx}{bx+1}\right)^m \log^k \left(\frac{abx}{bx+1}\right)}{x} - \frac{b\left(\frac{bx+1}{bx}\right)^m \log^k \left(\frac{a(bx+1)}{bx}\right)}{bx+1} \right) dx \tag{1}$$

where a, k, b and m are general complex numbers. This integral was of particular interest because it showcases two integrands which by themselves are divergent. However, if the two integrands are the same at large values, the difference of the integrands gives a convergent integral. In this work we provide a formal derivation for known and new integrals and tabulate these definite integrals in terms of special functions and fundamental constants. This could be viewed as a new entry table for books with table of integral formulae such

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as [7], [8], [12] and [4]. The derivations follow the method used by us in [10], [11] and [9]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dy. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant [11] has the same value at the end points of the contour. Then we multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

Definite Integral of the Contour Integral

We use the method in [11]. The variable of integration in the contour integral is $z = m + w$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we first replace y by $\log\left(\frac{abx}{bx+1}\right)$ then multiply by $\frac{\left(\frac{bx}{bx+1}\right)^m}{x}$ for the first equation and then y by $\log\left(\frac{a(bx+1)}{bx}\right)$ and multiply by $\frac{b\left(\frac{bx+1}{bx}\right)^m}{bx+1}$ to get the second equation. Then we subtract these two equations to get

$$\begin{aligned}
&\frac{1}{k!}\int_0^\infty \left(\frac{\left(\frac{bx}{bx+1}\right)^m \log^k\left(\frac{abx}{bx+1}\right)}{x} - \frac{b\left(\frac{bx+1}{bx}\right)^m \log^k\left(\frac{a(bx+1)}{bx}\right)}{bx+1}\right)dx \\
&= \frac{1}{2\pi i}\int_0^\infty\int_C \left(\frac{a^w w^{-k-1}\left(\frac{bx}{bx+1}\right)^{m+w}}{x} - \frac{ba^w w^{-k-1}\left(\frac{bx+1}{bx}\right)^{m+w}}{bx+1}\right)dwdx \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty \left(\frac{a^w w^{-k-1}\left(\frac{bx}{bx+1}\right)^{m+w}}{x} - \frac{ba^w w^{-k-1}\left(\frac{bx+1}{bx}\right)^{m+w}}{bx+1}\right)dxdw \\
&= \frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}\cot(\pi(m+w))dw
\end{aligned} \tag{3}$$

from Eq (3.217) in [12], where $0 < Re(m + w) < 1$ and $Re(b) > 0$, where the logarithmic function is defined in equation (4.1.2) in [1].

Definition of the Lerch Function

The Lerch function has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^{n} \tag{4}$$

where $|z|< 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{5}$$

where $Re(v) > 0$, and either $|z|\leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

Infinite Sum of the Contour Integral

In this section we will again use Cauchy's integral formula (2) and taking the infinite sum to derive equivalent sum representations for the contour integrals. First we replace y by $\log(a) + 2i\pi(y + 1))$ and multiply both sides by $-2i\pi e^{2i\pi m(y+1)}$ to get

$$\begin{aligned}&-\frac{ii^k(2\pi)^{k+1}e^{2i\pi my+2i\pi m}\left(-\frac{i\log(a)}{2\pi}+y+1\right)^k}{k!}\\&=-\frac{1}{2\pi i}\int_C 2i\pi w^{-k-1}\exp(w(\log(a)+2i\pi(y+1))+2i\pi m(y+1))dw\end{aligned} \tag{6}$$

Next we take the infinite sum over $y \in [0,\infty)$ and simplify in terms of the Lerch function to get

$$\begin{aligned}&-\frac{(2i\pi)^{k+1}e^{2i\pi m}\Phi\left(e^{2im\pi},-k,1-\frac{i\log(a)}{2\pi}\right)}{k!}\\&=-\sum_{y=0}^{\infty}\frac{1}{2\pi i}\int_C\left(2i\pi w^{-k-1}\exp(w(\log(a)+2i\pi(y+1))+2i\pi m(y+1))\right)dw\\&=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\left(2i\pi w^{-k-1}\exp(w(\log(a)+2i\pi(y+1))+2i\pi m(y+1))\right)dw\\&=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}\cot(\pi(m+w))+i\pi a^w w^{-k-1}dw\end{aligned} \tag{7}$$

from Eq (1.232.1) in [12], where $Im(m + w) > 0$ for the sum to converge and we replace w by $-w + \pi/2$.

