EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 4, 2021, 1237-1248
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Chatterjee and Extension of Chatterjee Fixed Point Theorems on Operators on Hilbert C*-Modules

Rashwan. A. Rashwan[1,*], Howida Adel AlFran[2], Asmaa Fangary[3], Saleh Omran[3]

[1] *Department of Mathematics, Faculty of Science, Assuit University, Assuit, Egypt*
[2] *Department of Mathematics, Al-Leith University College, Umm Al Qura University, Kingdom of Saudi Arabia*
[3] *Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt*

Abstract. In this paper we consider some fixed point theorem (such as Chatterjee and extension of Chatterjee) in operators of Hilbert C^*-modules, based on a definition of valued operator Hilbert C*-modules normed space. Also We give some examples to clear our definitions.

2020 Mathematics Subject Classifications: 47H10, 46L05, 46L08

Key Words and Phrases: Fixed Point Theorems, C^*-algebra, Operators on Hilbert C^*-modules

1. Introduction

Hilbert C^*-modules consider a mathematical objects where generalize the notion of a Hilbert space by allowing the inner product to take values in a (commutative, unital) C^*-algebra rather than in the field of complex numbers. Hilbert C^*-modules were first introduced in 1953 by Kaplansky [5]. Later, the theory was developed independently by Paschke [**12**] and Rieffel [**16**] where the research on Hilbert C^*-modules began in the 70's in the work of the induced representations of C^*-algebras by M. A. Rieffel [**16**] also Kasparov [**6**] introduced the definition of KK-theory by using Hilbert C^*-modules

C*-algebra is a main subject in the functional analysis and the operator theory which play fundamental role in noncommutative geometry and theoretical physics, especially the quantum mechanics

Ma and et al. [**20**], introduced the concept of C^*-algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i4.4071

Email addresses: rashwan10@gmail.com (R. A. Rashwan), hafran@uqu.edu.sa (H. Adel AlFran), Asmaa.fangary44@yahoo.com (A. Fangary), salehomran@yahoo.com (S. Omran)

of the set of real numbers. They presented some fixed point results for mapping under contractive or expansive conditions in these spaces. Later, Ma and et al. **[21]**, introduced the concept of C^*-algebra-valued b-metric spaces and proved some fixed point theorems such as Banach and Kannan type fixed point theorems.For other results on C^*-algebra-valued b-metric spaces and C^*-algebra-valued-metric spaces, see **[4, 13, 15, 18, 22]**.
An element $x \in \mathbb{A}$ is a positive element, denote it by $x \succeq 0$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, +\infty]$, where $\sigma(x)$ is the spectrum of x and $\mathbb{A}_h = \{x \in \mathbb{A} : x^* = x\}$. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq 0$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq 0\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

2. Preliminaries

In this section, we begin with some basic notations and definition C^*-algebra and fixed point theory that will be very important and useful in the sequal.

Definition 1. **[9]** *A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|ab\| \leq \|a\|\|b\| \quad (\text{for all} \quad a, b \in \mathbb{A})$. A C^*algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2 \quad (\text{for all} \quad a \in \mathbb{A})$.*

Definition 2. **[9]** *An element $a \in \mathbb{A}$ is positive element, if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}^+$, where $\sigma(a)$ is the spectrum of a, we denote $\mathbb{A}_+$ the set of all positive element in $\mathbb{A}$.*

Definition 3. **[8, 19]** *A pre-Hilbert C^*-module $\mathcal{E}$ over a C^*-algebra $\mathbb{A}$, is a right $\mathbb{A}$-module together with an $\mathbb{A}$-valued inner product $< ., . >: \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{A}$ satisfying the conditions:*

(1) $< x, x > \succeq 0$ *for all* $x \in \mathcal{E}$;

(2) $< x, x >= 0$ *if and only if* $x = 0$;

(3) $< x, \alpha y + \beta z >= \alpha < x, y > +\beta < x, z >$ *for all* $x, y, z \in \mathcal{E}, \alpha, \beta \in \mathbb{C}$;

(4) $< x, ya >=< x, y > a$ *for all* $x, y \in \mathcal{E}, a \in \mathbb{A}$;

(5) $< x, y >^* =< y, x >$ *for all* $x, y \in \mathcal{E}$.

Definition 4. **[8]** *The norm of an element $e \in \mathcal{E}$ is defined as*

$$\|x\|_{\mathcal{E}} := \sqrt{\| < x, x > \|_{\mathbb{R}}}, \text{ where } \|.\|_{\mathbb{R}} \text{ is the } \mathbb{R}\text{-valued norm.}$$

If a pre-Hilbert $\mathbb{A}$ -module is complete with respect to its norm,it is said to be a Hilbert $\mathbb{A}$ -module.

Example 1. *Every C^*-algebra $\mathbb{A}$ is a Hilbert $\mathbb{A}$-module over itself when equipped with the $\mathbb{A}$-valued inner product given simply by*

$$< a, b >= a^*b, \ (a, b \in \mathbb{A}).$$

Definition 5. **[19]** *Let $\mathcal{E}$ be a Hilbert $\mathbb{A}$-module. A map $T : \mathcal{E} \longrightarrow \mathcal{E}$ is said to be adjointable if there exists a map $T^* : \mathcal{E} \longrightarrow \mathcal{E}$ satisfying*

$$< x, Ty >=< T^{*}x, y >$$

for all $x, y \in \mathcal{E}$.

