On Restrained Strong Resolving Domination in Graphs

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Abstract. A set $S \subseteq V(G)$ is a *restrained strong resolving dominating set* in G if S is a strong resolving dominating set in G and $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The *restrained strong resolving domination number* of G, denoted by $\gamma_{rsR}(G)$, is the smallest cardinality of a restrained strong resolving dominating set in G. In this paper, we present characterizations of the restrained strong resolving dominating sets in the join, corona and lexicographic product of two graphs and determine the exact value of the restrained strong resolving domination number of each of these graphs.

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1. Introduction

Domination in graphs was first introduced by C. Berge in 1958 [1]. There are many studies involving domination and its variations. Slater [8] introduced and studied the concept of resolving set. In 2003, Robert Brigham et al. [18] linked the concepts of resolving and domination. In their article, they defined a resolving dominating set as a set that is both resolving and dominating. Resolving sets and resolving dominating sets were further studied in [2, 3]. Oellermann, O. R, and Peters-Fransen, J. [6] introduced and studied the concept of strong resolving set.

Domke et. al [7] introduced and investigated the concept of restrained domination in graphs.

Khuller, et. al. [11] introduced the concept of metric dimension and this has grown to become an interesting topic in graph theory. In line with this, Sebo and Tannier [13] introduced the concept of strong metric dimension, a concept which is more restrictive than

metric dimension. After its introduction, Slater in [8] studied it and discovered its usefulness when working with the United States sonar and Coast Guard Loran (long range aids to navigation) stations. Its applications arise in many diverse fields including chemistry, for representing chemical compounds [10], the robot navigation [11] and geographical routing protocols [12], to name a few. In [13], an invariant called the strong metric dimension, was presented where the authors illustrated its application to combinatorial search. Along with the increasing discovery of its applications, theoretical studies on this invariant also appear in several number of other papers including [14], [15], [16], [17]. This paper intends to generate additional theoretical results and help widen the pool of existing studies from where new researchers may draw new insights and directions for further investigation.

In this study, we investigate the concept of restrained strong resolving domination in the join, corona, and lexicographic product of graphs. Readers are referred to [9] for elementary Graph Theory concepts.

Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* of $v \in V(G)$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. An element $u \in N_G(v)$ is called a *neighbor* of v. The *closed neighborhood* $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of $v \in V(G)$ is given by $deg_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

A *clique* in a graph G is a complete induced subgraph. A set $C \subseteq V(G)$ is called a *superclique* in G if $\langle C \rangle$ is a clique and for every pair of distinct vertices $u, v \in C$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique C is maximum in G if $|C| \geq |C^*|$ for all supercliques C^* in G. The superclique number $\omega_S(G)$ of G is the cardinality of a maximum superclique in G. A superclique C is called a dominated superclique if for every $u \in C$ there exists $v \in V(G) \setminus C$ such that $uv \in E(G)$. The dominated superclique number $\omega_{DS}(G)$, of G is the cardinality of a maximum dominated superclique in G.

A vertex x of a graph G is said to *resolve two vertices* u and v of G if $d_G(x, u) \neq d_G(x, v)$. For an ordered set $W = \{x_1, ..., x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector*

$$r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), ..., d_G(v, x_k))$$

is called the representation of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The *metric dimension* of G, denoted by $dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $dim(G)$ is called a *basis*.

A set $S \subseteq V(G)$ of vertices of G is a *dominating set* if every $u \in V(G) \setminus S$ is adjacent to at least one vertex $v \in S$. The *domination number* of a graph G, denoted by $\gamma(G)$, is given by $\gamma(G) = min\{|S| : S$ is a dominating set of $G\}$.

A subset $S \subseteq V(G)$ is a *strong resolving dominating set* of G if S is a dominating set and for every pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $u \in I_G[v, w]$ or $v \in I_G[u, w]$. The smallest cardinality of a strong resolving dominating set of G is called the *strong resolving domination number* of G and is denoted by $\gamma_{sR}(G)$. A strong resolving dominating set of cardinality $\gamma_{sR}(G)$ is called a γ_{sR}-set of G.

