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On Eigenvectors of Nilpotent Lie Algebras of Linear Operators

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Abstract. We give a condition ensuring that the operators in a nilpotent Lie algebra of linear operators on a finite dimensional vector space have a common eigenvector.

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1. Introduction

Throughout this paper V is a vector space of positive dimension over a field f and $\gg$ is a nilpotent Lie algebra over f of linear operators on V. An element $u \in V$ is an *eigenvector for* $S \subset\gg$ if u is an eigenvector for every operator in S. If V has a basis $(e_1, \ldots, e_n)$ representing each element of $\gg$ by an upper triangular matrix, then e_1 is an eigenvector for $\gg$. Such a basis exists when f is algebraically closed and $\gg$ is solvable (Lie's Theorem), and also when every element of $\gg$ is a nilpotent operator (Engel's Theorem). Our results are further conditions guaranteeing existence of eigenvectors.

The minimal and characteristic polynomials of a linear operator A on V are denoted respectively by $\pi_A, \mu_A \in f[t]$ = the ring of polynomials over f. The cardinality of a set S is written $\#S$.

Let k be a Galois extension field of f of degree $d := [k : f]$, and define $\mathbf{M} \subset$ to be the additive monoid generated by zero and the prime divisors d.

Consider the conditions:

(C1) μ_A splits in k for every $A \in\gg$

(C2) $\dim V \notin \mathbf{M}$

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2. Results

Our main result is:

Theorem 1. *If(**C1**) and(**C2**) hold then* $\gg$ *has an eigenvector.*

The proof is preceded by some applications.

When(**C1**) holds, Theorem 1 shows that there is an eigenvector in every invariant subspace whose dimension is not in $\mathbf{M}$. This is exploited to yield the following two results:

Corollary 1. *If a nilpotent Lie algebra of linear operators on* n *does not have an eigenvector, every nontrivial invariant subspace has odd dimension.*

Proof. When f is the real field and k is the complex field , $\mathbf{M}$ consists of the positive even integers.

Corollary 2. *Let(**C1**) hold. Assume* $\gg$ *preserves a direct sum decomposition* $V = \oplus_i W_i$*, and let* $D \subset$ *denote the set of dimensions of the subspaces* W_i*.*

(i) If $\gg$ *does not have an eigenvector then* $D \subset \mathbf{M}$*.*

(ii) If $V' \subset V$ *is a maximal subspace spanned by eigenvectors of* $\gg$ *then* $\dim(V') \geq \#\{D \setminus \mathbf{M}\}$*.*

Proof. Assertion (i) follows from Theorem 1. To prove (ii) order the W_i so that $W_1, \ldots, W_m$ are the only summands whose dimensions are not in $\mathbf{M}$. For each $j \in \{1, \ldots, m\}$ we choose an eigenvector $e_j \in W_j$ by Theorem 1. The e_j are linearly independent and belong to V' by maximality of V', whence (ii).

Example 1. *Assume* $n \notin \mathbf{M}$ *and let* $\alpha \in f[t]$ *be a monic polynomial that splits in* $k[t]$*. Denote by* $A(\alpha)$ *the set of* $n \times n$ *matrices* T *over* f *such that* $\alpha(T) = 0$*. Then every pairwise commuting family* $T \subset A(\alpha)$ *has an eigenvector in* f^n*. This follows from Theorem 1 applied to the Lie algebra* $\gg$ *of linear operators on* f^n *generated by* T*. Being abelian,* $\gg$ *can be triangularized over* k*, hence(**C1**) holds.*

Example 2. *The assumption that* $n \in \mathbf{M}$ *is essential to Theorem 1. For instance, take* $f =$*,* $k =$*,* $V = 2$*. The abelian Lie algebra of* 2×2 *of real skew symmetric matrices. does not have an eigenvector in* 2*.*