Definition 6. **[3]** *An element* $T \in l(\mathcal{E})$ *is positive if for every* $x \in \mathcal{E}$ *we have* $< Tx, x >_{\mathbb{A}} \succeq 0$ *and we write it by* $T \succeq 0$ *and we denote the set* $l(\mathcal{E})_{+} = \{T \in \mathcal{E} \quad ; \quad T \succeq 0\}$, *we define a partial ordering relation on* $l(\mathcal{E})_{+}$ *as*

$$\text{if } T_1, T_2 \in l(\mathcal{E}), T_1 \preceq_{l(\mathcal{E})} T_2 \text{ if and only if } T_2 - T_1 \in l(\mathcal{E})_{+}$$

Definition 7. **[3]** $l(\mathcal{E}) = \{T : \mathcal{E} \longrightarrow \mathcal{E}\}$ *is the set of all adjiontable linear operators with* $\|T\| = sup\{\|Tx\|_{\mathcal{E}}; \|x\|_{\mathcal{E}} \leq 1\}$ *is a* C^{*}*-algebra.*

3. Main Results

Definition 8. *Let* $l(\mathcal{E})_{+}$ *be a subset of* $l(\mathcal{E})$. $l(\mathcal{E})_{+}$ *is called Cone of* $l(\mathcal{E})$ *if and only if :*

(1) $l(\mathcal{E})_{+} \cap (-l(\mathcal{E})_{+}) = \{0_{l(\mathcal{E})}\}$, *(*$0_{l(\mathcal{E})}$ *is the zero vector);*

(2) $l(\mathcal{E})_{+}$ *is closed in* $l(\mathcal{E})$;

(3) $Ta + Sb \in l(\mathcal{E})_{+}$; $aT + bS \in l(\mathcal{E})_{+}$ $a, b \in A$, $T\lambda + S\beta \in l(\mathcal{E})_{+}$: $\lambda, \beta \in \mathbb{C}$;

(4) $l(\mathcal{E})_{+} \cdot l(\mathcal{E})_{+} \subseteq l(\mathcal{E})_{+}$.

Definition 9. *An* $l(\mathcal{E})$*-valued metric on a set* X *is a function* $d_{l(\mathcal{E})} : X \times X \longrightarrow l(\mathcal{E})$ *such that for all* x, y *and* z *in* X *the following conditions are hold:*

(1) $d_{l(\mathcal{E})}(x, y) \succeq 0$;

(2) $d_{l(\mathcal{E})}(x, y) = 0$ *if and only if* $x = y$;

(3) $d_{l(\mathcal{E})}(x, y) = d_{l(\mathcal{E})}(y, x)$;

(4) $d_{l(\mathcal{E})}(x, y) \preceq d_{l(\mathcal{E})}(x, z) + d_{l(\mathcal{E})}(z, y)$.

Then the triple $(X, l(\mathcal{E}), d_{l(\mathcal{E})})$ *is called an* $l(\mathcal{E})$*-valued metric space.*

Definition 10. **[20]** *Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \longrightarrow \mathbb{A}$ *satisfies:*

(1) $0_{\mathbb{A}} \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0_{\mathbb{A}}$ *if and only if* $x = y$.

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(3) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d *is called a* C^{*}*-algebra-valued metric on* X *and* $(X, \mathbb{A}, d)$ *is a* C^{*}*-algebra-valued metric space.*

Definition 11. *Let* $(X, l(\mathcal{E}), d_{l(\mathcal{E})})$ *be an* $l(\mathcal{E})$*- valued metric spacs. Suppose that* $x_n \subset X$ *and* $x \in X$ *If for any* $\varepsilon_{l(\mathcal{E})} \succ 0_{l(\mathcal{E})}$ *(where* $0_{l(\mathcal{E})}$ *is the zero element in* $l(\mathcal{E})$ *) there exists* $N \in \mathbb{N}$ *such that for all* $n > N$*,* $d_{l(\mathcal{E})}(x_n, x) \preceq \varepsilon_{l(\mathcal{E})}$*, then* $\{x_n\}$ *is said to be converge with respect to* $l(\mathcal{E})$*, and* $\{x_n\}$ *converges to* x *and* x *is the limit of* $\{x_n\}$*. We denote it by* $\lim_{n \longrightarrow +\infty} \{x_n\} = x$ *.*
If for any $\varepsilon_{l(\mathcal{E})} \succ 0_{l(\mathcal{E})}$ *there exists* $N \in \mathbb{N}$ *such that for all* $n, m > N$*,* $d(x_n, x_m) \preceq \varepsilon_{l(\mathcal{E})}$*, then* $\{x_n\}$ *is said to be a Cauchy with respect to* $l(\mathcal{E})$*.*
We say $(X, l(\mathcal{E}), d_{l(\mathcal{E})})$ *is a complete* $l(\mathcal{E})$*- valued metric spacs if every Cauchy sequence with respect to* $l(\mathcal{E})$ *is convergent.*

Lemma 1. *A sequence* $x_n \subset X$ *is convergence if* $\|x_n\| \longrightarrow 0 \quad forall \quad n > N$ *such that* $N \in \mathbb{N}$*.*