A set $S \subseteq V(G)$ is a *restrained dominating set* of G if S is a dominating set of G and for every $v \in V(G) \setminus S$ there exists $u \in (V(G) \setminus S) \cap N_G(v)$. Equivalently, a dominating subset S of $V(G)$ is a restrained dominating set of graph G if $S = V(G)$ or $\langle V(G) \setminus S \rangle$ has no isolated vertex. The *restrained domination number* of G, denoted by $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G. Any restrained dominating set of G of cardinality $\gamma_r(G)$ is referred to as a γ_r-set of G.

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge set $E(G + H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$. The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ such that $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1 v_1 \in E(G)$ or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

2. Preliminary Results

This section presents some of the important known results in strong resolving domination of graphs and some properties of restrained strong resolving dominating set in a graph.

Theorem 1. [3] Let G be a nontrivial connected graph of order n with $\gamma(G) \neq 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$; or $S = V(K_1 + G) \setminus C$ or $S = V(G) \setminus C^*$ where C is a superclique and C^* is dominated superclique in G.

Theorem 2. [3] Let G be a nontrivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong connected resolving dominating set of $K_1 + G$ if and only if $S = V(G)$ or $S = V(K_1 + G) \setminus C$ or

$$S = (V(G) \setminus C^*) \cup \{x \in C^* : deg(x) = n - 1\}$$

where C and C^* are superclique and dominated superclique, respectively in G.

Theorem 3. [3] Let $K_1 = \langle v \rangle$ and G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, m$. A proper subset S of $V(K_1 + G)$ is a strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$ or $S = V(G) \setminus C_i^*$ or $S = V(K_1 + G) \setminus C_i$ where C_i is a superclique in G_i, for $i = 1, 2, \ldots, m$ and C_i^* is a dominated superclique of G_i.

Theorem 4. [3] Let G be a nontrivial connected graph and H a connected graph. A proper subset S of $V(G \circ H)$ is a strong resolving dominating set of $G \circ H$ if and only if one of the following holds:

(i) $S = A \cup \left(\bigcup\limits_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$;

(ii) $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex v in G, where
$A \subseteq V(G) \setminus \{v\}$ and B_v is a strong resolving dominating set of H^v if $\gamma(H) = 1$ or B^v is
a strong resolving dominating set of $\langle v \rangle + H^v$
if $\gamma(H) \neq 1$.

Theorem 5. [3] Let G and H be non-trivial connected graphs of orders m and n, respectively. A proper subset S of $V(G+H)$ is a strong resolving dominating set of $G+H$ if and only if at least one of the following is satisfied:

(i) $S = V(G + H) \setminus C_G$ where C_G is a superclique of G.

(ii) $S = V(G + H) \setminus C_H$ where C_H is a superclique of G.

(iii) If $\gamma(G) = 1$ and $\gamma(H) = 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : deg_G(z) = m - 1\}$ or
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : deg_H(w) = n - 1\}$
where C_G and C_H are supercliques in G and H, respectively.

(iv) If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H)$
where C_G and C_H are supercliques in G and H, respectively.

Lemma 1. [2] Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving set of G if and only if $C = \varnothing$ or C is a superclique in G. In particular, $sdim(G) = |V(G)| - \omega_S(G)$.

Theorem 6. [2] Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. A subset S of $V(G[H])$ is a strong resolving set of $G[H]$ if and only $S = V(G[H]) \setminus (A \times C)$, where A is a subset of $V(G)$ and $C = \varnothing$ or C is a superclique in H.

Remark 1. Every restrained strong resolving dominating set of a connected graph G is a strong resolving dominating set. Hence, $\gamma_{sR}(G) \leq \gamma_{rsR}(G)$. Also, every restrained strong resolving dominating set of G is a restrained dominating set. Thus, $\gamma_r(G) \leq \gamma_{rsR}(G)$.

Remark 2. For any connected graph G of order n, $1 \leq \gamma_{rsR}(G) \leq n$. Moreover, $\gamma(G) = 1$ if and only if G is a non-trivial graph and $\gamma_{rsR}(K_n) = n$ for $n \geq 1$.

Proposition 1. Let G be a nontrivial connected graph with $diam(G) \leq 2$. Then $S \subseteq V(G)$ is a restrained strong resolving dominating set of G if and only if $S = V(G) \setminus C$ where $C = \varnothing$ or C is a nonsingleton dominated superclique in G. In particular,

$$\gamma_{rsR}(G) = |V(G)| - \omega_{DS}(G)$$

.

