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Group Splitting with SOR/AOR Methods for Solving Boundary Value Problems: A Computational Comparison

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Abstract. Many researchers are working on the explicit group methods as the alternative methods for solving several boundary value problems. These methods have been shown to be much faster than the other point iterative methods in solving the elliptic partial differential equations (EPDEs), which is due to the formers' overall lower computational complexities. This paper is concerned with the application of a suitable Explicit Group (EG) iterative method for solving EPDEs. This study will compare several iterative methods such that S5-point-SOR,4 Point-EGSOR, 5S-point-AOR, and 4 Point-EGAOR. Numerical experiments were carried out to confirm our results by using MATLAB software. The results reveal that 4 Point-EGAOR is the most superior method among these methods.

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Key Words and Phrases: Explicit Group (EG) method, Poisson equation, Successive Over-Relaxation (SOR), Accelerated Over-Relaxation (AOR

1. Introduction

Many physical phenomena in engineering, fluid dynamics, and static field problems, particularly in the electromagnetic field and the incompressible potential flow field, are described by partial differential equations (PDEs) [2, 5, 9, 13]. It has been known that using finite difference schemes to discretize PDEs typically results in a broad and sparse system of linear equations. Several studies were proposed on different iterative methods for solving any linear system of equations to speed up the convergence rate due to the wide range of linear systems. Therefore, Yousif and Evans [13] has also pioneered 4-point block iterative methods for solving large linear structures using the Explicit Group (EG) iterative method. The iterative point method proposed by Hadjidimos [5], together with

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two accelerated parameters called the Accelerated Over-Relaxation (AOR) method, has been developed. In this work, the formulation of efficient explicit group by combining the EG iterative method with the AOR method will be presented for the elliptic partial differential equations solution. The new applications of the EG AOR method will be conducted by MATLAB software .
We shall begin with the presentation of several preliminary relevant theorems and lemmas, which are needed for the proof of the convergence properties of the solution of the mentioned iterative methods. The spectral radius of a matrix is denoted by$\rho(.)$, which is defined as the largest of the moduli of the eigenvalues of the iteration matrix, which play an important role to study these convergence properties.

Theorem 1. *[11] If $A = M - N$ is a regular splitting of the matrix A and $A^{-1} \geq 0$, then*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1.$$

Thus, an iterative method with coefficient matrix $M^{-1}N$ is convergent for any initial vector $x^{(0)}$.

An accurate analysis of convergence properties of the SOR method is possible if the matrix A is consistently ordered in the following sense (see [10]).

Definition 1. *A matrix A is a generalized (q,r)-consistently ordered matrix (a GCO(q,r)-matrix) if $\Delta = \det(\alpha^q E + \alpha^{-r} F - kD)$ is independent of α for all $\alpha \neq 0$ and for all k. Here, $D = diag\ A$ and E and F are strictly lower and strictly upper triangular matrices, respectively, such that: $A = D - E - F$.*

Definition 2. *[10] A matrix A of the form (3.2) is said to be generally consistently ordered (π,q,r) or simply GCO (π,q,r), where q and r are positive integers, if for the partitioning π of A the diagonal submatrices $A_{(ii)}, i = 1, 2, .., p(\geq 2)$, are non-singular, and the eigenvalues of*

$$B_J(\alpha) = \alpha^r L + \alpha^{-q} U \tag{1}$$

are indpendent of α, for all $\alpha \neq 0$, where L and U are strict blocks lower and upper triangular parts of A, respectively.
For any matrix $C = (c_{ij}) in \not\subset^{n_i,n_i}_{\pi,p}$, let $|C|$ denote the block matrix in $\not\subset^{n_i,n_i}_{\pi,p}$ with entries $\left|C_{(ij)}\right|$. Given the matrix

$$B_J = L + U, \tag{2}$$

then $\bar{\mu}$ denotes the spectral radius of the matrix

$$|B_J| = |L + U|, \tag{3}$$

such that $\overline{\mu} := \rho(|B_J|)$.