Example 2. *Let* $X = \mathbb{A}^{\oplus n}, \mathcal{E} = \mathbb{A}^{\oplus n}$ *and* $l(\mathcal{E}) = \{T : \mathbb{A}^{\oplus n} \longrightarrow \mathbb{A}^{\oplus n} \ : T(a_1, a_2, ..., a_n) = (Ta_1, Ta_2, ..., Ta_n)\}$*. Define*

$$d((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = (\|Ta_1 - Tb_1\|_{\mathbb{R}}, \|Ta_2 - Tb_2\|_{\mathbb{R}}, ..., \|Ta_n - Tb_n\|_{\mathbb{R}}) I_{\mathbb{A}},$$

where $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \in \mathbb{A}^{\oplus n}$ *and* $I_{\mathbb{A}}$ *is the identity element of* $\mathbb{A}$ *. It is easy to verify that* $d_{l(\mathcal{E})}$ *is an* $l(\mathcal{E})$ *-valued metric space and* $(X, \mathbb{A}^{\oplus n}, d_{l(\mathcal{E})})$ *is a complete* $l(\mathcal{E})$ *-valued metric space, since* $\mathbb{A}$ *is complete.*

Definition 12. *let* $(X, l(\mathcal{E}))$ *is an* $l(\mathcal{E})$*-metric space, we define the open ball on* X

$$B_{l(\mathcal{E})}(a, \epsilon_{l(\mathcal{E})}) = \{x \in X; \|x - a\| \prec \epsilon_{l(\mathcal{E})}\}$$

Definition 13. *Suppose that* $(X, d_{l(\mathcal{E})})$ *is* $l(\mathcal{E})$*-metric space, let* $x \in X$ *then a neighhborhood of* x *is any set containing* $B_{l(\mathcal{E})}(x, \epsilon_{l(\mathcal{E})})$ *for some* $\epsilon_{l(\mathcal{E})} \succ 0_{l(\mathcal{E})}$*.*

Definition 14. *Suppose that* $(X, d_{l(\mathcal{E})})$ *is* $l(\mathcal{E})$*-metric space, a subset* $U \subset X$ *is open if for every* $x \in U$ *there exist an open ball* $B_{l(\mathcal{E})}(a, \epsilon_{l(\mathcal{E})})$ *such that* $x \in B_{l(\mathcal{E})}(x, \epsilon_{l(\mathcal{E})}) \subset U$*.*

Motivaied by the idea in **[7]**,**[17]**,**[9]**, we give the following definations.

Definition 15. *Let* X *be vector space, if the function* $\|.\|_{l(\mathcal{E})} : X \longrightarrow l(\mathcal{E})$ *has the following properties:*

(1) $\|x\|_{l(\mathcal{E})} \succeq 0$ *i.e* $\|x\|_{l(\mathcal{E})}$ *is a positive operator,* $\|x\|_{l(\mathcal{E})} = 0$ *if and only if* $x = 0$*;*

(2) $\|\lambda x\|_{l(\mathcal{E})} = |\lambda| \|x\|_{l(\mathcal{E})}$ *;* $\lambda \in \mathbb{C}$*;*

(3) $\|x + y\|_{l(\mathcal{E})} \preceq \|x\|_{l(\mathcal{E})} + \|y\|_{l(\mathcal{E})}$*.*

Then $\|.\|$ *is said to be* $l(\mathcal{E})$*-valued norm defined on* X*, and* $(X, \|.\|)$ *is said to be* $l(\mathcal{E})$*-valued normed* $l(\mathcal{E})$ *space.*
Also we will set the relation between $l(\mathcal{E})$*-valued metric space and* $l(\mathcal{E})$*-valued normed space as follow* $\quad d_{l(\mathcal{E})}(x, y) = \|x - y\|_{l(\mathcal{E})}$*.*

Definition 16. *Let X be a vector space over a field $(F = \mathbb{C}, \mathbb{R})$ we say that X is a right $l(\mathcal{E})$-vector space if satisfy:*

(1) $(x + y)T = xT + yT$;

(3) $x(T_1 + T_2) = xT_1 + xT_2$;

(3) $(xS)T = x(ST)$.

Where $x, y \in X$ and $S, T \in l(\mathcal{E})$.

Lemma 3.2 Let X be a right $l(\mathcal{E})$-vector space then,

$$\|xT\|_{l(\mathcal{E})} \preceq \|x\|\|T\|_{l(\mathcal{E})}.$$

Proof. $\|xT\|^2 = \sup_{\|x\|=1}\{< xT, xT >, x \in \mathcal{E}\} \leq \|x\|\|T\|_{l(\mathcal{E})}$.