Proof: Let S be a restrained strong resolving dominating set of G. Let $C = V(G) \setminus S$. Then, $S = V(G) \setminus C$. Since S is a strong resolving, by Lemma 1, $C = \varnothing$ or C is a superclique in G. Since S is restrained dominating, $S = V(G)$ or $V(G) \setminus S$ has no isolated vertex. Thus, $C = \varnothing$ or C is a singleton dominated superclique in G.

The converse follows immediately from Lemma 1, definition of dominated superclique and definition of restrained dominating set.

Suppose S is a γ_{rsR}-set of G. Then, $S = V(G) \setminus C$ where C is a nonsingleton dominated superclique in G and $|C| = \omega_{DS}(G)$. Thus,

$$\gamma_{rsR}(G) = |S| = |V(G)| - |C| = |V(G)| - \omega_{DS}(G). \qquad \square$$

3. Restrained Strong Resolving Domination in the Join of Graphs

Theorem 7. Let G be a nontrivial connected graph of order n with $\gamma(G) \neq 1$. Then $S \subseteq V(K_1 + G)$ is a restrained strong resolving dominating set of $K_1 + G$ if and only if $S = V(G) \setminus C$ or $S = V(K_1 + G) \setminus C^*$ where C is a non-singleton dominated superclique and $C^* = \varnothing$ or C^* is non-singleton dominated superclique of G.

Proof: Let S be a restrained strong resolving set of $K_1 + G$. Since S is strong resolving dominating set by Theorem 1, $S = V(G)$ or $S = V(G) \setminus C$ or $S = V(K_1 + G) \setminus C^*$ where C is a dominated superclique and C^* is a superclique in G. Since S is restrained dominating, $S = V(G + K_1)$ or $V(G + K_1) \setminus S$ has no isolated vertex. Hence, $S \neq V(G)$ and $S = V(G) \setminus C$ or $S = V(K_1 + G) \setminus C^*$ where C is dominated supeclique and $C^* = \varnothing$ or C^* is a nonsingleton superclique of G.

The converse follows immediately from Theorem 1, definitions of dominated superclique and restrained dominating set of a graph. $\qquad \square$

Theorem 8. Let G be a nontrivial connected graph of order n with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a restrained strong resolving dominating set of $K_1 + G$ if and only if $S = V(K_1 + G) \setminus C$ or $S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$ where $C = \varnothing$ or C is nonsingleton superclique of G and C^* is a superclique of G.

Proof: Let S be a restrained strong resolving dominating set of $K_1 + G$. Then by Theorem 2, $S = V(G)$ or $S = V(K_1 + G) \setminus C$ or $S = (V(G) \setminus C^*) \cup \{x \in C^* : deg_G(x) = n - 1\}$ where C and C^* are superclique and dominated superclique of G, respectively. Since S is restrained dominating, $S = V(K_1 + G)$ or $V(K_1 + G) \setminus S$ has no isolated vertex. Thus, $S \neq V(G)$, $C = \varnothing$ or C is non-singleton dominated superclique of G and C^* is a superclique of G.

The converse follows immediately from Theorem 2, definitions of superclique and restrained dominating set of a graph. $\qquad \square$

The next results follow immediately from Theorem 7.

Corollary 1. Let G be a nontrivial connected graph of order n. Then

$$\gamma_{rsR}(K_1 + G) = \begin{cases} n - \omega_S(G) + 1 \ , & \text{if } \gamma(G) = 1 \\ n - \omega_{DS}(G) \quad \ , & \text{if } \gamma(G) \neq 1. \end{cases}$$

Corollary 2. Let G be a connected graph with $diam(G) \leq 2$. Then

$$\gamma_{rsR}(K_1 + G) = \begin{cases} sdim(G) + 1 \, , & \text{if } \gamma(G) = 1 \\ \gamma_{sR}(G) & , \quad \text{if } \gamma(G) \neq 1. \end{cases}$$

Corollary 3. Let $P_n = [v_1, v_2, \ldots, v_n]$ and $C_m = [c_1, c_2, \ldots c_m, c_1]$ where $n, m \geq 3$.