The additional Contour Integral

Using Eq (2) and replacing y by using $\log(a)$ and multiply by πi to get

$$\frac{i\pi\log^k(a)}{k!}=\frac{1}{2\pi i}\int_C i\pi a^w w^{-k-1}dw \tag{8}$$

Definite Integral in terms of the Lerch Function

Since the right-hand side of Eq (3) is equal to the sum of Eq's (7) and (8) we can equate the left-hand sides to yield the definite integral given by

$$\int_0^\infty \left(\frac{\left(\frac{bx}{bx+1}\right)^m \log^k\left(\frac{abx}{bx+1}\right)}{x} - \frac{b\left(\frac{bx+1}{bx}\right)^m \log^k\left(\frac{a(bx+1)}{bx}\right)}{bx+1} \right) dx$$
$$= -(2i\pi)^{k+1} e^{2i\pi m} \Phi\left(e^{2im\pi}, -k, 1 - \frac{i\log(a)}{2\pi}\right) - i\pi \log^k(a) \tag{9}$$

Table of Definite Integrals

In this section we use Eq (9) to derive a Table of definite integrals in terms of fundamental constants and special functions.

Derivation of entry 3.217 in [12]

Using Eq (9) replacing b by q, m by p and setting $k = 0$ simplifying we get

$$\int_0^\infty \left(\frac{\left(\frac{qx}{qx+1}\right)^p}{x} - \frac{q\left(\frac{1}{qx}+1\right)^p}{qx+1} \right) dx = \pi\cot(\pi p) \tag{10}$$

from entry (2) in Table below (64:12:7) in [6].

Derivation of new entry 3.217.1 in [12]

Using Eq (9) replacing b by q, m by p and setting $k = a = 1$ simplifying we get

$$\int_0^\infty \left(\frac{q\left(\frac{1}{qx}+1\right)^p}{qx+1} + \frac{\left(\frac{qx}{qx+1}\right)^p}{x} \right) \log\left(\frac{qx}{qx+1}\right) dx = -\pi^2\csc^2(\pi p) \tag{11}$$

from entry (1) in Table below (64:12:7) in [6].

Derivation of new entry 3.217.2 in [12]

Using Eq (9) replacing b by q and setting $k = 2$, $a = 1$ and $m = 1/2$ simplifying we get

$$\int_0^\infty \left(\frac{\sqrt{\frac{qx}{qx+1}}\log^2\left(\frac{qx}{qx+1}\right)}{x} - \frac{q\sqrt{\frac{qx+1}{qx}}\log^2\left(\frac{qx+1}{qx}\right)}{qx+1} \right) dx = 0 \tag{12}$$

from entry (2) in Table below (64:12:7) in [6].

Derivation of new entry 3.217.3 in [12]

Using Eq (9) setting $k = 1$, $a = e$, $b = 1$ and $m = 1/2$ simplifying we get

$$\int_0^\infty \left(\frac{\sqrt{\frac{1}{x}+1}\left(\log\left(\frac{1}{x}+1\right)+1\right)}{x+1} + \frac{-\log(x)+\log(x+1)-1}{\sqrt{x}\sqrt{x+1}} \right) dx = \pi^2 \tag{13}$$

from entry (3) in Table below (64:12:7) in [6].

Derivation of new entry 3.217.4 in [12]

Using Eq (9) and setting $m = 1/2$ and $a = 1$ simplifying we get

$$\int_0^\infty \left(\frac{\sqrt{\frac{bx}{bx+1}}\log^k\left(\frac{bx}{bx+1}\right)}{x} - \frac{b\sqrt{\frac{bx+1}{bx}}\log^k\left(\frac{bx+1}{bx}\right)}{bx+1} \right) dx = \left(1-2^{k+1}\right)(2i\pi)^{k+1}\zeta(-k) \tag{14}$$

from entries (2) in Table below (64:7) and entry (3) in Table below (64:12:7) in [6]

Derivation of new entry 3.217.5 in [12]

Using Eq (9) we first set $a = -1$ and $m = 1/2$ followed by taking the first partial derivative with respect to k then setting $k = 0$ simplifying we get

$$\int_0^\infty \left(\frac{\sqrt{\frac{bx}{bx+1}}\log\left(\log\left(\frac{1}{bx+1}-1\right)\right)}{x} - \frac{\log\left(\log\left(-\frac{1}{bx}-1\right)\right)}{x\sqrt{\frac{1}{bx}+1}} \right) dx = 2i\pi\log\left(-\frac{2\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)}\right) \tag{15}$$

from equations (1.10.10) in [7], (25.14.2) in [2] and (64:13:3) in [6].