Definition 17. *Let $\mathbb{A}$ be C^*-algebra,and $l(\mathcal{E})$ be an $l(\mathcal{E})$-normed spac. We say that $l(\mathcal{E})$ is right $\mathbb{A}$-module if the mapping is right module multiplication $(a, T) \longmapsto xa$ of $\mathbb{A} \times l(\mathcal{E}) \longrightarrow l(\mathcal{E})$ such that the following axioms are satisfied:*

(1) *For each fixed $a \in \mathbb{A}$ the map $(a, T) \longrightarrow Ta$ is linear on $l(\mathcal{E})$: $T \in l(\mathcal{E})$;*

(2) *For each fixed $T \in l(\mathcal{E})$ the map $(a, T) \longrightarrow Ta$ is linear on $\mathbb{A}$;*

(3) *For all $a_1, a_2 \in \mathbb{A}$ and all $T \in l(\mathcal{E})$ we have that $(Ta_1)a_2 = T(a_1a_2)$.*

Example 3. *If we define the norm $\|x\|_{l(\mathcal{E})} = \|x\|I_{l(\mathcal{E})}$ (where $I_{l(\mathcal{E})}$ is the identity operator of $l(\mathcal{E})$) then we have that $l(\mathcal{E})$ with this norm is $l(\mathcal{E})$-norm.*

Lemma 2. *If T is positive if and only if T^* is positive.*

Proof. Let $* : \mathbb{A} \longrightarrow \mathbb{A}$ is $*$-homomorphism.

if T^* is positive implies $< T^*x, x >\succeq 0$ implies $< x, Tx >\succeq 0$ implies $< x, Tx >^*\succeq 0$ implies $< Tx, x >\succeq 0$ implies T is positive.

$\Longleftarrow$ if T is positive implies $< Tx, x >\succeq 0$ implies $< x, T^*x >\succeq 0$ implies $< x, T^*x >^*\succeq 0$ implies $< T^*x, x >\succeq 0$ implies $T^* \succeq 0$ implies T^* is positive.

Lemma 3. *If S is positive operator then for any operator T implies T^*ST is positive operator.*

Proof. Since $S \succeq 0$, we can write $S = R^*R$, for any $R \in (l_{\mathcal{E}})$ implies $T^*(R^*R)T = (T^*R^*)(RT) = (RT)^*(RT) \succeq 0$

Definition 18. *A sequence $\{x_n\}$ in X is said to be convergent if for every $\epsilon > 0$, there is a natural number N such that for $n > N$ we have*

$\|x_n - x\| \preceq_{l(\mathcal{E})} \epsilon I_{l(\mathcal{E})}$ *(where* $I_{l(\mathcal{E})}$ *the identity operator of* $l(\mathcal{E})$ *).*

Definition 19. *A sequence* $\{x_n\}$ *in* X *is said to be a Cuachy sequence if for every* $\epsilon > 0$*, there is a natural number* N *such that for* $n, m > N$ *we have*

$$\|x_n - x_m\| \preceq_{l(\mathcal{E})} \epsilon I_{l(\mathcal{E})}.$$

Lemma 4. *A sequence* $\{x_n\}$ *in* X *is convergence in* X *if* $\|x_n\|_{\mathbb{R}} \longrightarrow 0$ *at* $n \longrightarrow +\infty$*.*

Proof. Since in $l_(\mathcal{E})$- valued metric spacs. We say that a sequance $x_n \subset X$ converges to $x \in X$ If for any $\varepsilon_{l(\mathcal{E})} \succ 0_{l(\mathcal{E})}$ (where $0_{l(\mathcal{E})}$ is the zero element in $l(\mathcal{E})$) there exists $N \in \mathbb{N}$ such that for all $n > N$, $d_{l(\mathcal{E})}(x_n, x) \preceq \varepsilon_{l(\mathcal{E})}$, then this implies $\|d_{l(\mathcal{E})}(x_n, x)\|_{\mathbb{R}} < \mathcal{E}, \quad \mathcal{E} \in \mathbb{R}$.

Lemma 5. **[2, 9]** *Suppose that* $\mathbb{A}$ *is a unital* C^**-algebra with a unit* I*:*

(1) *for any* $x \in \mathbb{A}_+$ *we have* $x \preceq I$ *if and only if* $\|x\| \leq 1$*;*

(2) *If* $a \in \mathbb{A}_+$ *with* $\|a\| < \frac{1}{2}$*, then* $I - a$ *is invertable and* $\|a(I-a)^{-1}\| < 1$*;*

(3) *suppose that* $a, b \in \mathbb{A}$ *with* $a, b \succeq 0$ *and* $ab = ba$*,then* $ab \succeq 0$*.*

(4) *by* $\acute{\mathbb{A}}$ *we denote the set* $\{a \in \mathbb{A} : ab = ba \text{ forall } b \in \mathbb{A}\}$ *Let* $a \in \acute{\mathbb{A}}$*, if* $b, c \in \mathbb{A}$ *with* $b \succeq c \succeq 0$

$(I-a)^{-1}b \succeq (I-a)^{-1}c$.

Definition 20. *Let* $(X, l(\mathcal{E}), \|.\|_{l(\mathcal{E})})$ *be an* $l(\mathcal{E})$ *normed space. We call a mapping* $T : X \longrightarrow X$ *is* $l(\mathcal{E})$ *contractive mapping on* X *if there exists an* $M \in l(\mathcal{E})$ *with* $\|M\|_{l(\mathcal{E})} \leq 1$ *such that*

$$\|Tx - Ty\|_{l(\mathcal{E})} \preceq M^* \|x - y\|_{l(\mathcal{E})} M \ forall \quad x, y \in X.$$

Definition 21. *An* $l(\mathcal{E})$*- Banach space is a complete* $l(\mathcal{E})$*-normed space* $(X, \|.\|_{l(\mathcal{E})})$*.*