Lemma 1. *[10] Let $|B_J|$ of (4.3) be a GCO (q,r)-matrix and $p = q + r$. Then for any real nonnegative constant α, β, and γ with $\gamma \neq 0$ satisfying: $\alpha^r \beta^q \overline{\mu} \prec \gamma^p$, the matrix $A'' := \gamma I - \alpha |L| - \beta |U|$ is such that $A^{n-1} \geq 0$.*

Lemma 2. *[8] Suppose $A = I–L–U$ is a GCO(π,q,r), where $–L$ and $–U$ are strictly lower and upper triangular matrices, respectively. Let B_{ℓ_w} be the block iteration matrix of the SOR method given by (2.2). If $0 < w < 2$, then the block SOR method converges, i.e. $\rho(B_{\ell_w}) < 1$.*

Theorem 2. *[8] Suppose $A = I–L–U$ is a GCO(π,q,r), where $–L$ and $–U$ are strictly lower and upper triangular matrices, respectively. Let B_{ℓ_w} and $\tilde{B}_{\ell_w}$ be the iteration matrices of the SOR method given by (2.2) and (2.6) respectively. If $0 < \omega < 2$, then*

(i) $\rho(\tilde{B}_{\ell_w}) < \rho(B_{\ell_w})$ *if* $\rho(B_{\ell_w}) < 1$

(ii) $\rho(\tilde{B}_{\ell_w}) = \rho(B_{\ell_w})$ *if* $\rho(B_{\ell_w}) = 1$

(iii) $\rho(\tilde{B}_{\ell_w}) > \rho(B_{\ell_w})$ *if* $\rho(B_{\ell_w}) > 1$.

Theorem 3. *[3] Let the matrix $A = D–L–U$ be a pcyclic consistently ordered one with non-singular diagonal submatrices A_{jj}, $j = 1, 2, ..p$. If all the eigenvalues of the pth power of the associated Jacobi matrix $T\left(T = I - D_A^{-1}A, D_A = diag\left(A_{11}, A_{22}, \ldots, A_{PP}\right)\right)$ are real and nonnegative and $0 \leq (\rho(T)) < 1$ then with $\hat{\omega}_p$ defined by $(\rho\omega)^P = P^P(P-1)^{1-P}(\omega - 1)$, when $\rho = \rho(T)$, it is $\rho\left(L_\omega\right) > \rho\left(L_{\hat{\omega}_p}\right) = (p-1)\left(\hat{\omega}_p - 1\right)$ for all $\omega \neq \hat{\omega}_p$.*

This paper is organized as follows: In section 2, we describe the five-point finite difference (SOR) and EG(SOR). In Section 3, we provide an overview of the iterative five-point differential formula (AOR) and EG(AOR) for the resolution of Poisson's Equation. The numerical findings are presented in Section 4 to demonstrate the EGAOR method's efficiency. The final observations and conclusions are given in Section 5.

2. Explict Group SOR Method

The SOR (successive over-relaxation) method is a standard iterative method for solving linear systems of equations, particularly large sparse systems arising from partial differential equations. In the SOR method, the parameter ω must be determined where a suitable value of ω can lead to dramatic convergence improvements. Therefore, the method of SOR became popular and was chosen as a method in computer codes to solve major practical problems such as weather prediction and diffusion of the nuclear reactor [1, 4, 7, 12].
Consider the elliptic equation that is linear and self-adjoint,

$$a\frac{\partial^2 U}{\partial x^2} + b\frac{\partial^2 U}{\partial x \partial y} + c\frac{\partial^2 U}{\partial y^2} + d\frac{\partial U}{\partial x} + e\frac{\partial U}{\partial y} + fU + g = 0,$$

where a, b, c, d, e, f and g may be a function of the independent variables x and y of the dependent variables U. Elliptic equations describe problems in a closed region. The Poisson and Laplace equations are examples of the elliptic equations. Only boundary conditions are considered because the dependent variable does not depend on time. In this work, we will consider the Poisson equation.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), (x,y) \in \Omega \tag{4}$$

with specific Dirichlet boundary conditions

$$u(x,y) = g(x,y), (x,y) \in \partial\ \Omega.$$

At this point, (x_i, y_j) the Poisson equation (4) may in several ways be approximated. Assume $u(x_i, y_j)$ is a function of independent variables x and y. The grid lines are equally spaced in both directions, x, and y. then, we can set $h = \Delta x = \Delta y$ where the grid point P is (x_i, y_j) with $x_i = i\Delta x, i = 1, 2, ..., n-1$, and $y_i = i\Delta x, j = 1, 2, ..., n-1$. The basic five-point formula is obtained by discretizing Eq. (4) and using the finite difference approximation.