Derivation of new entry 3.217.6 in [12]

Using Eq (9) and setting $a = 1$ and simplifying we get

$$\int_0^\infty \left(\frac{\left(\frac{bx}{bx+1}\right)^m\log^k\left(\frac{bx}{bx+1}\right)}{x} - \frac{b\left(\frac{1}{bx}+1\right)^m\log^k\left(\frac{1}{bx}+1\right)}{bx+1} \right) dx = -(2i\pi)^{k+1}\mathrm{Li}_{-k}\left(e^{2im\pi}\right) \tag{16}$$

from equation (1.11.14) in [7].

Derivation of new entry 3.217.7 in [12]

Using Eq (9) and setting $k = -1$ simplifying we get

$$\begin{aligned}
&\int_0^\infty \left(\frac{\left(\frac{bx}{bx+1}\right)^m}{x \log\left(\frac{abx}{bx+1}\right)} - \frac{b\left(\frac{1}{bx}+1\right)^m}{(bx+1)\log\left(\frac{a}{bx}+a\right)} \right) dx \\
&= -e^{2i\pi m}\Phi\left(e^{2im\pi}, 1, 1-\frac{i\log(a)}{2\pi}\right) - \frac{i\pi}{\log(a)} \\
&= \frac{2\pi}{2\pi - i\log(a)}\, {}_2F_1\left(1, 1-\frac{i\log(a)}{2\pi}; 2-\frac{i\log(a)}{2\pi}; e^{2im\pi}\right)
\end{aligned} \tag{17}$$

from Eq (9.559) in [12].

Derivation of new entry 3.217.8 in [12]

Using Eq (9) setting $m = 1/2$, $k = -2$ and $a = 1-$ simplifying we get

$$\int_0^\infty \left(\frac{\sqrt{\frac{bx}{bx+1}}}{x \log^2\left(\frac{1}{bx+1}-1\right)} - \frac{1}{x\sqrt{\frac{1}{bx}+1}\log^2\left(-\frac{1}{bx}-1\right)} \right) dx = \frac{i(2G-1)}{\pi} \tag{18}$$

where G is Catalan's constant given by Eq (9.73) in [12].

Derivation of new entry 3.217.9 in [12]

Using Eq (9) we first set $m = 1/2$ and $a = 1$, then we take the first partial derivative with respect to k to get

$$\begin{aligned}
&\int_0^\infty \left(\frac{\sqrt{\frac{bx}{bx+1}}\log\left(\log\left(\frac{bx}{bx+1}\right)\right)\log^k\left(\frac{bx}{bx+1}\right)}{x} - \frac{\log\left(\log\left(\frac{1}{bx}+1\right)\right)\log^k\left(\frac{1}{bx}+1\right)}{x\sqrt{\frac{1}{bx}+1}} \right) dx \\
&= -2i\pi i^k \left(2^{2k+1}\pi^k \log(2)\zeta(-k) + \left(2^{k+1}-1\right)(2\pi)^k\left(-\zeta'(-k) + \log(2i\pi)\zeta(-k)\right)\right)
\end{aligned} \tag{19}$$

Next we apply L'Hopitals' rule to the left-hand side as $k \to -1$ simplifying to get

$$\begin{aligned}
&\int_0^\infty \left(\frac{\sqrt{\frac{bx}{bx+1}}\log\left(\log\left(\frac{bx}{bx+1}\right)\right)}{x\log\left(\frac{bx}{bx+1}\right)} - \frac{\log\left(\log\left(\frac{1}{bx}+1\right)\right)}{x\sqrt{\frac{1}{bx}+1}\log\left(\frac{1}{bx}+1\right)} \right) dx \\
&= \frac{1}{2}\log(2)\left(-2\gamma + i\pi + \log\left(8\pi^2\right)\right)
\end{aligned} \tag{20}$$

where γ is Euler's constant given by Eq (9.73) in [12].

Derivation of new entry 3.217.10 in [12]

Using Eq (19) and setting $k = -2$ simplifying we get

$$\begin{aligned}\int_0^\infty &\left(\frac{\sqrt{\frac{bx}{bx+1}}\log\left(\log\left(\frac{bx}{bx+1}\right)\right)}{x\log^2\left(\frac{bx}{bx+1}\right)} - \frac{\log\left(\log\left(\frac{1}{bx}+1\right)\right)}{x\sqrt{\frac{1}{bx}+1}\log^2\left(\frac{1}{bx}+1\right)}\right)dx \\ &= \frac{1}{48}i\pi(-24\log(A)+2\gamma-i\pi+\log(4))\end{aligned} \tag{21}$$

where A is the Glaisher-Kinkelin constant given by Eq (2.15) in [3].