Many results on fixed point theorems have been extended from metric spaces to C^*-algebra valued metric spaces with different contraction conditions (see for example **[20]**,**[9]**,**[10]**,**[11]**,**[14]**)

Theorem 1. *(Chatterjee Type theorem* **[1]***) Let* $(X, l(\mathcal{E}), \|.\|_{l(\mathcal{E})})$ *be an* $l(\mathcal{E})$ *complete normed space and* $T : X \longrightarrow X$ *be a self mapping satisfy the following contraction condition*

$$\|Tx - Ty\|_{l(\mathcal{E})} \preceq \tfrac{M}{2}[\quad \|Tx - y\|_{l(\mathcal{E})} + \|Ty - x\|_{l(\mathcal{E})}],$$

where $M \in (l(\mathcal{E}))^{\grave{}}_+$ *with* $\|M\|_{l(\mathcal{E})} < 1$*, Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{+\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$

$$\|x_{n+1}-x_n\|_{l(\mathcal{E})} = \|Tx_n - Tx_{n-1}\|_{l(\mathcal{E})}$$
$$\preceq \frac{M}{2}[\quad \|Tx_n - x_{n-1}\|_{l(\mathcal{E})} + \|Tx_{n-1}-x_n\|_{l(\mathcal{E})} \quad]$$
$$= \frac{M}{2}[\quad \|x_{n+1} - x_{n-1}\|_{l(\mathcal{E})} + \|x_n - x_n\|_{l(\mathcal{E})} \quad]$$
$$\preceq \frac{M}{2}[\quad \|x_{n+1} - x_n\|_{l(\mathcal{E})} + \|x_n - x_{n-1}\|_{l(\mathcal{E})} \quad]$$
$$\preceq \frac{M}{2}\|x_{n+1}-x_n\|_{l(\mathcal{E})} + \frac{M}{2}\|x_n - x_{n-1}\|_{l(\mathcal{E})}.$$

Thus,

$(I_{l(\mathcal{E})} - \frac{M}{2})\|x_{n+1}-x_n\|_{l(\mathcal{E})} \preceq \frac{M}{2}\|x_n - x_{n-1}\|_{l(\mathcal{E})}$.
Since $M \in (l(\mathcal{E}))\grave{}_+$ with $\|\frac{M}{2}\|_{l(\mathcal{E})} \preceq \frac{1}{2}$, one have $(I_{l(\mathcal{E})} - \frac{M}{2})^{-1} \in (l(\mathcal{E}))\grave{}_+$, and furthermore $\frac{M}{2}(I_{l(\mathcal{E})} - \frac{M}{2})^{-1} \in (l(\mathcal{E}))\grave{}_+$ with $\|\frac{M}{2}(I_{l(\mathcal{E})} - \frac{M}{2})^{-1}\|_{l(\mathcal{E})} \leq 1$. Therefore,

$$\|x_{n+1}-x_n\|_{l(\mathcal{E})} \preceq (\frac{\frac{M}{2}}{I_{l(\mathcal{E})}-\frac{M}{2}})\|x_n - x_{n-1}\|_{l(\mathcal{E})}$$
$$\preceq (\frac{\frac{M}{2}}{I_{l(\mathcal{E})}-\frac{M}{2}})^2\|x_{n-1} - x_{n-2}\|_{l(\mathcal{E})}$$
$$\vdots$$
$$\preceq (\frac{\frac{M}{2}}{I_{l(\mathcal{E})}-\frac{M}{2}})^n\|x_1 - x_0\|_{l(\mathcal{E})}.$$

Let $t = \frac{M}{2}(I_{l(\mathcal{E})} - \frac{M}{2})^{-1}, B = \|x_1 - x_0\|_{l(\mathcal{E})}$.
Implies $\|x_{n+1}-x_n\|_{l(\mathcal{E})} \preceq t^n B$
For $n+1 > m$

$$\|x_{n+1}-x_m\|_{l(\mathcal{E})} \preceq \|x_{n+1}-x_n\|_{l(\mathcal{E})} + \|x_n - x_{n-1}\|_{l(\mathcal{E})} + \cdots + \|x_{m+1}-x_m\|_{l(\mathcal{E})}$$
$$\preceq t^n B + t^{n-1}B + \cdots + t^m B$$
$$\preceq (t^n + t^{n-1} + \cdots + t^m)B$$
$$= \sum_{k=m}^{n} t^k B$$
$$= \sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}}$$
$$= \sum_{k=m}^{n} B^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}}$$
$$= \sum_{k=m}^{n} (t^{\frac{k}{2}} B^{\frac{1}{2}})^*(t^{\frac{k}{2}} B^{\frac{1}{2}})$$
$$= \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2$$

$$\preceq \| \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2 \|_{l(\mathcal{E})} I_{l(\mathcal{E})}$$

$$\preceq \sum_{k=m}^{n} \|B^{\frac{1}{2}}\|_{l(\mathcal{E})}^2 \|t^{\frac{k}{2}}\|_{l(\mathcal{E})}^2 I_{l(\mathcal{E})}$$

$$= \|B\|_{l(\mathcal{E})} \sum_{k=m}^{n} \|t\|_{l(\mathcal{E})}^k I_{l(\mathcal{E})}$$

