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Solutions of Different Types of the linear and Non-linear Higher-Order Boundary Value Problems by Differential Transformation Method

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Abstract. In [12], a numerical comparison between the differential transform method and Adomian decomposition method for solving fourth-order boundary value problems was presented. In this article, we use the differential transformation method (DTM) to solve the linear and non-linear higher-order boundary value problems (HOBVPs). The method proved to be very successful and powerful in computing such elements. The specific problems chosen for this purpose is that of the different types of higher order (e.g. fifth, sixth, ninth, tenth and twelfth) boundary value problems. The differential transformation (DT) solutions are compared with the theoretical solution. It is shown that the solutions obtained from the technique have a very high degree of accuracy.

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1. Introduction

Recently a great deal of interest has been focused on the applications of the (DTM) to solve various scientific models, see Refs.[2],[3],[4],[5-6],[7-8],[9],[13-15], [16-17], [18] and [22]. The (DTM) has also been applied to solve linear and non-linear higher-order initial value problems, for example [15] and [16].

In this paper, we are interested in the differential transformation method (DTM) to solve linear and non-linear higher-order boundary value problems (HOBVPs). Many numerical techniques, such as finite difference method [10] and decomposition method [19-21] have been implemented to solve (HOBVPs) numerically. The differential transformation method (DTM) is a numerical method for solving boundary value problem [17]. The concept of differential transformation was first proposed by Zhou [23] in 1986, and it was applied to solve linear and non-linear initial value problems in electric circuit analysis. The method can be used to evaluates the approximating solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. The basic definitions of the differential transformation are introduced in Section 2. The mathematical background of the higher-order boundary value problems is described in Section 3. Analysis of higher-order boundary value problems is illustrated by differential transformation method (DTM) in Section 4. Numerical examples are used to illustrate the effectiveness of the proposed method in Section 5.

2. The Differential Transformation Method (DTM)

An kth- order differential transformation (DT) of a function $y(x) = f(x)$ is defined about a point $x = x0$ as:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}, \tag{2.1}$$

where k belongs to the set of non-negative integers, denoted as the K-domain.

The function $y(x)$ may be expressed in terms of the differential transforms (DT), $Y(k)$ as:

$$y(x) = \sum_{k=0}^{\infty}\left[\frac{(x-x_0)^k}{k!}\right]Y(k). \tag{2.2}$$

Upon combining (2.1) and (2.2), we obtain

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^k \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}, \tag{2.3}$$

which is actually the Taylor's series for $y(x)$ about $x = x_0$.

From the basic definition of the differential transforms (DT), one can obtain certain laws of transformational operations, some of these, are listed in the following.

(1): If $z(x) = u(x) \pm v(x)$ then, $Z(k) = U(k) \pm V(k)$.

(2): If $z(x) = \alpha u(x)$ then, $Z(k) = \alpha U(k)$. Here α is a constant.

(3): If $z(x) = \frac{du(x)}{dx}$ then, $Z(k) = (k+1)U(k+1)$.

(4): If $z(x) = \frac{d^2u(x)}{dx^2}$ then, $Z(k) = (k+1)(k+2)U(k+2)$.

(5): If $z(x) = \frac{d^m u(x)}{dx^m}$ then, $Z(k) = (k+1)(k+2)\cdots(k+m)U(k+m)$.

(6): If $z(x) = u(x)v(x)$ then, $Z(k) = \sum_{l=0}^{k} V(l)U(k-l)$.

(7): If $z(x) = x^m$ then, $Z(k) = \delta(k-n)$ where, $\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$

(8): If $z(x) = \exp(\lambda x)$ then, $Z(k) = \frac{\lambda^k}{k!}$.

(9): If $z(x) = (1+x)^m$ then, $Z(k) = \frac{m(m-1)\cdots(m-k+1)}{k!}$.

(10): If $z(x) =$sin$(\omega x + \alpha)$ then, $Z(k) = \frac{\omega^k}{k!}\sin(\frac{\pi k}{2} + \alpha)$.

