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On Resolving Hop Domination in Graphs

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Abstract. A set S of vertices in a connected graph G is a *resolving hop dominating set* of G if S is a resolving set in G and for every vertex $v \in V(G) \setminus S$ there exists $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of such a set S is called the resolving hop domination number of G. This paper presents the characterizations of the resolving hop dominating sets in the join, corona and lexicographic product of two graphs and determines the exact values of their corresponding resolving hop domination number.

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Key Words and Phrases: Resolving hop dominating set, resolving hop domination number, join, corona, lexicographic product

1. Introduction

Domination in graphs was first introduced by C. Berge in 1958 [3]. There are now many studies involving domination and its variations. Natarajan and Ayyaswamy [9] introduced and studied the concept of hop domination in graphs. Hop domination in graphs were also studied in [6, 10, 11].

Slater [12] introduced and studied the concept of resolving set. Resolving sets and resolving dominating sets were studied in [1, 2, 4, 7, 8].

This paper combines the idea of resolving and hop domination sets by introducing the concept of resolving hop domination in graphs. Resolving hop dominating sets in graphs can have real world applications. One possible application is in the minimization problem with specific conditions. For example, a company that makes electric cars with a smart navigation feature, wants to build the least number of charging stations in a given city, such that any car with a low charge, from any area, can reach a charging station before

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running out of either or both its remaining charge and auxiliary power. However, fully charging a car takes time, which at some point can overwhelm a station's capacity. To reduce the chance of this from happening, the company may require, as much as possible, that no such two cars from different areas arrive at the same station at about the same time. Assuming that both the remaining low charge and auxiliary power can each cover the same travel distance d, the graph-theoretic model for this scenario could be that vertices represents the areas, and adjacency of vertices represent a connected route of distance d. Resolving hop domination in graphs can be used to determine the minimum number of charging stations and where to build them in such a manner that cars from different areas have relatively distinct distances from these stations.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [5] for elementary Graph Theory concepts.

Let $G = \big(V(G), E(G)\big)$ be a graph. $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is a *neighborhood* of v. An element $u \in N_G(v)$ is called a *neighbor* of v. $N_G[v] = N_G(v) \cup \{v\}$ is a *closed neighborhood* of v. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

A connected graph G is said to be *point determining* if distinct vertices have distinct neighborhoods, that is, $N_G(a) \neq N_G(b)$ whenever $a, b \in V(G)$ and $a \neq b$.

A connected graph G of order $n \geq 3$ is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

A vertex x of a graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, ..., x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector*

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), ..., d_G(v, x_k))$$

is called the representation of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The *metric dimension* of G, denoted by $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called *basis*.

A set $S \subseteq V(G)$ of vertices of G is a *dominating set* if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = min\{|S| : S$ is a dominating set of $G\}$.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

A set $S \subseteq V(G)$ is a *locating set* of G if for every two distinct vertices u and v of $V(G) \backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The *locating number* of G, denoted by $ln(G)$, is the smallest cardinality of a locating set of G. A locating set of G of cardinality $ln(G)$ is referred to as a *ln-set* of G. A set $S \subseteq V(G)$ is a *strictly locating set* of G if it is a locating set of G and $N_G(u) \cap S \neq S$ for all $u \in V(G) \backslash S$. The *strictly locating number* of G, denoted by $sln(G)$, is the smallest cardinality of a strictly locating set of G. A strictly locating set of G of cardinality $sln(G)$ is referred to as a *sln-set* of G.

A set $S \subseteq V(G)$ is a *resolving hop dominating set* of G if S is both a resolving set and a hop dominating set. The minimum cardinality of a resolving hop dominating set of G, denoted by $\gamma_{Rh}(G)$, is called the *resolving hop domination number* of G. Any resolving hop dominating set with cardinality equal to $\gamma_{Rh}(G)$ is called a γ_{Rh}-set.

2. Preliminary Results

Remark 1. For any connected graph G of order $n \geq 2$, $2 \leq \gamma_{Rh}(G) \leq n$. Moreover, $\gamma_{Rh}(P_2) = 2$ and $\gamma_{Rh}(K_n) = n$.

Proposition 1. For any connected graph G of order $n \geq 2$. Then, $\gamma_{Rh}(G) = n$ if and only if $G = K_n$.

