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New Topologies between the usual topology and the half-disc

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Abstract. We generate new topologies on the closed upper half plane which lie between the usual topology and the half-disc topology. We study some of their fundamental properties and weaker versions of normality.

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We generate new topologies on the closed upper half plane which lie between the usual metric topology and the half-disc topology. These new spaces may work as counterexamples in Topology and help in study of some advances topological properties. We study some of their fundamental properties and weaker versions of normality. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a T_1 completely regular space. We do not assume T_2 in the definition of compactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. If two topologies τ and τ' on a set X are considered, we denote the interior of A in (X , τ) by $\text{int}_\tau A$ and the closure of A in (X , τ') by $\overline{A}^{\tau'}$.

We start by state some definitions and fix some notations. Let $X = \{ \langle x, y\rangle \in \mathbb{R}^2 : y \geq 0 \}$ be the closed upper half plane. $K = \{ \langle x, y\rangle \in \mathbb{R}^2 : y > 0 \}$, so the x-axis is $L = X \setminus K$. Denote the usual metric topology on X by $\mathcal{U}$ and the half-disc topology on X be $\mathcal{H}$. For every $\langle a, b\rangle \in X$ and $r > 0$ where $r \in \mathbb{R}$, let $U_r(\langle a, b\rangle)$ be the set of all points in X inside the circle of radius r centered at $\langle a, b\rangle$. So, $U_r(\langle a, b\rangle) = \{ \langle x, y\rangle \in X :$

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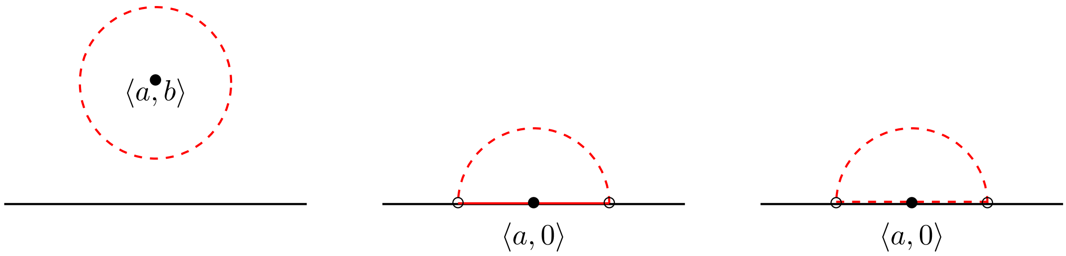
$\sqrt{(x-a)^2+(y-b)^2} < r\,\}$. For every $\langle a,0\rangle \in L$, let $C(\langle a,0\rangle, r)$ be the set of all points of K inside the circle of radius r centered at $\langle a,0\rangle$. So, $C(\langle a,0\rangle, r) = U_r(\langle a,0\rangle)\cap K$. Let $C_r(\langle a,0\rangle) = C(\langle a,0\rangle, r)\cup\{\langle a,0\rangle\}$. Recall that the half-disc topology $\mathcal{H}$ on X [8, Example 78] is generated by the following neighborhood system: For every $\langle a,0\rangle \in L$, let $\mathfrak{B}(\langle a,0\rangle) = \{\,C_r(\langle a,0\rangle) : r>0\,\}$ and for every $\langle a,b\rangle \in K$, let $\mathfrak{B}(\langle a,b\rangle) = \{\,U_r(\langle a,b\rangle) : r > 0\,\}$. Observe that K as a subspace of X with the usual metric topology coincides with K as a subspace of X with the half-disc topology.