(11): If $z(x) = \cos(\omega x + \alpha)$ then, $Z(k) = \frac{\omega^k}{k!}\cos(\frac{\pi k}{2} + \alpha)$.

3. The Higher-order Boundary Value Problems

(i) *even-order boundary value problems*

Consider the special $(2m)$-order BVP of the form

$$y^{(2m)}(x) = f(x, y), \qquad 0 < x < b, \tag{3.1}$$

with boundary conditions

$$y^{(2j)}(0) = \alpha_{2j}; \qquad j = 0, 1, 2, \ldots, (m-1), \tag{3.2}$$

$$y^{(2j)}(b) = \beta_{2j}; \qquad j = 0, 1, 2, \ldots, (m-1), \tag{3.3}$$

(ii) *odd-order boundary value problems*

Consider the special $(2m+1)$-order BVP of the form

$$y^{(2m+1)}(x) = f(x, y), \qquad 0 < x < b, \tag{3.4}$$

with boundary conditions

$$y^{(2j+1)}(0) = \gamma_{2j+1}; \qquad j = 0, 1, 2, \ldots, m, \tag{3.5}$$

$$y^{(2j+1)}(b) = \gamma_{2j+1}; \qquad j = 0, 1, 2, \ldots, m, \tag{3.6}$$

It is interesting to point out that $y(x)$ and $f(x, y)$ are assumed real and as many times differentiable as required for $x \in [0, b]$ and α_{2j} and β_{2j} , $j = 0, 1, 2, \ldots (m-1)$ describe the even-order are real finite constants [11], moreover, the conditions α_{2j} , $j = 0, 1, 2, \ldots, (m-1)$ describe the even-order derivatives at the boundary $x = 0$, while γ_{2j+1} and γ_{2j+1}, $j = 0, 1, 2, \ldots, m$ describe the odd-order are real finite constants and the conditions γ_{2j+1}, $j = 0, 1, 2, \ldots m$ describe the odd-order derivatives at the boundary $x = b$.

Theorems which list the conditions for existence and uniqueness solution of such problems are contained in a comprehensive survey in a book by Agarwal [1], though no numerical methods are contained therein for solving HOBVP's of higher order.

4. Analysis of Higher-order Boundary Value Problems by Differential Transformation

Let the differential transform of the deflection function $y(x)$ be defined from Eq. (2.1) as:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}, \tag{4.1}$$

where $x_0 = 0$. Also the deflection function may be expressed in terms of $Y(k)$ from Eq. (2.2) as:

$$y(x) = \sum_{k=0}^{\infty}\left[\frac{x^k}{k!}\right]Y(k). \tag{4.2}$$

Now, using the transformational operations which has been formed in Sec.2, one can obtain by taking the differential transform of Eq. (3.1) and (3.4) respectively and some simplification, the following recurrence equations as $m = 0, 1, 2, \ldots$.

$$Y(2m+k) = \sum_{k=0}^{\infty}\left[\frac{(2m)!}{(2m+k)!}\right]Y(.,.), \tag{4.3}$$

$$Y(2m+k+1) = \sum_{k=0}^{\infty}\left[\frac{(2m+1)!}{(2m+k+1)!}\right]Y(.,.), \tag{4.4}$$

where $Y(.,.)$ denotes the transformed function of linear or nonlinear function $f(x, y)$. It may be noted that Eq. (4.2) is independent of the boundary conditions.