Proof: If $G = K_n$, then $\gamma_{Rh}(G) = n$. Suppose $\gamma_{Rh}(G) = n$ and $G \neq K_n$. Then there exists $x, y \in V(G)$ such that $d(x, y) = 2$. Let $S = V(G) \setminus \{y\})$. Then S is a resolving hop dominating set of G. Hence, $\gamma_{Rh}(G) \leq |S| = n - 1$, a contradiction. □

Remark 2. Let G be a connected graph and $S \subseteq V(G)$. Then for any two distinct vertices $x, y \in V(G) \setminus S$ with $N_G(x, 2) \cap S \neq N_G(y, 2) \cap S$, we have $r_G(x/S) \neq r_G(y/S)$.

Remark 3. Every resolving hop dominating set of a connected graph G is a resolving set of G. Thus, $dim(G) \leq \gamma_{Rh}(G)$.

Proposition 2. Let G be a connected graph of order 4. Then $\gamma_{Rh}(G) = 2$ if and only if $G = C_4$ or $G = P_4$.

Proof: If $G = C_4$ or P_4, then $\gamma_{Rh}(G) = 2$. Suppose that $\gamma_{Rh}(G) = 2$. Let $W = \{x_1, x_2\}$ be a γ_{Rh}-set of G. Since W is a hop dominating set, possible representations of distinct vertices $u, v \in V(G) \setminus W$ are (1,2), (2,1) or (2,2). Clearly (2,2) cannot be a representation of vertex u or v since G is of order 4. Thus we consider the following cases:
Case 1. $r_G(u/W) = (1, 2)$ and $r_G(v/W) = (2, 1)$
Case 2. $r_G(u/W) = (2, 1)$ and $r_G(v/W) = (1, 2)$

For case 1, ux_1, $vx_2 \in E(G)$ and either $x_1x_2 \in E(G)$ or $uv \in E(G)$ or both $x_1x_2, uv \in E(G)$. Hence, $G = [u, x_1, x_2, v]$ or $G = [x_1, u, v, x_2]$ or $G = [u, x_1, x_2, v, u]$. Thus, G is either a path P_4 or a cycle C_4. Similarly, if case 2 holds, then $G = P_4$ or $G = C_4$. □

Proposition 3. Let n be a positive number.

(*i*) For a path P_n on n vertices, $n > 1$

$$\gamma_{Rh}(P_n) = \begin{cases} 2 & \text{if} \quad n = 2,3,4,5 \\ 2r & \text{if} \quad n = 6r \\ 2r+1 & \text{if} \quad n = 6r+1 \\ 2r+2 & \text{if} \quad n = 6r+s; \quad 2 \le s \le 5. \end{cases}$$

(ii) For a cycle C_n of length n,

$$\gamma_{Rh}(C_n) = \begin{cases} 2 & \text{if} \quad n = 4,5 \\ 2r & \text{if} \quad n = 6r \\ 2r+1 & \text{if} \quad n = 6r+1 \\ 2r+2 & \text{if} \quad n = 6r+s; \quad 2 \le s \le 5. \end{cases}$$

3. On Resolving Hop Domination in the Join of Graphs

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G+H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 1. [7, 8] Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving set of $G+H$ if and only if $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are locating sets of G and H, respectively, where W_G or W_H is a strictly locating set.

Theorem 2. Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving hop dominating set of $G+H$ if and only if $W = W_G \cup W_H$ where W_G and W_H are strictly locating sets of G and H, respectively.

Proof: Suppose that W is a resolving hop dominating set of $G+H$. Then W is a resolving set of $G+H$. By Theorem 1, $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are locating sets of G and H, respectively. Suppose W_G or W_H is not strictly locating set, say W_G is not strictly locating. Then there exists $v \in V(G) \setminus W_G$ such that $N_G(v) \cap W_G = W_G$. Hence, $v \in V(G+H) \setminus W$ and $d_{G+H}(v,w) = 1$ for all $w \in W$. This contradicts the assumption that W is a hop dominating set of $G+H$. Similarly, if W_H is not strictly locating, then a contradiction follows. Hence, W_G and W_H are both strictly locating.