$$\preceq \|B\|_{l(\mathcal{E})} \frac{\|t\|_{l(\mathcal{E})}^m}{1-\|t\|_{l(\mathcal{E})}^m} I_{l(\mathcal{E})} \longrightarrow 0_{l(\mathcal{E})} (m \longrightarrow +\infty),$$

where $I_{l(\mathcal{E})}$ the unite element in $l(\mathcal{E})$, Therefore $\{x_n\}$ is a Cauchy sequence with respect to $l(\mathcal{E})$. By the completeness of$(X, l(\mathcal{E}), \|.\|_{l(\mathcal{E})})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = \lim_{n\longrightarrow+\infty} Tx_{n-1} = x$.
Since

$$\|Tx - x\|_{l(\mathcal{E})} \preceq \|Tx - Tx_n\|_{l(\mathcal{E})} + \|Tx_n - x\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{2}(\|Tx - x_n\|_{l(\mathcal{E})} + \|Tx_n - x\|_{l(\mathcal{E})}) + \|Tx_n - x\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{2}(\|Tx - x\|_{l(\mathcal{E})} + \|x - x_n\|_{l(\mathcal{E})} + \|Tx_n - x\|_{l(\mathcal{E})})$$

$$+\|Tx_n - x\|_{l(\mathcal{E})}$$

$$= \frac{M}{2}\|Tx - x\|_{l(\mathcal{E})} + \frac{M}{2}\|x - x_n\|_{l(\mathcal{E})} + \frac{M}{2}\|Tx_n - x\|_{l(\mathcal{E})}$$

$$+\|Tx_n - x\|_{l(\mathcal{E})}.$$

Implies $\|Tx - x\|_{l(\mathcal{E})} \preceq \frac{\frac{M}{2}}{I_{l(\mathcal{E})} - \frac{M}{2}} \|Tx_n - x\|_{l(\mathcal{E})} + \frac{\frac{M}{2}}{I_{l(\mathcal{E})} - \frac{M}{2}} \|x - x_n\|_{l(\mathcal{E})} + \frac{1}{I_{l(\mathcal{E})} - \frac{M}{2}} \|Tx_n - x\|_{l(\mathcal{E})}$

$\|Tx - x\|_{l(\mathcal{E})} \preceq \frac{\frac{M}{2}}{I_{l(\mathcal{E})} - \frac{M}{2}} \|x_{n+1} - x\|_{l(\mathcal{E})} + \frac{1}{I_{l(\mathcal{E})} - \frac{M}{2}} \|x_{n+1} - x\|_{l(\mathcal{E})} \longrightarrow 0 (n \longrightarrow +\infty)$

Implies $\|Tx - x\|_{l(\mathcal{E})} = 0$ implies $Tx = x$.
To prove the uniquness suppose that $y (\neq x)$ is another fixed point of T, then

$$0 \preceq \|x - y\|_{l(\mathcal{E})} = \|Tx - Ty\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{2}(\|Tx - y\|_{l(\mathcal{E})} + \|Ty - x\|_{l(\mathcal{E})})$$

Implies $\|x - y\|_{l(\mathcal{E})} \preceq \frac{\frac{M}{2}}{I_{l(\mathcal{E})} - \frac{M}{2}} \|x - y\|_{l(\mathcal{E})}$

Implies $\| \|x - y\|_{l(\mathcal{E})} \|_{l(\mathcal{E})} \preceq \| \frac{\frac{M}{2}}{I_{l(\mathcal{E})} - \frac{M}{2}} \|_{l(\mathcal{E})} \| \|x - y\|_{l(\mathcal{E})} \|_{l(\mathcal{E})} \prec \| \|x - y\|_{l(\mathcal{E})} \|_{l(\mathcal{E})}$

This means that
$\|x - y\|_{l(\mathcal{E})} = 0$ implies $x = y$.
Therefore the fixed point is unique.

Theorem 2. *(Extension of Chatterjee Type Theorem) Let $(X, l(\mathcal{E}), \|.\|_{l(\mathcal{E})})$ be an $l(\mathcal{E})$ complete normed space and $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition*

$$\|Tx - Ty\|_{l(\mathcal{E})} \preceq \frac{M}{3}[\quad \|x - y\|_{l(\mathcal{E})} + \|Tx - y\|_{l(\mathcal{E})} + \|Ty - x\|_{l(\mathcal{E})}],$$

where $M \in (l(\mathcal{E}))\grave{}_{+}$ with $\|M\|_{l(\mathcal{E})} < \frac{3}{4}$, Then T has a unique fixed point.

Proof. Le $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{+\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n$.