$\langle a,b\rangle \in K$, $U_r(\langle a,b\rangle)$ $\quad$ $\langle a,0\rangle \in L$, $U_r(\langle a,0\rangle)$ $\quad$ $\langle a,0\rangle \in L$, $C_r(\langle a,0\rangle)$

1. H-generated topology.

Definition 1. *Let A be a non-empty proper subset of the x-axis L. For each $\langle a,b\rangle \in K\cup A$, let $\mathfrak{B}(\langle a,b\rangle) = \{\,U_r(\langle a,b\rangle) : r>0\,\}$. For each $\langle a,0\rangle \in L\setminus A$, let $\mathfrak{B}(\langle a,0\rangle) = \{\,C_r(\langle a,0\rangle) : r>0\,\}$. So, the points in $K\cup A$ will have the same local base as in $(\,X\,,\mathcal{U}\,)$. The points in $L\setminus A$ will have the same local base as in $(\,X\,,\mathcal{H}\,)$. We call the topology on X generated by the neighborhood system $\{\,\mathfrak{B}(\langle a,b\rangle) : \langle a,b\rangle \in X\,\}$ the H-generated* topology on X from $\mathcal{U}$ and $\mathcal{H}$, shortly *H-topology,* and denote it by $\mathcal{U}_A\mathcal{H}$. We call X with this H-generated topology an *H-space* and denote it by $(\,X\,,\mathcal{U}_A\mathcal{H}\,)$.

Observe that if $A=\emptyset$, then the H-generated topology on X,$\mathcal{U}_A\mathcal{H}$ is just the half-disc topology $\mathcal{H}$ and if $A=L$, then the H-generated topology on X, $\mathcal{U}_A\mathcal{H}$ is just the usual metric topology $\mathcal{U}$. So, from now on, when we consider an H-space $(\,X\,,\mathcal{U}_A\mathcal{H}\,)$, we are assuming that A is a non-empty proper subset of the x-axis L. In Definition 1, if we interchange the local bases as follows: The points in $K\cup(L\setminus A)$ will have the same local base as in $(\,X\,,\mathcal{U}\,)$. The points in A will have the same local base as in $(\,X\,,\mathcal{H}\,)$. Then we get the H-space $(\,X\,,\mathcal{H}_A\mathcal{U}\,)$ and it is easy to see that $(\,X\,,\mathcal{H}_A\mathcal{U}\,) \cong (\,X\,,\mathcal{U}_{L\setminus A}\mathcal{H}\,)$. So, in this paper, we will study the H-spaces of Definition 1, $(\,X\,,\mathcal{U}_A\mathcal{H}\,)$. Observe that K as a subspace of X with the usual metric topology coincides with K as a subspace of X with the H-topology.

2. Basic Properties of an H-space.

Since any basic open set in $(\,X\,,\mathcal{U}\,)$ is also open in $(\,X\,,\mathcal{U}_A\mathcal{H}\,)$, we get the following fact.

Theorem 1. *The usual topology* $\mathcal{U}$ *on* X *is coarser than the* H*-topology* $\mathcal{U}_A\mathcal{H}$ *and the* H*-topology* $\mathcal{U}_A\mathcal{H}$ *is coarser than the half-disc topology* $\mathcal{H}$*. That is,* $\mathcal{U} \subset \mathcal{U}_A\mathcal{H} \subset \mathcal{H}$*.*