Example 3. *The hypothesis of Theorem 1 cannot be weakened to* $\gg$ *being merely solvable. For a counterexample with* $f =, k =$*, take* $\gg$ *to be the solvable 3-dimensional real Lie algebra with basis* (X, U, V) *such that* $[X, U] = -V,\ [X, V] = U,\ [U, V] = 0$*.*

A Lie algebra β over f is *supersolvable* if the spectrum of the linear map $ad\ A : \beta \to \beta$ lies in f for all $A \in \beta$. If β is not supersolvable it need not have an eigenvector, as is shown by Example 3. We don't know if Theorem 1 extends to supersolvable Lie algebras, except for the following special case:

Theorem 2. *A supersolvable Lie algebra β of linear transformations of* 3 *has an eigenvector.*

Proof. Lacking an algebraic proof, we use a dynamical argument. Let $G \subset GL(3,)$ be the connected Lie subgroup having Lie algebra β. The natural action of G on the projective plane ¶2 of lines in 3 through the origin fixes some $L \in$ ¶2. This follows from supersolvability because dim(¶2) = 2, the action on ¶2 is effective and analytic, and the Euler characteristic of ¶2 is nonzero (Hirsch & Weinstein [1]). The nonzero points of L are eigenvectors for β.

2.1. Proof of Theorem 1

We rely on Jacobson's *Primary Decomposition Theorem* [2, II.4, Theorem 5]. This states that V has a $\gg$-invariant direct sum decomposition $\oplus V_i$ where each *primary component* V_i has the following property: For each $A \in\gg$ the minimal polynomial of $A|V_i$ is a prime power in $f[t]$.

Condition**(C2)** implies the dimension of some primary component is $\notin \mathbf{M}$. To prove Theorem 1 it therefore suffices to apply the following result to such a primary component:

Theorem 3. *Assume**(C1)** and**(C2)**. If π_A is a prime power in $f[t]$ for each $A \in\gg$ then the following hold:*

(a) $\pi_A(t) = (t - r_A)^n$, $r_A \in f$

(b) there is a basis putting $\gg$ in triangular form

Assertion (a) is equivalent to π_A having a root $r_A \in f$. Therefore (a) follows from:

Lemma 1. *Let $\alpha \in f[t]$ be a polynomial of degree n that splits in $k[t]$. If $n \notin \mathbf{M}$ then α has a root in f, and the sum of the multiplicities of such roots is $\notin \mathbf{M}$.*

Proof. Let $R \subset k$ denote the set of roots of π, and $R_j \subset R$ the set of roots of multiplicity j.

The Galois group Γ has order $[k : f]$ and acts on R by permutations. The cardinality of each orbit divides $[k : f]$, and $R \cap f$ is the set of fixed points of this action.

Each R_j is a union of orbits, as is $R_j \backslash f$. It follows that $\#(R_j \setminus f) \in \mathbf{M}$.

Let $k \leq n$ denote the sum of the multiplicities of the roots that are not in f. Then

$$k = \sum_{j=2}^{n} j \cdot \#(R_j \setminus f)$$

Therefore $k \in \mathbf{M}$ because $\mathbf{M}$ is closed under addition.

By hypothesis $n \notin \mathbf{M}$, hence $n - k \notin \mathbf{M}$ and $n - k > 0$. As $n - k$ is the sum of the multiplicities of the roots in f, the conclusion follows.

Now that (a) of Theorem 3 is proved, assertion (b) is a consequence of the following result:

Lemma 2. *Let be a nilpotent Lie algebra of linear operators on* V*. Assume that for all* $A \in$ *there exists* $r_A \in f$ *such that* $\pi_A(t) = (t - r_A)^n$*.*

Then V *has a basis putting in triangular form.*

Proof. Every $A \in$ can be written uniquely as $r_A I + N_A$ with N_A nilpotent and I the identity map of V. It is easy to see that the set comprising the N_A is closed under commutator brackets. Therefore V has a basis triangularizing all the N_A (Jacobson [2, II.2, Theorem 1′]), and such a basis triangularizes .

This completes the proof of Theorem 1.

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