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathcal{E})} &= \|Tx_n - Tx_{n-1}\|_{l(\mathcal{E})} \\
&\preceq \frac{M}{3}[\|x_n - x_{n-1}\|_{l(\mathcal{E})} + \|Tx_n - x_{n-1}\|_{l(\mathcal{E})} + \|Tx_{n-1} - x_n\|_{l(\mathcal{E})}] \\
&= \frac{M}{3}[\|x_n - x_{n-1}\|_{l(\mathcal{E})} + \|x_{n+1} - x_{n-1}\|_{l(\mathcal{E})} + \|x_n - x_n\|_{l(\mathcal{E})}] \\
&\preceq \frac{M}{3}[\|x_n - x_{n-1}\|_{l(\mathcal{E})} + \|x_{n+1} - x_n\|_{l(\mathcal{E})} + \|x_n - x_{n-1}\|_{l(\mathcal{E})}] \\
&= \frac{M}{3}[\quad 2\|x_n - x_{n-1}\|_{l(\mathcal{E})} + \|x_{n+1} - x_n\|_{l(\mathcal{E})} \quad] \\
&= \frac{2M}{3} \quad \|x_n - x_{n-1}\|_{l(\mathcal{E})} + \frac{M}{3}\|x_{n+1} - x_n\|_{l(\mathcal{E})} \quad .
\end{aligned}$$

Thus,

$(I_{l(\mathcal{E})} - \frac{M}{3})\|x_{n+1} - x_n\|_{l(\mathcal{E})} \preceq \frac{2M}{3}\|x_n - x_{n-1}\|_{l(\mathcal{E})}$.

Since $M \in (l(\mathcal{E}))\grave{}_{+}$ with $\|\frac{M}{3}\|_{l(\mathcal{E})} \leq \frac{1}{4}$, one have $(I_{l(\mathcal{E})} - \frac{M}{3})^{-1} \in (l(\mathcal{E}))\grave{}_{+}$, and furthermore $\frac{M}{3}(I - \frac{M}{3})^{-1} \in (l(\mathcal{E}))\grave{}_{+}$ with $\|\frac{M}{3}(I_{l(\mathcal{E})} - \frac{M}{3})^{-1}\|_{l(\mathcal{E})} \leq \frac{1}{2}$,we have that $\|2(\frac{M}{3}(I_{l(\mathcal{E})} - \frac{M}{3})^{-1})\|_{l(\mathcal{E})} \leq 1$. Therefore,

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathcal{E})} &\preceq 2\Big(\frac{\frac{M}{3}}{I_{l(\mathcal{E})} - \frac{M}{3}}\Big)\|x_n - x_{n-1}\|_{l(\mathcal{E})} = t\|x_n - x_{n-1}\|_{l(\mathcal{E})} \\
&\preceq t^2\|x_{n-1} - x_{n-2}\|_{l(\mathcal{E})} \\
&\vdots \\
&\preceq t^n\|x_1 - x_0\|_{l(\mathcal{E})},
\end{aligned}$$

where $t = 2(\frac{M}{3}(I_{l(\mathcal{E})} - \frac{M}{3})^{-1})$.
For $n + 1 > m$.

$$\|x_{n+1}-x_m\|_{l(\mathcal{E})} \preceq \|x_{n+1}-x_n\|_{l(\mathcal{E})} + \|x_n - x_{n-1}\|_{l(\mathcal{E})} + \cdots + \|x_{m+1}-x_m\|_{l(\mathcal{E})}$$

$$\preceq (t^n + t^{n-1} + \cdots + t^m)\|x_1 - x_0\|_{l(\mathcal{E})}.$$

Let $B = \|x_1 - x_0\|_{l(\mathcal{E})}$

$$\Rightarrow \|x_{n+1}-x_m\|_{l(\mathcal{E})} = \sum_{k=m}^{n} t^k B$$

$$= \sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}}$$

$$= \sum_{k=m}^{n} B^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}}$$

$$= \sum_{k=m}^{n} (t^{\frac{k}{2}} B^{\frac{1}{2}})^* (t^{\frac{k}{2}} B^{\frac{1}{2}})$$

$$= \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2$$

$$\preceq \| \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2 \|_{l(\mathcal{E})} I_{l(\mathcal{E})}$$

$$\preceq \sum_{k=m}^{n} \|B^{\frac{1}{2}}\|^2_{l(\mathcal{E})} \|t^{\frac{k}{2}}\|^2_{l(\mathcal{E})} I_{l(\mathcal{E})}$$

$$= \|B\|_{l(\mathcal{E})} \sum_{k=m}^{n} \|t\|^k_{l(\mathcal{E})} I_{l(\mathcal{E})}$$

$$\preceq \|B\|_{l(\mathcal{E})} \frac{\|t\|^m_{l(\mathcal{E})}}{1-\|t\|^m_{l(\mathcal{E})}} I_{l(\mathcal{E})} \longrightarrow 0_{l(\mathcal{E})} (m \longrightarrow +\infty),$$

where $I_{l(\mathcal{E})}$ the unite element in $l(\mathcal{E})$, Therefore $\{x_n\}$ is a Cauchy sequence with respect to $l(\mathcal{E})$. By the completeness of$(X, l(\mathcal{E}), \|.\|_{l(\mathcal{E})})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = \lim_{n\longrightarrow+\infty} Tx_{n-1} = x$.
Since

$$\|Tx - x\|_{l(\mathcal{E})} \preceq \|Tx - Tx_n\|_{l(\mathcal{E})} + \|Tx_n - x\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{3}(\|x - x_n\|_{l(\mathcal{E})} + \|Tx - x_n\|_{l(\mathcal{E})} + \|Tx_n - x\|_{l(\mathcal{E})}) + \|Tx_n - x\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{3}(\|x - x_n\|_{l(\mathcal{E})} + \|Tx - x_n\|_{l(\mathcal{E})} + \|x_{n+1} - x\|_{l(\mathcal{E})}) + \|Tx_n - x\|_{l(\mathcal{E})}.$$

Implies $\|Tx - x\|_{l(\mathcal{E})} \preceq \frac{\frac{M}{3}}{I_{l(\mathcal{E})} - \frac{M}{3}}(2\|x - x_n\|_{l(\mathcal{E})} + \|x_{n+1} - x\|_{l(\mathcal{E})}) + \frac{1}{I_{l(\mathcal{E})} - \frac{M}{3}}\|x_{n+1} - x\|_{l(\mathcal{E})} \longrightarrow 0(at \quad n \longrightarrow +\infty)$.