(i) The sets $V(P_n) \setminus \{v_i, v_{i+1}\}$ for $i = 2, 3, \ldots, n - 2$, $V(P_n) \setminus \{v_j\}$ for $j = 1, 2, \ldots, n$ and $V(P_n) \cup \{v\} \setminus \{v_k, v_{k+1}\}$ for $k = 1, 2, \ldots, n - 1$ are the restrained strong resolving dominating sets of $\langle v \rangle + P_n$.

(ii) The sets $V(C_m) \setminus \{c_i, c_{i+1}\}$, $(V(C_m) \cup \{v\}) \setminus \{c_i, c_{i+1}\}$, $V(C_m) \setminus \{v_i\}$, $V(C_m) \setminus \{v_m\}$, $V(C_m) \setminus \{c_i, c_m\}$ and $(V(C_m) \cup \{v\} \setminus \{c_1, c_m\})$ for $i = 1, 2, \ldots, m - 1$, are the restrained strong resolving dominating sets of $\langle v \rangle + C_n$.

Theorem 9. Let G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, n$. A subset S of $V(K_1 + G)$ is a restrained strong resolving dominating set of $K_1 + G$ if and only if $S = V(G) \setminus C_i$ or $S = V(K_1 + G) \setminus C_i^*$ where C_i is a dominated superclique of G_i and $C_i^* = \varnothing$ or C_i^* is a nonsingleton superclique of G.

Proof: Let S be a restrained strong resolving dominating set of $K_1 + G$. Then by Theorem 3 $S = V(G)$ or $S = V(G) \setminus C_i$ or $S = V(K_1 + G) \setminus C_i^*$ where C_i is a dominated superclique of G_i and $C_i^* = \varnothing$ or C_i^* is a nonsingleton superclique in G_i. Since S is restrained dominating, $S = V(K_1 + G)$ or $V(K_1 + G) \setminus S$ has no isolated vertex. Hence, $S \neq V(G)$ and $C_i^* = \varnothing$ or C_i^* is nonsingleton. Therefore, $S = V(G) \setminus C_1$ or $S = V(K_1 + G) \setminus C_i^*$ where C_i is a dominated superclique in G_i and C_i^* is a nonsingleton superclique in G_i.

The converse follows immediately from Theorem 3, definitions of dominated superclique and restrained dominating set of a graph. $\qquad \square$

Corollary 4. Let G_i be connected graphs of order n_i and G be a disconnected graph whose components are G_i for $i = 1, 2, \ldots, m$. Then, $\gamma_{rsR}(K_1 + G) = \sum_{i=1}^{m} n_i - r_G$ where $r_G = \max \{\max\{\omega_{DS}(G_i)\}, \max\{\omega_S(G_i) + 1\} : i = 1, 2, \ldots, m\}$.

In the join of two graphs G and H, the results Theorem 7 and Theorem 8 have already considered the case when G or H is trivial. Hence, the next results, considered the characterizations of the restrained strong resolving dominating sets of nontrivial connected graphs G and H.

Theorem 10. Let G and H be nontrivial connected graphs of orders m and n, respectively. A subset S of $V(G + H)$ is a restrained strong resolving dominating set of $G + H$ if and only if at least one of the following is satisfied:

(i) $S = V(G + H) \setminus C_G$ where C_G is a nonsingleton superclique of G.

(ii) $S = V(G + H) \setminus C_H$ where C_H is a nonsingleton superclique of G.

(iii) If $\gamma(G) = 1$ and $\gamma(H) = 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : deg_G(z) = m - 1\}$ or
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : deg_H(w) = n - 1\}$
where C_G and C_H are supercliques in G and H, respectively.

(iv) If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H)$
where C_G and C_H are supercliques in G and H, respectively.

Proof: Let S be a restrained strong resolving dominating set of $G + H$. By Theorem 5

(a) $S = V(G + H) \setminus C_G$ where C_G is a superclique of G.

(b) $S = V(G + H) \setminus C_H$ where C_H is a superclique of G.