The differential transforms of the boundary conditions at $x = 0$ are obtained from Eqs. (3.2) and (3.5) in the cases even-order (odd-order) boundary value problems respectively, with the definition (4.1) as:

$$Y(2j) = \frac{1}{(2j)!}\alpha_{2j}, \qquad j = 0, 1, 2, \ldots, (2m-1), \tag{4.5}$$

$$Y(2j+1) = \frac{1}{(2j+1)!}\gamma_{2j+1}, \qquad j = 0, 1, 2, \ldots, 2m, \tag{4.6}$$

Substituting from (4.5) and (4.6) into (4.3) or (4.4) and using (4.2), yields for $j = 0, 1, 2, \ldots, (m-1)$,

$$y(x) = \sum_{k=0}^{\infty}\left[\frac{1}{(2j)!}\alpha_{2j}\right]Y(k)x^k, \tag{4.7}$$

and for $j = 0, 1, 2, \ldots, m$,

$$y(x) = \sum_{k=0}^{\infty} \left[\frac{1}{(2j+1)!} \gamma_{2j+1} \right] Y(k) x^k. \tag{4.8}$$

Noting that $y^{(2r+1)}(0) = A_r$, $r = 0, 1, 2, \ldots, (m-1)$, and $y^{(2r)}(0) = B_r$, $r = 0, 1, 2, \ldots, m$, are constants that will be approximated at the end point $x = b$.

5. Numerical Examples

In this section, linear and nonlinear HOBVPs will be tested by using the differential transformation method, (see [12]).

Example 5.1. *We first consider the following linear fifth-order BVP , which is also solved by Adomian decomposition method (ADM) in the study of [21]*

$$y^{(v)}(x) = y(x) - 15e^x - 10xe^x, \qquad 0 < x < 1, \tag{5.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y(1) = 0, \quad y'(1) = -e. \tag{5.2}$$

Applying the operations of (DT) to Eq. (5.1), the following recurrence relation is obtained:

$$Y(k+5) = \frac{k! \left[Y(k) - \frac{15}{k!} - 10 \left(\sum_{l=0}^{k} \left[\frac{\delta(k-l-1)}{l!} \right] \right) \right]}{(km+5)!} \tag{5.3}$$

By using Eqs. (2.1) and (5.2) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 0, \quad Y(1) = 1, \quad Y(2) = 0, \tag{5.4}$$

where, according to Eq. (2.1), $a = \frac{y'''(0)}{3!} = Y(3)$ and $b = \frac{y^{(iv)}(0)}{4!} = Y(4)$. Utilizing the recurrence relation in Eq. (5.3) and the transformed B.C.'s in Eq. (5.4), $Y(k)$ for $k \geq 5$ are easily obtained.

The constants a and b are evaluated from the B.C.'s given in Eq. (5.2) for x =1, by taking $N = 13$, to obtain the system:

$$\frac{148284463}{148262400}a + \frac{3632669041}{3632428800}b = -\frac{4541061529}{5448643200},$$
$$\frac{239595841}{79833600}a + \frac{5148284463}{37065600}b = -e + \frac{15028547}{129729600}.$$

This in turn gives $a = -0.3333315065$ and $b = -0.5000018268$.

For $N = 24$, these values are $a = -0.333315040$ and $b = -0.5000018292$. Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to $N = 24$:

$$\begin{aligned} y(x) \;=\; & x - 0.5000018292x^3 - 0.3333315040x^4 - 0.125x^5 - 0.03333333333x^6 \\ & -0.6944444444E - 2\ x^7 - 0.1190476463E - 2\ x^8 - 0.1736109901E - 3\ x^9 \\ & -0.2204585538E - 4x^{10} - 0.2480158730E - 5x^{11} - 0.2505210839E - 6x^{12} \\ & -0.2296443445E - 7x^{13} - 0.192708757E - 8x^{14} - 0.1491196928E - 9x^{15} \\ & -0.1070602922E - 10x^{16} - 0.7169215999E - 12x^{17} - 0.2655264824E - 13x^{18} \\ & -0.2655264824E - 14x^{19} - 0.1479714344E - 15x^{20} - 0.7809603484E - 17x^{21} \\ & -0.3914588213E - 18x^{22} - 0.1868326235E - 19x^{23} - 0.8509973668E - 21x^{24}. \end{aligned}$$