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{ij}. \tag{5}$$

If we use the standard five-point approximation scheme, SOR iterative method can be written as equations (5)

$$u_{ij}^{(k+1)} = \frac{\omega}{4}(u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} - h^2 f_{ij}) + (1-\omega)u_{ij}^{(k)}. \tag{6}$$

Each of equations (5), and (6) has local order truncation errors $O(h^2)$. In the (EG) method, the solution domain is divided into groups of four-points. The mesh points are grouped in blocks of four points, and the standard five-point equation is implemented at each of these points, and this will result in a (4×4) system of equations,

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{ij} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{bmatrix} = \begin{bmatrix} u_{i-1,j} + u_{i,j-1} - h^2 f_{ij} \\ u_{i+2,j} + u_{i+1,j-1} - h^2 f_{i+1j} \\ u_{i+2,j+1} + u_{i+1,j+2} - h^2 f_{i+1j+1} \\ u_{i-1,j+1} + u_{i,j+2} - h^2 f_{i,j+1} \end{bmatrix}. \tag{7}$$

Equation (7) can be easily inverted to produce a four-point explicit group equation:

$$\begin{bmatrix} u_{ij} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 7 & 2 & 1 & 2 \\ 2 & 7 & 2 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} u_{i-1,j} + u_{i,j-1} - h^2 f_{ij} \\ u_{i+2,j} + u_{i+1,j-1} - h^2 f_{i+1j} \\ u_{i+2,j+1} + u_{i+1,j+2} - h^2 f_{i+1j+1} \\ u_{i-1,j+1} + u_{i,j+2} - h^2 f_{i,j+1} \end{bmatrix}$$

whose each explicit equations can be written in the form

$$\begin{aligned} u_{ij} &= \frac{1}{24}[7r_1 + 2(r_2 + r_4) + r_3], \\ u_{i+1,j} &= \frac{1}{24}[7r_2 + 2(r_1 + r_3) + r_4], \\ u_{i+1,j+1} &= \frac{1}{24}[7r_3 + 2(r_2 + r_4) + r_1], \\ u_{i,j+1} &= \frac{1}{24}[7r_4 + 2(r_1 + r_3) + r_2] \end{aligned}, \tag{8}$$

where

$$r_1 = u_{i-1,j} + u_{i,j-1} - h^2 f_{ij} \qquad r_2 = u_{i+2,j} + u_{i+1,j-1} - h^2 f_{i+1,j}$$
$$r_3 = u_{i+2,j+1} + u_{i+1,j+2} - h^2 f_{i+1,j+1} \qquad r_4 = u_{i-1,j+1} + u_{i,j+2} - h^2 f_{i,j+1}.$$

The approach uses the collection of equations to iteratively evaluate the solution in blocks of four points around the entire solution domain (4). Until convergence is achieved, the process is continuous.

3. Explicit Group AOR Method

The AOR method, $Au = b$, is a well-known iterative method for solving linear systems of equations with two real parameters r and ω. When the two parameters involved in the (AOR) method take on some particular values, it can be thought of as an extrapolation of the (SOR) method [4, 5]. A matrix A can be decomposed as

$$A = D - L - U, \tag{9}$$

D is a diagonal matrix, and L and U are strictly lower and upper triangular matrices. The AOR iterative method can be written as:

$$u^{(k+1)} = L_{r,\omega} u^{(k)} + \omega (D - rL)^{-1} b \tag{10}$$

where
$$L_{r,\omega} = (I - rD^{-1}L)^{-1}[(1-\omega)I + (\omega - r)D^{-1}L + \omega D^{-1}U].$$