(c)   If $\gamma(G) = 1$ and $\gamma(H) = 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : deg_G(z) = m - 1\}$ or
$S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : deg_H(w) = n - 1\}$
where $\langle C_G \rangle$ and C_H are supercliques in G and H, respectively.

(d)   If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$,
$S = [V(G + H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H)$
where C_G and C_H are supercliques in G and H, respectively.

Since S is a restrained dominating set of $G + H$, (i), (ii), (iii), (iv) hold.
The converse immediately follows from Theorem 5 and from definition of restrained dominating set of a graph. □

Corollary 5. Let G and H be nontrivial connected graphs of orders m and n, respectively. Then
$$sdim(G + H) = \begin{cases} (m - \omega_s(G)) + (n - \omega_s(H)) + 1, & \text{if } \gamma(G) = 1 \\ & \text{or } \gamma(H) = 1 \\ (m - \omega_s(G)) + (n - \omega_s(H)), & \text{if } \gamma(G) \neq 1 \\ & \text{and } \gamma(H) \neq 1. \end{cases}$$

4. Restrained Strong Resolving Domination in the Corona of Graphs

This section gives characterization of the restrained strong resolving dominating sets in the corona of graphs as well as its restrained strong resolving domination number.

Theorem 11. Let G be a nontrivial connected graph and H a connected graph. A proper subset $S \subseteq V(G \circ H)$ of a restrained strong resolving dominating set of $G \circ H$ if and only if one of the following holds:

(i) $S = A \cup \left(\bigcup_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$ and $V(G) \setminus A$ has no isolated vertex.

(ii) $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex v in G, where $A = V(G) \setminus \{v\}$
or $V(G) \setminus (A \cup \{v\})$ has no isolated vertex and B_v is a strong resolving dominating set
of $H^v + \langle v \rangle$ if $v \in S$ and B_v is a strong resolving set where $V(H^v) \setminus B_v$ has no isolated
vertex if $v \in S$.

Proof: Suppose S is a restrained resolving dominating set of $G \circ H$. Then S is a strong
resolving dominating and by Theorem 4 one of the following holds:

(a) $S = A \cup \left(\bigcup\limits_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$;

(b) $S = A \cup \left(\bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique vertex v in G, where $A \subseteq V(G) \setminus \{v\}$
        and B_v is a strong resolving dominating set of H^v if $\gamma(H) = 1$ or B_v is a strong
        resolving dominating set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

   Suppose (a) holds. Since S is a proper restrained dominating subset of $G \circ H$, $V(G \circ H) \setminus S = V(G) \setminus A$ has no isolated vertex. Thus, (i) holds. On the other   hand   if   (b)
holds,   then   since S is   a   restrained   dominating   set   and $V(G \circ H) \setminus S = (V(G) \setminus A) \cup (V(H^v + \langle v \rangle) \setminus B_v)$, $A = V(G) \setminus \{v\}$ or $V(G) \setminus (A \cup \{v\})$ has no isolated vertex
and B_v is a strong resolving dominating set of $H^v + \langle v \rangle$. Since $v \in S$, $V(H^v) \setminus B_v$ has no
isolated vertex and B_v is a strong resolving dominating set of $H^v + \langle v \rangle$ if $v \in S$ and B_v
is strong resolving set of H^v and $V(H^v) \setminus B_v$ has no isolated vertex if $v \in S$. Hence, (ii)
holds.
   Conversely,   suppose   (i) and (ii)   hold.   By   Theorem   4, S is a   strong   resolving
dominating   set   of $G \circ H$. If (i)   holds,   then $V(G \circ H) \setminus S = V(G) \setminus A$ has   no
isolated   vertex. If (ii) holds then $V(G \circ H) \setminus S = (V(G) \setminus A) \cup (V(H^v + \langle v \rangle) \setminus B_v)$. Since
$A = V(G) \setminus \{v\}$ or $V(G) \setminus (A \cup \{v\})$ has no isolated vertex, $V(G + H) \setminus S$ has no isolated
vertex. In either case, $V(G \circ H) \setminus S$ has no isolated vertex. Therefore S is a restrained
strong resolving dominating set of $G \circ H$. $\square$

Corollary 6. Let G and H be nontrivial connected graphs of orders m and n, respectively.
Then, $\gamma_{rsR}(G \circ H) = (m-1)n + \gamma_{sR}(H + K_1)$.