For the converse, suppose that $W = W_G \cup W_H$ where $W_G \subseteq V(G)$, $W_H \subseteq V(H)$ and both W_G and W_H are strictly locating sets of G and H, respectively. Since W_G and W_H are locating sets by Theorem 1, W is a resolving set of $G+H$. Let $v \in V(G+H) \setminus W$. If $v \in V(G)$, then $v \notin W_G$. Since W_G is strictly locating there exists $u \in W_G \setminus N_G(v)$. Hence, $d_{G+H}(v,u) = 2$. Similarly, if $v \in V(H)$, then $v \notin W_H$ and there exists $w \in W_G \setminus N_G(v)$. Thus, $d_G(v,w) = 2$. Therefore W is a hop dominating set of $G+H$.

Accordingly, W is a resolving hop dominating set of $G+H$. □

The next result follows immediately from Theorem 2.

Corollary 1. Let G and H be non-trivial connected graphs. Then

$$\gamma_{Rh}(G+H) = sln(G) + sln(H).$$

4. On Resolving Hop Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$.

Theorem 3. [7, 8] Let G and H be non-trivial connected graphs. Then $W \subseteq V(G \circ H)$ is a resolving set of $G \circ H$ if and only if $W \cap V(H^v) \neq \varnothing$ for all $v \in V(G)$ and $W = A \cup B$, where $A \subseteq V(G)$, and $B = \cup\{B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v\}$.

Theorem 4. Let G and H be non-trivial connected graphs. Then $W \subseteq V(G \circ H)$ is a resolving hop dominating set of $G \circ H$ if and only if $W \cap V(H^v) \neq \varnothing$ for every $v \in V(G)$ and $W = A \cup B \cup D$ where $A \subseteq V(G)$,

$$B = \cup\{B_v : v \in V(G) \cap N_G(A) \text{ and } B_v \text{ is a locating set of } H^v\} \text{ and}$$
$$D = \cup\{D_u : u \in V(G) \setminus N_G(A) \text{ and } D_u \text{ is a strictly locating set of } H^u\}.$$

Proof: Suppose W is a resolving hop dominating set of $G \circ H$. Then by Theorem 3, $W \cap V(H^v) \neq \varnothing$ for every $v \in V(G)$. Let $A = W \cap V(G)$,

$$B_v = W \cap V(H^v) \text{ for each } v \in V(G) \cap N_G(A) \text{ and}$$
$$D_u = W \cap V(H^u) \text{ for each } u \in V(G) \setminus N_G(A).$$

Set $B = \bigcup B_v$ and $D = \bigcup D_u$. Then $W = A \cup B \cup D$ where $A \subseteq V(G)$. By Theorem 3, B_v and D_u are locating sets of H^v and H^u, respectively. Let $x \in V(H^u) \setminus D_u$. Then $x \in V(G \circ H) \setminus W$. Since W is a hop dominating set of $G \circ H$, there exists $y \in W$ such that $d_{G \circ H}(x, y) = 2$. Since $u \in V(G) \setminus N_G(A)$, $y \in V(H^u) \cap D_u$. Hence, $y \in D_u \setminus N_{H^u}(x)$. Thus, $N_{H^u}(x) \cap D_u \neq D_u$, showing that D_u is strictly locating.

For the converse, suppose that $W \cap V(H^v) \neq \varnothing$ for every $v \in V(G)$ and $W = A \cup B \cup D$ where A, B and D satisfy the given conditions. Let $x \in V(G \circ H) \setminus W$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Suppose $x = v$. Then $v \notin A$. Let $u \in V(G) \cap N_G(v)$. Since $W \cap V(H^u) \neq \varnothing$, there exists $y \in W \cap V(H^u)$ and $d_{G \circ H}(x, y) = 2$. Suppose $x \neq v$. If $v \in N_G(A)$, then there exists $z \in A \cap N_G(v)$. Hence, $z \in W$ and $d_{G \circ H}(x, z) = 2$. Suppose $v \notin N_G(A)$. Then $x \in V(H^v) \setminus D_v$. Since D_v is strictly locating there exists $y \in D_v \setminus N_{H^v}(x)$. Thus, $y \in W$ and $d_{G \circ H}(x, y) = 2$. This shows that W is a hop dominating set of $G \circ H$. Since B_v or D_v is a locating set for each $v \in V(G)$, by Theorem 3, W is a resolving set of $G \circ H$.

Accordingly, W is a resolving hop dominating set of $G \circ H$. □

Corollary 2. Let G be a non-trivial graph of order m and H be any graph. Then the following statements hold.

(*i*) $\gamma_{Rh}(G \circ H) \leq m(1 + ln(H))$.

(*ii*) If $sln(H) = ln(H)$, then $\gamma_{Rh}(G \circ H) = m(sln(H))$.