By Theorem 1, we conclude that any H-space $(X, \mathcal{U}_A\mathcal{H})$ is T_0, T_1, T_2 Hausdorff, and $T_{2\frac{1}{2}}$ Urysohn (completely Hausdorff) [1]. Now, take any $\langle x, 0\rangle \in L \setminus A$. For any $0 < \epsilon < r$, we have that $\overline{C_\epsilon(\langle x, 0\rangle)} \not\subset C_r(\langle x, 0\rangle)$, thus any H-space $(X, \mathcal{U}_A\mathcal{H})$ is not regular nor zero-dimensional, hence neither normal, T_4, nor metrizable. We conclude also that any H-space $(X, \mathcal{U}_A\mathcal{H})$ cannot be Tychonoff $T_{3\frac{1}{2}}$ hence has no compactification and is neither paracompact, as any T_2 paracompact space is T_4, nor locally compact, as any T_2 locally compact space is Tychonoff. The subset $D = \{ \langle x, y\rangle \in X : x, y \in \mathbb{Q} \}$ is a countable dense subset, thus any H-space $(X, \mathcal{U}_A\mathcal{H})$ is separable. For $\langle x, y\rangle \in K \cup A$, the family $\mathfrak{B}(\langle x, y\rangle) = \{ U_{\frac{1}{n}}(\langle x, y\rangle) : n \in \mathbb{N} \}$ is a countable local base for $(X, \mathcal{U}_A\mathcal{H})$ at $\langle x, y\rangle$. For $\langle x, 0\rangle \in L \setminus A$, the family $\mathfrak{B}(\langle x, 0\rangle) = \{ C_{\frac{1}{n}}(\langle x, 0\rangle) : n \in \mathbb{N} \}$ is a countable local base for $(X, \mathcal{U}_A\mathcal{H})$ at $\langle x, 0\rangle$. Therefore, any H-space $(X, \mathcal{U}_A\mathcal{H})$ is first countable.

Theorem 2. *An* H*-space* $(X, \mathcal{U}_A\mathcal{H})$ *is second countable if and only if* $L \setminus A$ *is countable.*

Proof. If $L \setminus A$ is uncountable, then $L \setminus A$ is an uncountable discrete subspace of the H-space $(X, \mathcal{U}_A\mathcal{H})$, thus cannot be second countable. Assume that $L \setminus A$ is countable. Since $X = K \cup A \cup (L \setminus A)$ and $K \cup A$ is a separable metrizable subspace from $(X, \mathcal{U})$, then $K \cup A$ is second countable.

Let $\mathfrak{B}'$ be a countable base for $K \cup A$. Let $\mathfrak{B}^\star = \{\mathfrak{B}(\langle x, 0\rangle) = \{ C_{\frac{1}{n}}(\langle x, 0\rangle) : n \in \mathbb{N} \} : \langle x, 0\rangle \in L \setminus A \}$. We show that $\mathfrak{B} = \mathfrak{B}' \bigcup \mathfrak{B}^\star$ is a countable base for the H-space $(X, \mathcal{U}_A\mathcal{H})$. Let W be an arbitrary non-empty open set in $(X, \mathcal{U}_A\mathcal{H})$ and pick an arbitrary $\langle x, y\rangle \in W$.

Case 1: If $\langle x, y\rangle \in K \cup A$, then there exists $r > 0$ such that $U_r(\langle x, y\rangle) \subseteq W$. Since $U_r(\langle x, y\rangle)$ is open in $K \cup A$ containing $\langle x, y\rangle$, then there exists $B \in \mathfrak{B}' \subset \mathfrak{B}$ such that $\langle x, y\rangle \in B \subseteq U_r(\langle x, y\rangle) \subseteq W$.

Case 2: If $\langle x, y\rangle \in L \setminus A$, so $y = 0$, then there exists $r > 0$ such that $C_r(\langle x, 0\rangle) \subseteq W$, thus there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$, thus $\langle x, 0\rangle \in C_{\frac{1}{n}}(\langle x, 0\rangle) \subseteq C_r(\langle x, 0\rangle) \subseteq W$, where $C_{\frac{1}{n}}(\langle x, 0\rangle) \in \mathfrak{B}^\star$.

Therefore, $\mathfrak{B}$ is a base for the H-space $(X, \mathcal{U}_A\mathcal{H})$.

For each $n \in \mathbb{N}$, let $G_n = \mathbb{R} \times [0, n)$, then the family $\{ G_n : n \in \mathbb{N} \}$ is a countable open cover for X which has no finite subcover. Thus any H-space $(X, \mathcal{U}_A\mathcal{H})$ is neither compact nor countably compact. Since any second countable is Lindelöf, by Theorem 2, we conclude the following.