Summary of Results

In this section we generate a table of definite integrals which can be included in [12].

$f(x)$	$\int_0^\infty f(x)dx$
$\frac{\left(\frac{qx}{qx+1}\right)^p}{x} - \frac{q\left(\frac{1}{qx}+1\right)^p}{qx+1}$	$\pi \cot(\pi p)$
$\left(\frac{q\left(\frac{1}{qx}+1\right)^p}{qx+1} + \frac{\left(\frac{qx}{qx+1}\right)^p}{x}\right) \log\left(\frac{qx}{qx+1}\right)$	$-\pi^2 \csc^2(\pi p)$
$\frac{\sqrt{\frac{qx}{qx+1}} \log^2\left(\frac{qx}{qx+1}\right)}{x} - \frac{q\sqrt{\frac{qx+1}{qx}} \log^2\left(\frac{qx+1}{qx}\right)}{qx+1}$	0
$\frac{\sqrt{\frac{1}{x}+1}\left(\log\left(\frac{1}{x}+1\right)+1\right)}{x+1} + \frac{-\log(x)+\log(x+1)-1}{\sqrt{x}\sqrt{x+1}}$	π^2
$\frac{\sqrt{\frac{bx}{bx+1}} \log^k\left(\frac{bx}{bx+1}\right)}{x} - \frac{b\sqrt{\frac{bx+1}{bx}} \log^k\left(\frac{bx+1}{bx}\right)}{bx+1}$	$\left(1-2^{k+1}\right)(2i\pi)^{k+1}\zeta(-k)$
$\frac{\sqrt{\frac{bx}{bx+1}} \log\left(\log\left(\frac{1}{bx+1}-1\right)\right)}{x} - \frac{\log\left(\log\left(-\frac{1}{bx}-1\right)\right)}{x\sqrt{\frac{1}{bx}+1}}$	$2i\pi \log\left(-\frac{2\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)}\right)$
$\frac{\left(\frac{bx}{bx+1}\right)^m \log^k\left(\frac{bx}{bx+1}\right)}{x} - \frac{b\left(\frac{1}{bx}+1\right)^m \log^k\left(\frac{1}{bx}+1\right)}{bx+1}$	$-(2i\pi)^{k+1}\mathrm{Li}_{-k}\left(e^{2im\pi}\right)$
$\frac{\left(\frac{bx}{bx+1}\right)^m}{x\log\left(\frac{abx}{bx+1}\right)} - \frac{b\left(\frac{1}{bx}+1\right)^m}{(bx+1)\log\left(\frac{a}{bx}+a\right)}$	$\frac{2\pi\, {}_2F_1\left(1,1-\frac{i\log(a)}{2\pi};2-\frac{i\log(a)}{2\pi};e^{2im\pi}\right)}{2\pi-i\log(a)}$
$\frac{\sqrt{\frac{bx}{bx+1}} \log\left(\log\left(\frac{bx}{bx+1}\right)\right)}{x\log\left(\frac{bx}{bx+1}\right)} - \frac{\log\left(\log\left(\frac{1}{bx}+1\right)\right)}{x\sqrt{\frac{1}{bx}+1}\log\left(\frac{1}{bx}+1\right)}$	$\frac{1}{2}\log(2)\left(-2\gamma+i\pi+\log\left(8\pi^2\right)\right)$
$\frac{\sqrt{\frac{bx}{bx+1}} \log\left(\log\left(\frac{bx}{bx+1}\right)\right)}{x\log^2\left(\frac{bx}{bx+1}\right)} - \frac{\log\left(\log\left(\frac{1}{bx}+1\right)\right)}{x\sqrt{\frac{1}{bx}+1}\log^2\left(\frac{1}{bx}+1\right)}$	$\frac{1}{48}i\pi(-24\log(A)+2\gamma-i\pi+\log(4))$

Discussion

In this paper we have derived a table of definite integrals known and new in terms of special functions and fundamental constants. Table of definite integrals of the logarithmic function provide a useful reference for research into various topics such as Perturbative and Non-perturbative Aspects of Quantum Field Theory [5] etc. We have devoted this note to provide an extended listing of such integrals in [4] and [12] for potential future work. The present paper should be seen as an extension of these results.

Conclusion

In this paper, we have presented a novel method for deriving some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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