Proof: (*i*) Set $A_1 = V(G)$ and let B_v be an ln-set of H for each $v \in V(G)$. Then $W_1 = A_1 \cup \left(\bigcup_{v \in V(G)} B_v\right)$ is a resolving hop dominating set of $G \circ H$ by Theorem 4, Hence,

$$\gamma_{Rh}(G \circ H) \leq |W_1| = |V(G)| + |V(G)||B_v| = m(1 + ln(H)).$$

(*ii*) Suppose that $sln(H) = ln(H)$. Set $A_2 = \varnothing$ and let D_u be an sln-set of H for each $u \in V(G)$. Then $W_2 = A_2 \cup \left(\bigcup_{u \in V(G)} D_u\right)$ is a resolving hop dominating set of $G \circ H$ by Theorem 4. Thus,

$$\gamma_{Rh}(G \circ H) \leq |W_2| = |A_2| + |V(G)||D_u| = m(sln(H)).$$

Now, let $W_0 = A_0 \cup \left(\bigcup_{u \in V(G) \setminus S_0} B_v\right) \cup \left(\bigcup_{u \in S_0} D_u\right)$ be a γ_{Rh}-set of $G \circ H$. By Theorem 4, $A_0 \subseteq V(G)$, $S_0 = \{x \in V(G) : x \notin N_G(A_0)\}$, B_v is a locating set of H^v for each $v \in V(G) \setminus S_0$ and D_u is a strict locating set of H^u for each $u \in S_0$. Thus,

$$\begin{aligned}
\gamma_{Rh}(G \circ H) &= |W_0| \\
&= |A_0| + |V(G) \setminus S_0| \; |B_v| + |S_0| \; |D_u| \\
&\geq |V(G) \setminus S_0| \; ln(H) + |S_0| \; sln(H) \\
&= (|V(G)| - |S_0|) \;\; sln(H) + |S_0| \; sln(H) \\
&= m(sln(H)).
\end{aligned}$$

Therefore, $\gamma_{Rh}(G \circ H) = m(sln(H))$. □

5. On Resolving Hop Domination in the Lexicographic Product of Graphs

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Theorem 5. [7, 8] Let G and H be non-trivial connected graphs with $\triangle(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving set of $G[H]$ if and only if

(*i*) $S = V(G)$;

(*ii*) T_x is a locating set for every $x \in V(G)$;

(*iii*) T_x or T_y is a strictly locating set of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$; and

(*iv*) T_x or T_y is a (locating) dominating set of H whenever x and y are nonadjacent vertices of G with $N_G(x) = N_G(y)$.

Theorem 6. Let G and H be non-trivial connected graphs with $\triangle(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving hop dominating set of $G[H]$ if and only if

- (i) $S = V(G)$;
- (ii) T_x is a locating set for every $x \in V(G)$;
- (iii) T_x or T_y is a strictly locating set of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$;
- (iv) T_x or T_y is a (locating) dominating set of H whenever x and y are nonadjacent vertices of G with $N_G(x) = N_G(y)$; and
- (v) T_x is a strictly locating set of H for each $x \in S \setminus N_G(S, 2)$.

Proof: Suppose W is a resolving hop dominating set of $G[H]$. Then W is a resolving set. By Theorem 5, (i) to (iv) hold. Let $x \in S \setminus N_G(S, 2)$. If $T_x = V(H)$, then T_x is a strictly locating set of H. So suppose that $T_x \neq V(H)$ and let $a \in V(H) \setminus T_x$. Since W is hop dominating and $(x, a) \notin W$, there exists $(y, b) \in W$ such that $d_{G[H]}((x, a), (y, b)) = 2$. The condition $x \in S \setminus N(S, 2)$ would imply that $y = x$ and $b \in (V(H) \setminus N_H(a)) \cap T_x$. Hence, T_x is a strictly locating set of H.

Conversely, supose that W satisfies (i) to (v). By Theorem 5, W is a resolving set. Let $(x, a) \in V(G[H]) \setminus W$. Since $S = V(G)$, $a \in V(H) \setminus T_x$. If $x \in N_G(S, 2)$, then there exists $z \in N_G(x, 2)$. Let $b \in T_z$. Then $(z, b) \in W \cap N_{G[H]}((x, a), 2)$. Suppose $x \in S \setminus N_G(S, 2)$. By (v), T_x is a strictly locating set of H. Hence, there exists $p \in [V(H) \setminus N_H(a)] \cap T_x$. This implies that $(x, p) \in W \cap N_{G[H]}((x, a), 2)$. Therefore, W is a hop dominating set of $G[H]$.