Theorem 3. *The* H*-spaces* $(X, \mathcal{U}_A\mathcal{H})$ *is Lindelöf if* $L \setminus A$ *is countable.*

3. Other Properties of an H-space.

Definition 2. *A subset A of a space X is called a closed domain* of X [1, 1.1.C] (also called *regularly closed, κ-closed*) if $A = \overline{\text{int}A}$. A space X is called *mildly normal* [7] (also called *κ-normal* [10]) if for any two disjoint closed domains A and B of X there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [2, 5]. A space X is called *almost normal* [6] if for any two disjoint closed subsets A and B of X one of which is closed domain, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [4].

It is clear from the definitions that

normal $\Rightarrow$ almost normal $\Rightarrow$ mildly normal.

Each implication above is not reversible, see [2, 4].

Lemma 1. *Let D be any non-empty closed domain in $(X, \mathcal{U}_A\mathcal{H})$, then D is a closed set in $(X, \mathcal{U})$.*

Proof. If $D = X$, we are done. Assume $X \setminus D \neq \emptyset$. Let $\langle x, y\rangle \in X \setminus D$ be arbitrary.

Case 1: $\langle x, y\rangle \in K \cup A$. Since D is closed in $(X, \mathcal{U}_A\mathcal{H})$, then $X \setminus D$ is open in $(X, \mathcal{U}_A\mathcal{H})$, thus there exists $r > 0$ such that $U_r(\langle x, y\rangle) \subseteq X \setminus D$. But $U_r(\langle x, y\rangle)$ is also a basic open set in $(X, \mathcal{U})$.

Case 2: $y = 0$ and $\langle x, 0\rangle \in L \setminus A$. There exists $r > 0$ such that

$$C_r(\langle x, 0\rangle) \subseteq X \setminus D \;.\;.\;.\;(\star)$$

Suppose that for all $0 < \varepsilon \leq r$ we have $U_\varepsilon(\langle x, 0\rangle) \nsubseteq X \setminus D$. That is, $U_\varepsilon(\langle x, 0\rangle) \cap D \neq \emptyset$. Fix such an ε, then there exists $z \in (x - \varepsilon, x + \varepsilon)$; $z \neq x$ such that $\langle z, 0\rangle \in D = \overline{\text{int}_{\mathcal{U}_A\mathcal{H}} D}^{\mathcal{U}_A\mathcal{H}}$. For all $0 < \delta < \varepsilon$, we have $C_\delta(\langle z, 0\rangle) \cap \text{int}_{\mathcal{U}_A\mathcal{H}} D \neq \emptyset$ if $\langle z, 0\rangle \in L \setminus A$ or $U_\delta(\langle z, 0\rangle) \cap \text{int}_{\mathcal{U}_A\mathcal{H}} D \neq \emptyset$ if $\langle z, 0\rangle \in A$. If $\langle z, 0\rangle \in L \setminus A$, then$\langle z, 0\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}} D$, because $C_\delta(\langle z, 0\rangle) \cap L = \{\langle z, 0\rangle\}$ and $C_\delta(\langle z, 0\rangle) \cap K \subseteq X \setminus D$. Now, $\langle z, 0\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}} D$ means that there exists $\delta' < \delta$ such that $C_{\delta'}(\langle z, 0\rangle) \subseteq D$ which contradicts $(\star)$, because $C_{\delta'}(\langle z, 0\rangle) \cap K \subseteq C_r(\langle x, 0\rangle) \subseteq X \setminus D$. If $\langle z, 0\rangle \in A$, and $U_\delta(\langle z, 0\rangle) \cap \text{int}_{\mathcal{U}_A\mathcal{H}} D \neq \emptyset$, then pick $\langle u, 0\rangle \in U_\delta(\langle z, 0\rangle) \cap \text{int}_{\mathcal{U}_A\mathcal{H}} D$, thus $\langle u, 0\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}} D$. Then there exists $\delta' < \delta$ such that $U_{\delta'}(\langle u, 0\rangle) \subseteq D$ if $\langle u, 0\rangle \in A$ or $C_{\delta'}(\langle u, 0\rangle) \subseteq D$, if $\langle u, 0\rangle \in L \setminus A$. In both cases, we get a contradiction to $(\star)$, because $U_{\delta'}(\langle u, 0\rangle) \cap K \subseteq C_r(\langle x, 0\rangle) \subseteq X \setminus D$ and $C_{\delta'}(\langle u, 0\rangle) \cap K \subseteq C_r(\langle x, 0\rangle) \subseteq X \setminus D$.