Proof: Let S be a γ_{rsR}-set of $G \circ H$. Then   by   Theorem   11   (ii),

$$S = \bigcup\limits_{u \in V(G) \setminus \{v\}} V(H^u) \bigcup B_v$$

for a unique vertex v in G and B_v is a strong resolving dominating set of H^v. Hence,

$$\gamma_{rsR}(G \circ H) = |S| = |V(H)||V(G) \setminus \{v\}| + |B_v|$$
$$\geq (m-1)n + \gamma_{sR}(H)$$

Let C_v be a minimum strong resolving dominating set of H^v. For a unique vertex $v \in V(G)$, let $\langle B_v \rangle \cong \langle C_v \rangle$. Then by Theorem 11 $S = \left(\bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \bigcup B_v$ is a restrained strong resolving dominating set of $G \circ H$. Thus,

$$\gamma_{rsR}(G \circ H) \leq |S|$$

$$= \left| \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right| + |B_v|$$

$$= (m-1)(n) + |C_v|$$

$$= (m-1)(n) + \gamma_{sR}(H)$$

Therefore, $\gamma_{rsR}(G \circ H) = (m-1)n + \gamma_{sR}(H)$. $\qquad\square$

5. Restrained Strong Resolving Domination in the Lexicographic Product of Graphs

Lemma 2. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. Then $A \times C \subseteq V(G[H])$ is a dominated superclique in $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H.

Proof: Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. Suppose $A \times C \subseteq V(G[H])$ is a dominated superclique in $G[H]$. By a Lemma in [3], A is a nonempty subset of $V(G)$ and C is a superclique in H.

Conversely, suppose $A \subseteq V(G)$, $A \neq \varnothing$ and C is a superclique in H. Then $A \times C \subseteq V(G[H])$ is a superclique in $G[H]$ by a Lemma in [3]. We show that $A \times C$ is a dominated superclique . Let $(a,b) \in A \times C$. Then $a \in A$ and $b \in C$. Since $\gamma(H) \neq 1$, there exists $d \in V(H) \setminus N_H(b)$. Hence, $d \notin C$. Since $G = K_n$ for $n > 1$, a vertex $v \in V(K_n) \setminus \{a\}$ exists where $av \in E(K_n)$. Thus, $(v,d) \notin A \times C$ and $(a,b)(v,d) \in E(G[H])$. Therefore, $A \times C$ is a dominated superclique in $G[H]$. $\qquad\square$

Theorem 12. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) \neq 1$. A subset S of $V(G[H])$ is a restrained strong resolving dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$ and one of the following is satisfied:

(i) $A \subseteq V(G)$ and $C = \varnothing$

(ii) A is singleton subset of $V(G)$ and C is a nonsingleton suerclique in H.

(iii) A is nonempty nonsingleton subset of $V(G)$ and C is a superclique in H.

Proof: Let S be a restrained strong resolving dominating set of $G[H]$. By Theorem 6, $S = V(G[H]) \setminus (A \times C)$ where A is a subset of $V(G)$ and $C = \varnothing$ or C is a superclique in H. Since S is restrained strong resolving dominating, $S = V(G[H])$ or $V(G[H]) \setminus S$ has

no isolated vertex. If $S = V(G[H])$ then $A \times C = \varnothing$, showing that $A \subseteq V(G)$ and $C = \varnothing$. Thus, *(i)* holds. If $V(G[H]) \setminus S$ has no isolated vertex, then $A \times C$ is a nonsingleton dominated superclique in $G[H]$. This implies that A is a singleton subset of $V(G)$ and C is a nonsingleton superclique in H or A is nonempty nonsingleton subset of $V(G)$ and C is a superclique in H. Hence *(ii)* or *(iii)* holds.