Accordingly, W is a resolving hop dominating set of $G[H]$. □

Corollary 3. Let G and H be non-trivial connected graphs. Then

$$\gamma_{Rh}(G[H]) \leq |V(G)|sln(H).$$

If G is totally point determining graph and $\gamma(G) \neq 1$, then

$$\gamma_{Rh}(G[H]) = |V(G)|ln(H).$$

Proof: Let $S = V(G)$ and let T_x be an *sln*-set of H. By Theorem 6, $W = \bigcup_{x\in S}[\{x\} \times T_x]$ is a resolving hop dominating set of $G[H]$. It follows that

$$\gamma_{Rh}(G[H]) \leq |W| = |V(G)||T_x| = |V(G)|sln(H).$$

Next, supppose that G is totally point determining graph and $\gamma(G) \neq 1$. Let $S = V(G)$ and let R_x be an *ln*-set of H for each $x \in S$. Since $\gamma(G) \neq 1$, $x \in N_G(S, 2)$ for each $x \in S$. By Theorem 6, $W = \bigcup_{x\in S}[\{x\} \times R_x]$ is a resolving hop dominating set of $G[H]$. It follows that

$$\gamma_{Rh}(G[H]) \leq |W| = |V(G)||R_x| = |V(G)|ln(H).$$

Now, if $W_0 = \bigcup_{x\in S_0}[\{x\} \times T_x]$ is a γ_{Rh}-set of $G[H]$, then $S_0 = V(G)$ and T_x is a locating set of H for each $x \in V(G)$ by Theorem 6. Hence,

$$\gamma_{Rh}(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)|ln(H).$$

Therefore, $\gamma_{Rh}(G[H]) = |V(G)|ln(H)$. □

Corollary 4. Let G and H be non-trivial connected graphs. If G is totally point determining and $\gamma(G) = 1$, then

$$\gamma_{Rh}(G[H]) = sln(H) + (|V(G)| - 1)ln(H).$$

Proof: Let $D_G = \{v \in V(G) : \{v\}$ is a dominating set of $G\}$. Since G is totally point determining, it follows that $|D_G| = 1$. Set $S = V(G)$. Let T_v be an sln-set of H for $v \in D_G$ and let T_x be an ln-set of H for each $x \in V(G) \setminus \{v\}$. Then by Theorem 6, $W = \bigcup_{x\in S\setminus\{v\}}[\{x\} \times T_x] \cup (\{v\} \times T_v)$ is a resolving hop dominating set of $G[H]$. Hence,

$$\gamma_{Rh}(G[H]) \leq |W| = (|V(G)| - 1)ln(H) + sln(H).$$

Supose now that $W^* = \bigcup_{x\in S^*}[\{x\} \times R_x]$ is a γ_{Rh}-set of $G[H]$. Then there exists a unique vertex v such that $\{v\}$ is a dominating set of G. By Theorem 6, $S^* = V(G)$, R_v is a strictly locating set of H and R_x is a locating set of H for each $x \in V(G) \setminus \{v\}$. Thus,

$$\begin{aligned}\gamma_{Rh}(G[H]) =&|W^*|\\ =&|R_v| + \sum_{x\in S^*\setminus\{v\}} |R_x|\\ \geq& sln(H) + (|V(G)| - 1)ln(H).\end{aligned}$$

Therefore,

$$\gamma_{Rh}(G[H]) = sln(H) + (|V(G)| - 1)ln(H). \quad \square$$

Corollary 5. Let H be a non-trivial connected graph and let $n \geq 2$ be an integer. Then

$$\gamma_{Rh}(K_n[H]) = n(sln(H)).$$

Proof: Let $G = K_n$. Then v is a dominating vertex of G for each $v \in V(G)$. Thus, if $W_0 = \bigcup_{x\in S_0}(\{x\} \times T_x)$ is a γ_{Rh}-set of $G[H]$, then $S_0 = V(G)$ and T_x is an sln-set of H for each $x \in S_0$, by Theorem 6. Hence,

$$\gamma_{Rh}(K_n[H]) = |W_0| = |V(K_n)|sln(H) = n(sln(H)). \quad \square$$

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