Thus $X \setminus D$ is open in the usual metric topology. Therefore, D is a closed set in $(X, \mathcal{U})$.

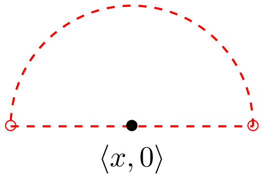
Lemma 2. *Let D be any non-empty closed domain in $(X, \mathcal{U}_A\mathcal{H})$, then* $(\text{int}_{\mathcal{U}_A\mathcal{H}} D) \cap (K \cup A) = (\text{int}_{\mathcal{U}} D) \cap (K \cup A)$.

Proof. Let $\langle u, v\rangle \in K \cup A$.

$\langle u, v\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}} D$ if and only if there exists $r > 0$ such that $U_r(\langle u, v\rangle) \subseteq D$ if and only if $\langle u, v\rangle \in \text{int}_{\mathcal{U}} D$

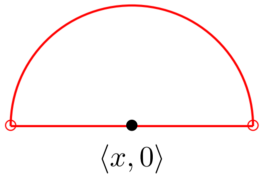
Lemma 3. *Let D be any non-empty closed domain in $(X, \mathcal{U}_A\mathcal{H})$, and $\langle x,0\rangle \in (L\setminus A)\cap (\text{int}_{\mathcal{U}_A\mathcal{H}}D)$, then there exists $r>0$ such that $U_r(\langle x,0\rangle)\subseteq D$.*

Proof. Since $\langle x,0\rangle \in L\setminus A$ and $\langle x,0\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}}D$, then there exists $r>0$ such that $C_r(\langle x,0\rangle)\subseteq D$



$\subseteq D$

Now $\overline{C_r(\langle x,0\rangle)}^{\mathcal{U}_A\mathcal{H}} \subseteq \overline{D}^{\mathcal{U}_A\mathcal{H}} = D$ as D is closed.



$\subseteq D$

But $U_r(\langle x,0\rangle)\subseteq \overline{C_r(\langle x,0\rangle)}^{\mathcal{U}_A\mathcal{H}} \subseteq D$.

Theorem 4. *Let D be any non-empty closed domain in $(X, \mathcal{U}_A\mathcal{H})$, then D is a closed domain in $(X, \mathcal{U})$.*

Proof. Assume $D=\overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}} \neq \emptyset$, we show $D=\overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$.Since $\text{int}_{\mathcal{U}}D\subseteq D$, then $\overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}\subseteq \overline{D}^{\mathcal{U}}=D$, by Lemma 1.
Now, we show $D\subseteq \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$. Let $\langle x,y\rangle\in D$ arbitrary. If $\langle x,y\rangle \in \text{int}_{\mathcal{U}}D$, then clearly $\langle x,y\rangle \in \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$. So, assume that $\langle x,y\rangle \in D\setminus \text{int}_{\mathcal{U}}D$. To show $\langle x,y\rangle \in \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$ we have to show that for all $r>0$, we have $U_r(\langle x,y\rangle)\cap \text{int}_{\mathcal{U}}D\neq\emptyset$

Case 1: $\langle x,y\rangle\in K\cup A$.