For the converse, suppose $S = V(G[H]) \setminus (A \times C)$, where A and C satisfy *(i),(ii)* or *(iii)*. Then, either $A \times C = \varnothing$ or by Lemma 2, $A \times C$ is a nonsingleton dominated superclique in $G[H]$. By Theorem 6, S is a strong resolving set in $G[H]$. Since $A \times C$ is a dominated superclique, S is a strong resolving dominating set of $G[H]$. If *(i)* is true, then $A \subseteq V(G)$ and $C = \varnothing$, that is, $A \times C = \varnothing$ and $S = V(G[H])$. If *(ii)* or *(iii)* is satisfied, then $V(G[H]) \setminus S$ has no isolated vertex. Therefore, S is a restrained strong resolving dominating set $G[H]$. $\square$

Lemma 3. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) = 1$. Then $A \times C \subseteq V(G[H])$ is a dominated superclique of $G[H]$ if and only if A is a nonempty subset of $V(G)$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.

Proof: Suppose $A \times C \subseteq V(G[H])$ is a dominated superclique in $G[H]$. Then $A \times C \neq \varnothing$. By a Lemma in [3], A is a nonempty subset of $V(G)$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.

Conversely, suppose $A \subseteq V(G)$ and $A \neq \varnothing$ and C is a superclique in H such that $|A| = 1$ whenever $C \cap C^* \neq \varnothing$ for some γ-set C^* of H. Then by a Lemma in [3], $A \times C$ is a superclique in $G[H]$. We show that $A \times C$ is a dominated superclique in $G[H]$. Let $(a, b) \in A \times C$. Then $a \in A$ and $b \in C$. If $C \cap C^* = \varnothing$ for all γ-set C^* of H, then there exists $d \in V(H) \setminus N_H(b)$. Thus, $d \notin C$. Since $G = K_n$ for $n > 1$, a vertex $y \in (V(K_n) \setminus \{a\}) \cup N_G(a)$ exists. This implies that

$$(y, d) \in [V(G[H]) \setminus (A \times C)] \cap N_{G[H]}((a, b)).$$

Suppose, that $C \cap C^* \neq \varnothing$ for some γ-set C^* of H. Then $|A| = 1$. Let $w \in A$ and $p \in C \cap C^*$. Since $G = K_n$ for $n > 1$, there exists $u \in (V(K_n) \setminus \{w\}) \cap N_G(w)$ and $(u, v) \notin (A \times C)$ for all $v \in V(H)$. Thus, $(w, p)(u, v) \in E(G[H])$. Therefore $A \times C$ is a dominated superclique in $G[H]$. $\square$

Using Lemma 3, the proof of the next result will just be similar to that of Theorem 12.

Theorem 13. Let $G = K_n$ for $n > 1$ and H a nontrivial connected graph with $\gamma(H) = 1$. A subset S of $V(G[H])$ is a restrained strong resolving dominating set of $G[H]$ if and only if $S = V(G[H]) \setminus (A \times C)$ and one of the following is satisfied:

(i) $A \subseteq V(G)$ and $C = \varnothing$

(ii) A is nonempty nonsingleton subset of $V(G)$ and C is a superclique in H if $C \cap C^* = \varnothing$ for all γ-set C^* in H.

(iii) A is a singleton subset of $V(G)$ and C is a nonsingleton superclique in H if $C \cap C^* \neq \varnothing$ for some γ-set C^* of H.

Corollary 7. Let $G = K_n$ for $n > 1$ and H a non-trivial connected graph of order m. Then $\gamma_{rsR}(G[H]) = mn - \omega_{DS}(G[H])$.

Proof: Let S be a γ_{rsR}-set of $G[H]$. Then S is a restrained strong resolving dominating set of $G[H]$. By Theorem 12 or Theorem 13, $S = V(G[H]) \setminus (A \times C)$ where $A \times C$ is a dominated superclique in $G[H]$. Since S is a γ_{rsR}-set, $A \times C$ is a maximum dominated superclique of $G[H]$. Hence,

$$\begin{aligned}
\gamma_{rsR}(G[H]) &= |S| \\
&= |V(G[H])| - |A \times C| \\
&= mn - \omega_{DS}(G[H]). \qquad \square
\end{aligned}$$

By Lemma 2 and Lemma 3, the dominated superclique $A \times C$ of $K_n[H]$ is maximum if $A = V(G)$ and C is a maximum superclique in H. Thus, the next result follows.

Corollary 8. Let $G = K_n$ for $n > 1$ and H be a connected graph of order $m > 1$. Then $\gamma_{rsR}(G[H]) = n(m - \omega_S(H))$.

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