Then this implies that $Tx = x$ i.e., x is fixed point of T .
To prove the uniquencess suppose that $y(\neq x)$ is another fixed point of T, then

$$0 \leq \|x - y\|_{l(\mathcal{E})} = \|Tx - Ty\|_{l(\mathcal{E})}$$

$$\preceq \frac{M}{3}(\|x-y\|_{l(\mathcal{E})}+\|Tx-y\|_{l(\mathcal{E})}+\|Ty-x\|_{l(\mathcal{E})})$$

$$\preceq M\|x-y\|_{l(\mathcal{E})},$$

Implies $0 \leq \|\|x-y\|\|_{l(\mathcal{E})} \leq \|M\|x-y\|\|_{l(\mathcal{E})} < \|\|x-y\|\|_{l(\mathcal{E})}$

This is contradiction implies $x = y$.

Therefore the fixed point is unique.

4. Conclusions

In this paper, we introduced the notions of metric space valued-operator of Hilbert C^*-module. We define some contraction mapping and prove some fixed point theorems (such as Chatterjee and extension of Chatterjee) for a self mappings T on the Banach space $l(\mathcal{E})$.

References

[1] Chatterjee, SK. Fixed-point theorems. C. R. Acad. Bulgare Sci. **25**, 727-730 (1972).

[2] Douglas, RG. Banach Algebra Techniques in Operator Theory. Springer, Berlin (1998).

[3] Gilbert helmberg. Introduction to spectral theory in Hilbert space. Technological University Eindhoven .

[4] Kadelburg et al, Z . Remarks on the paper "Fixed point theorems for cyclic contractions in C*-algebra-valued b-metric spaces. Adv. Oper. Theory. no. 1, 93-104,1 (2016).

[5] Kaplansky, I. Modules over operator algebras. Amer J. Math., **75**, 839-858, (1953).

[6] Kasparov, G.G. Hilbert C^*-modules. Theorems of Stinespring and Voiculescu, J. Operator Theory **4**, 133-150,(1980).

[7] Karapınar, E. Fixed point theorems in cone Banach spaces. Fixed Point Theory Appl. 2009 Article ID 609281, 9 pages, 2009, doi:10.1155/2009/609281

[8] Lance, E. C. Hilbert C^*-Modules. A Toolkit for Operator Algebraists. Cambridge, England: Cambridge University Press, 1995.

[9] Murphy, G. J. C^*-algebras and operator theory. Academic Press, 1990.

[10] Mustafa, R. Omran, S. Ngoc, Qn. Fixed point theory using ψ contractive mapping In C^*-Algebra valued b-metric space. Mathematics , 9, 92,(2021).

[11] Özer, Ö. Omran, S. Common fixed point In C^*-Algebra b-valued metric space. AIP Conference Proceedings 1773 (1) 050005(2016).

[12] Paschke, W. L. Inner product modules over B^*-algebras. Trans Amer. Math. Soc.**182** , 443-468,(1973).

[13] Priyobarta et al, N. Fixed point theorems on parametric A-metric space, American Journal of Applied Mathematics and Statistics, Vol.6, No. 1, 1-5, 2018.

[14] Qiaoling, X. Lining, J. Zhenhua, Ma. Common fixed point theorems in C^*-algebra-valued metric spaces[J]. Journal of Hubei Normal University(Natural Science), 2015.

[15] Radenović et al,S. Coupled fixed point theorems in C*-algebra-valued b-metric spaces. Scientific publications of the state University of Novi Pazar, Ser. A: Appl. Math. Inform. and Mech. vol. 9,81-90,1 (2017).

[16] Rieffel, M. A. Induced representations of C^*-algebras. Adv in Math.,**13**(2), 176-257,(1974).

[17] Turkoglu, D. Abuloha, M. and Abdeljawad, T. Some theorems and examples of cone Banach spaces. J. Comput. Anal. Appl. 12 **4**, 739-753, (2010).

[18] Vesna Todorčević, Harmonic Quasiconformmal Mappings and Hyperbolic Type Metrics, Springer Nature Switzerland AG 2019.

[19] Wegge-Olsen, N. E. K-Theory and C^*-Algebras. A Friendly Approach. Oxford, England: Oxford University Press, 1993.

[20] Zhenhua, Ma. Lining, J. Hongkai, S. C^*-Algebras-valued metric spaces and related fixed point theorems. Fixed Point Theory Appl. 2014, 206 (2014).

[21] Zhenhua, Ma. Lining, J. C^*-Algebras-valued b-metric spaces and related fixed point theorems. Fixed Point Theory Appl.1,1-12, (2015).

[22] Zoran Kadelburg, Stojan Radenović, Critical remarks on some recent fixed points results in C*-algebra-valued metric spaces, Fixed Point Theory Appl. (2016) 2016:53.