According to equation (6) in section 2, the AOR iterative scheme for the standard five-point can be written as we have to determine the expressions consist in the lower triangular matrix . The values of these expressions have been computed. From equation (6), we have already computed $u_{i-1,j}^{(k+1)}$ and $u_{i,j-1}^{(k+1)}$. So, we have to change $u_{i-1,j}^{(k+1)}$ and $u_{i,j-1}^{(k+1)}$ to $u_{i-1,j}^{(k)}$ and $u_{i,j-1}^{(k)}$. Then, add $\frac{r(u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)})}{4}$, and $\frac{r(u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)})}{4}$ [6]. The coefficient is $\frac{1}{4}$, corresponds to the equation (6).

$$\begin{aligned} u_{ij}^{(k+1)} &= r\Big(\frac{u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)} + u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)})}{4}\Big) + \omega\Big(\frac{u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{ij})}{4}\Big) \\ &\quad + (1-\omega) u_{ij}^{(k)}. \end{aligned} \tag{11}$$

Unlike SOR, however, the optimal values of r and ω for which the minimum number of iterations is given are not generally specified. According to [5], the r is typically chosen to be close to the ω value of the respective SOR; then, a certain ω range will be executed by the numerical experiment.

For the four-point group AOR method, we can use the following iterative scheme

$$\begin{aligned} u_{ij}^{(k+1)} &= \frac{1}{4}[\omega(7b_1 + s_2 + b_4) + r(7t_1 + 2t_2)] + (1-\omega)u_{ij}^{(k)}, \\ u_{i+1,j}^{(k+1)} &= \frac{1}{24}[\omega(7b_2 + s_1 + b_3) + r(7c_4 + 2t_1 + c_3)] + (1-\omega)u_{i+1,j}^{(k)}, \\ u_{i,j+1}^{(k+1)} &= \frac{1}{24}[\omega(7b_3 + s_1 + b_2) + r(7c_3 + 2t_1 + c_4)] + (1-\omega)u_{i,j+1}^{(k)}, \\ u_{i+1,j+1}^{(k+1)} &= \frac{1}{24}[\omega(7b_4 + s_2 + b_1) + r(2t_2 + t_1)] + (1-\omega)u_{i,j+1}^{(k)}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &b_1 = u_{i,j-1}^{(k)} + u_{i-1,j}^{(k)} - h^2 f_{i,j}, && b_2 = u_{i+1,j-1}^{(k)} + u_{i+2,j}^{(k)} - h^2 f_{i+1,j}, \\ &b_3 = u_{i-1,j-1}^{(k)} + u_{i,j+2}^{(k)} - h^2 f_{i,j+1}, && b_4 = u_{i+1,j+2}^{(k)} + u_{i+2,j+1}^{(k)} - h^2 f_{i+1,j+1}, \\ &c_1 = u_{i-1,j}^{(k+1)} + u_{i-1,j}^{(k)}, && c_2 = u_{i,j-1}^{(k+1)} + u_{i,j-1}^{(k)}, && c_3 = u_{i-1,j+1}^{(k+1)} + u_{i-1,j+1}^{(k)}, \\ &c_4 = u_{i+1,j-1}^{(k+1)} + u_{i+1,j-1}^{(k)}, && s_1 = 2(b_1 + b_4), && s_2 = 2(b_2 + b_3), \\ &t_1 = c_1 + c_2, && t_2 = c_3 + c_4. \end{aligned}$$

4. Numerical Experiments And Results

In order to compare the standard five-point and four-point group SOR iterative methods, some numerical experiments have been performed. These methods were implemented to Poisson equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)e^{xy}, \tag{13}$$