Let $r>0$ be arbitrary, we have $U_r(\langle x,y\rangle)\cap \text{int}_{\mathcal{U}_A\mathcal{H}}D\neq\emptyset$. By Lemma 2, we have $(\text{int}_{\mathcal{U}_A\mathcal{H}}D)\cap(K\cup A)=(\text{int}_{\mathcal{U}}D)\cap(K\cup A)$, since $\langle x,y\rangle\in K\cup A$, then $U_r(\langle x,y\rangle)\cap \text{int}_{\mathcal{U}}D\neq\emptyset$.

Case 2: $\langle x,y\rangle\in L\setminus A$, then $y=0$. We want to show that for any $r>0$ we have $U_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}}D\neq\emptyset$. Suppose that there exists $r>0$ such that

$$U_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}}D=\emptyset\ldots(\star).$$

Since $C_r(\langle x,0\rangle)\subseteq U_r(\langle x,0\rangle)$, then $C_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}}D=\emptyset$.

Claim: $C_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}_A\mathcal{H}}D=\emptyset$.

If we prove the claim, we get $\langle x,0\rangle\in D\setminus \text{int}_{\mathcal{U}_A\mathcal{H}}D$, but $\langle x,0\rangle\notin \overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}}$, thus D is not a closed domain in $(X, \mathcal{U}_A\mathcal{H})$, which is a contradiction.

Proof of Claim: Suppose $C_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}_A\mathcal{H}}D \neq \emptyset$. Pick $\langle u,v\rangle \in C_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}_A\mathcal{H}}D$. If $v>0$, then $\langle u,v\rangle \in K$, since $C_r(\langle x,0\rangle) \subseteq U_r(\langle x,0\rangle)$, and $(\text{int}_{\mathcal{U}_A\mathcal{H}}D)\cap K = (\text{int}_{\mathcal{U}}D)\cap K$, then $\langle u,v\rangle \in U_r(\langle x,0\rangle)\cap \text{int}_{\mathcal{U}}D$, which is a contradicts $(\star)$. Then $v=0$, so $\langle u,v\rangle = \langle x,0\rangle$, hence $\langle x,0\rangle \in \text{int}_{\mathcal{U}_A\mathcal{H}}D$, therefore there exists $0<s<r$ such that $C_s(\langle x,0\rangle) \subseteq \text{int}_{\mathcal{U}_A\mathcal{H}}D \subset D$. By Lemma 3 $U_s(\langle x,0\rangle) \subseteq \text{int}_{\mathcal{U}_A\mathcal{H}}D$, then $\langle x,0\rangle \in \text{int}_{\mathcal{U}}D$, but $\langle x,0\rangle \in D\setminus \text{int}_{\mathcal{U}}D$, which is a contradiction. So, claim is proved.

Theorem 5. *Any H-spaces $(X, \mathcal{U}_A\mathcal{H})$ is mildly normal.*

Proof. Let E and F be any tow disjoint closed domains in $(X, \mathcal{U}_A\mathcal{H})$, by Theorem 4, E and F are closed domains in $(X,\mathcal{U})$, and $(X,\mathcal{U})$ is mildly normal, then there exists U and V in $\mathcal{U}$ such that $E\subset U$, $F\subset V$ and $U\cap V=\emptyset$. Since $\mathcal{U}\subseteq \mathcal{U}_A\mathcal{H}$, then U and V are both open in H-space, thus $(X,\mathcal{U}_A\mathcal{H})$ is mildly normal.

Theorem 6. *Any closed domain in usual metric space $(X, \mathcal{U})$ is closed domain in H-spaces $(X,\mathcal{U}_A\mathcal{H})$.*

Proof. Let D be a closed domain in usual metric space, we want to show that $\overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}} = \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$. We have $\overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}} \subseteq \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$.

Claim : $\overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}} \subseteq \overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}}$.