with $u(x,0) = u(0,y) = 1, u(x,1) = e^x, u(1,y) = e^y, 0 \le x, y \le 1$.
The exact solution for this problem is $u(x,y) = e^{xy}$. In this experimental work, we choose the value of tolerance $\varepsilon = 10^{-5}$. The computer processing unit was Intel(R) Core(TM) i7- 7500U CPU with a memory of 8 Gb and the software used to implement and generate the results was MATLAB. We have computed the average absolute errors and record the number of iterations for convergence for different sizes of grids. Due to the MATLAB program's use, a larger number of processors were used in this experiment 12 ,46 ,86 , 106, 146, 186, 226, 350, and 426. Table 1 represents the comparison of S5 and 4-point EGSOR, and Table 2 represents the comparison of S5 and 4-point EGAOR, where N the number of squares, ω the SOR parameter, r the second parameter of AOR, k the number of iterations, e the maximum errors and t represents CPU time. The convergence of the iteration methods relies on the spectral radius, which is defined as the largest of the moduli of the iteration matrix's eigenvalues. It is stated and proven that a linear system with a smaller spectral radius will have a better convergence rate [6]. Therefore, the spectral radius of the resulted iteration matrices was calculated through these numerical experiments to verify that the superior iterative method has the smallest spectral radius. We can also observe that the resulted spectral radius of all methods used is strictly less

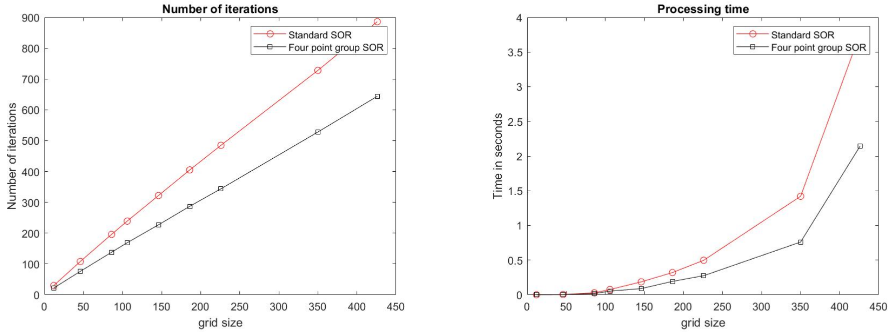
than 1 which coincides with the theoretical parts and previous studies. Through the figures 1,2 and 3, it becomes clear the progress of the four-point EGAOR iterative method in reducing time and number of iteration among the other studied methods.

Table 1: Comparison of standard five-point and four-point EGSOR iterative methods.

	Standard Five-Point					Four-Point Group (SOR)				
N	ω	k	t	$\rho(J)$	E	ω	k	t	$\rho(J)$	E
12	1.5604	30	0.0076	0.9595	$6.2000e-06$	1.3520	22	0.021156	0.9184	$6.2312e-06$
46	1.8696	108	0.0460	0.9976	$8.6000e-06$	1.7846	76	0.040845	0.9951	$9.6553e-06$
86	1.9287	196	0.1747	0.9993	$9.5000e-06$	1.8797	138	0.106233	0.9986	$9.3103e-06$
106	1.9419	239	0.3154	0.9996	$9.7000e-06$	1.9015	169	0.174041	0.9991	$8.1532e-06$
146	1.9576	323	0.4282	0.9998	$9.8000e-06$	1.9277	227	0.181077	0.9995	$9.1567e-06$
186	1.9666	405	0.7470	0.9999	$9.9000e-06$	1.9429	287	0.227092	0.9997	$9.7462e-06$
226	1.9725	487	0.9530	0.9999	$9.6000e-06$	1.9528	344	0.309206	0.9998	$9.6445e-06$
350	1.9822	732	2.8480	0.9999	$9.7000e-06$	1.9693	528	0.796697	0.9999	$9.5333e-06$
426	1.9853	882	6.4068	0.9999	$9.4000e-06$	1.9747	644	2.142877	0.9999	$9.3794e-06$

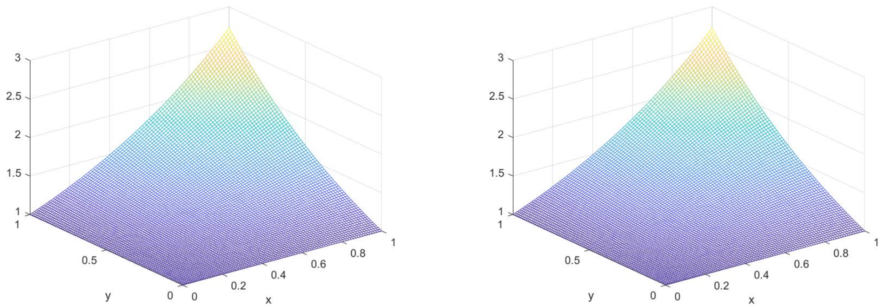
Table 2: Comparison of standard five-point and four-point EGAOR iterative methods.

	Standard Five-Point					
N	ω	r	k	t	$\rho(J)$	E
12	1.5604	1.5709	26	$0.0032s$	0.9595	$8.8565e-06$
46	1.8696	1.8749	100	$0.0146s$	0.9976	$9.5039e-06$
86	1.9287	1.9329	181	$0.0678s$	0.9993	$9.8956e-06$
106	1.9419	1.9449	222	$0.1072s$	0.9996	$9.9577e-06$
146	1.9576	1.9592	306	$0.2163s$	.9998	$9.9646e-06$
186	1.9666	1.9690	395	$0.6039s$	0.9998	$9.9828e-06$
226	1.9699	1.9742	466	$0.67701s$	0.9999	$9.4652e-06$
350	1.9818	1.9823	724	$2.1866s$	0.9999	$9.8868e-06$
426	1.9845	1.9856	869	$4.90743s$	.9999	$9.91628e-06$
	Four-Point Group (AOR)					
N	ω	r	k	t	$\rho(J)$	E
12	1.4331	1.4537	17	$0.0085s$	0.9184	$9.7737e-06$
46	$1.792-1.763$	1.8270	69	$0.0261s$	0.9951	$9.8840e-06$
86	$1.847-1.853$	1.9050	124	$0.0920s$	0.9986	$9.9985e-06$
106	$1.880-1.867$	1.9220	152	$0.0990s$	0.9991	$9.8045e-06$
146	$1.903-1.843$	1.9430	208	$0.1782s$	0.9995	$9.8531e-06$
186	$1.919-1.917$	1.9550	263	$0.2521s$	0.9997	$9.9905e-06$
226	$1.931-1.927$	1.9620	318	$0.4317s$	0.9998	$9.9684e-06$
350	$1.954-1952$	1.9753	488	$0.9133s$	0.9999	$9.9444e-06$
426	$1.935-1.929$	1.9800	587	$1.9163s7$	0.9999	$9.9413e-06$



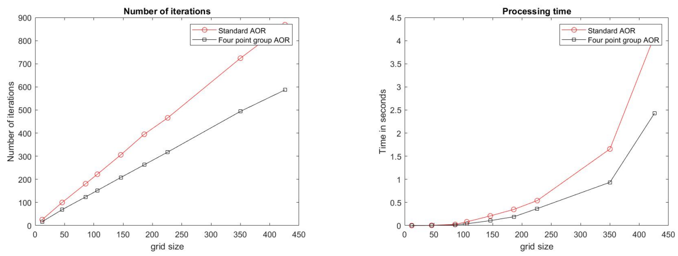
(a) (b)

Figure 1: Comparison of the CPU time (t) and the number of iterations (k) for Standard Five-Point SOR and Foure point EGSOR iterative methods.



(a) (b)

Figure 2: (a) approximation solution of EGSOR for N=86,(b) approximation solution EGAOR for N=86.



(a) (b)

Figure 3: Comparison of the CPU time (t) and the number of iterations (k) for Standard Five-Point AOR and Foure point EGAOR iterative methods.

5. Conclusions

The results reported in Tables 1 and 2, clearly show that the 4- EGAOR process outperforms the corresponding 4-EGSOR, point SOR, and AOR methods. The number of iterations and execution time has also decreased. In all of the cases studied, it is evident that the 4- EGAOR approach needs the least amount of computational effort, which corresponds to the pattern of timing results obtained in our experiments. This is due in large part to the AOR method's reduction of computational complexity and acceleration parameters. This research can be continued in the future to look at the use of another group iterative methods combined with AOR scheme as a smoother with a complexity reduction approach.

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