Proof of claim: Let $\langle x,y\rangle \in \overline{\text{int}_{\mathcal{U}}D}^{\mathcal{U}}$ be arbitrary, then for all $r>0$ we have $U_r(\langle x,y\rangle)\cap \text{int}_{\mathcal{U}}D \neq \emptyset$, by Lemma 2 we have $U_r(\langle x,y\rangle)\cap \text{int}_{\mathcal{U}_A\mathcal{H}}D \neq \emptyset$, therefore $\langle x,y\rangle \in \overline{\text{int}_{\mathcal{U}_A\mathcal{H}}D}^{\mathcal{U}_A\mathcal{H}}$.

Recall that a space X is *semiregular* if it has a base consisting of open domains, [1, 1.7.8 (a)], see also [9]. Now, let $(X, \mathcal{T})$ be a T_2 space. Generate a coarser topology $\mathcal{T}'\subseteq \mathcal{T}$ on X by the base of all open domains in $(X, \mathcal{T})$. Then $(X, \mathcal{T}')$ is semiregular and the two spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ have the same open domains. $(X, \mathcal{T}')$ is called the *semiregularization* of $(X, \mathcal{T})$ [1, 1.7.8 (b)], see also [9]. Since any closed domain in an H-space $(X, \mathcal{U}_A\mathcal{H})$ is a closed domain in the usual metric space $(X, \mathcal{U})$, see Theorem 4 and Theorem 6, we conclude that any open domain in an H-space $(X,\mathcal{U}_A\mathcal{H})$ is an open domain in the usual metric space $(X, \mathcal{U})$. Thus the semiregularization of an H-space $(X, \mathcal{U}_A\mathcal{H})$ is $(X, \mathcal{U})$.

We can conclude more interesting result from Theorem 4. Since an H-space $(X, \mathcal{U}_A\mathcal{H})$ and the usual metric space $(X, \mathcal{U})$ are having the same closed domain, then any H-space $(X, \mathcal{U}_A\mathcal{H})$ is κ-metrizable. Let us recall the definitions. Denote the family of all closed domains in X by $R[X]$. A *κ-metric* on a T_3 space is a non-negative real-valued function $\phi(x,C)$ of two variables, $x\in X$ and $C\in R[X]$, with the requirements:

(i) (K1) (*membership axiom*)
For every $x\in X$ and $C\in R[X]$, $\phi(x,C)=0 \Leftrightarrow x\in C$.

(ii) (K2) (*monotonicity*)
If $C,C'\in R[X]$ and $C\subset C'$, then $\phi(x,C)\geq \phi(x,C')$, for all $x\in X$.

(iii) (K3) (*continuity*)
For every $C\in R[x]$, $\phi(x,C)$ is continuous in x.

(iv) (K4) (*union axiom*)

$$\phi\left(x, \overline{\bigcup_{\alpha\in\Lambda} C_\alpha}\right) = \inf\{\phi(x, C_\alpha) : \alpha \in \Lambda\}$$

For every increasing transfinite sequence $\{C_\alpha \in R[X] : \alpha \in \Lambda\}$.

A space on which there exists a κ-metric on it is said to be *κ-metrizable* [11]. The concept of κ-metrizability is a generalization of metrizability, in the sense that every metric is a κ-metric, and every metrizable space is κ-metrizable. In [11], Ščepin proved that "*any κ-metrizable space is mildly normal*", which gives another proof that any H-spaces $(X,\mathcal{U}_A\mathcal{H})$ is mildly normal. Also, in [11], Ščepin proved that "*κ-metrizability is countable multiplicative*", we get the following corollary.

Corollary 1. *If Λ is countable and for each $\alpha \in \Lambda$, A_α is a non-empty proper subset of the x-axis L, then $\prod_{\alpha\in\Lambda}(X,\mathcal{U}_{A_\alpha}\mathcal{H})$ is κ-metrizable, hence mildly normal.*

Here are some open problems about this H-space.

(i) Is any H-space $(X,\mathcal{U}_A\mathcal{H})$ almost normal, [6] ?

(ii) Is any H-space $(X,\mathcal{U}_A\mathcal{H})$ CC-normal, [3] ?

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