Example 5.2. *We next consider the following non-linear fifth-order BVP*

$$y^{(v)}(x) = e^{-x}y^2(x), \qquad 0 < x < 1, \tag{5.5}$$

subject to the boundary conditions

$$y(0) = 0 = y'(0) = \ y''(0) = 1, \quad y(1) = y'(1) = e. \tag{5.6}$$

Applying the operations of (DT) to Eq. (5.6), the following recurrence relation is obtained:

$$Y(k+5) = \tfrac{k!}{(k+5)!}\sum_{l=0}^{k}\sum_{s=0}^{l}\tfrac{(-1)^s}{s!}Y(l-s)Y(k-l). \tag{5.7}$$

By using Eqs. (2.1) and (5.7) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 1, \quad Y(1) = 1, \quad Y(2) = \tfrac{1}{2}, \tag{5.8}$$

where, according to Eq. (2.1), $a_1 = \frac{y'''(0)}{3!} = Y(3)$ *and* $a_2 = \frac{y^{(iv)}(0)}{4!} = Y(4)$*. Utilizing the recurrence relation in Eq. (5.3) and the transformed B.C.'s in Eq. (5.4),* $Y(k)$ *for* $k \geq 5$ *are easily obtained.*

The constants a_1 *and* a_2 *are evaluated from the B.C.'s given in Eq. (5.7) for* $x = 1$*, by taking* $N = 12$*, to obtain the system:*

$$\begin{aligned} e - \frac{80149541}{31933440} &= \frac{98589}{98560}a_1 + \frac{1996097}{1996097}a_2 + \frac{1}{133056}a_1^2 + \frac{1}{47520}a_1a_2, \\ e - \frac{5848303}{2851200} &= \frac{285343}{95040}a_1 + \frac{665471}{166320}a_2 + \frac{1}{13860}a_1^2 + \frac{1}{3960}a_1a_2. \end{aligned}$$

This in turn gives $a_1 = 0.666611767$ *and* $a_2 = 0.0416703271$.

For $N = 20$*, these values are* $a_1 = 0.1666611892$ *and* $a_2 = 0.0416703549$*. Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to* $N = 20$*:*

$$\begin{aligned} y(x) = {} & 1 + x + 0.5x^2 + 0.1666611892x^3 + 0.04167031549x^4 + 0.8333333333E - 2x^5 \\ & +0.1388888889E - 2\ x^6 + 0.19841269843E - 3\ x^7 + 0.2479995710E - 4\ x^8 \\ & +0.2756214572E - 5x^9 + 0.2755731922E - 6x^{10} + 0.2505210920E - 7x^{11} \\ & +0.2087674900E - 8x^{12} + 0.1605698300E - 9x^{13} + 0.114745900E - 10x^{14} \\ & +0.3304339489E - 8x^{15} + 0.1987113795E - 8x^{16} + 0.2812690000E - 14x^{17} \\ & +0.1560220000E - 15x^{18} + 0.8235580000E - 17x^{19} + 0.143090494E - 12x^{20}. \end{aligned}$$

Numerical results for linear and non-linear of fifth- order BVP's, the differential transformation method (DTM) with comparison to the exact solution are given in Table 1.

Table 1:
Comparison of numerical result of **BVP's** (5.1)-(5.2) and (5.6)-(5.7) see Fig. 1.

	LN-5^{th} order BVP		**NLN-5^{th} order BVP**	
x	$DTM(N=24)^*$	$y_{exact} = x(1-x)e^x$	$DTM(N=20)^*$	$y_{exact} = e^x$
0.0	.000000000E+00	.000000000E+00	.100000000E+01	.100000000E+01
0.1	.994653900E-01	.994653900E-01	.110517100E+01	.110517100E+01
0.2	.195424400E+00	.195424400E+00	.122140300E+01	.122140300E+01
0.3	.283470300E+00	.283470400E+00	.134985900E+01	.134985900E+01
0.4	.358037900E+00	.358037900E+00	.149182500E+01	.149182500E+01
0.5	.412180200E+00	.412180300E+00	.164872100E+01	.164872100E+01
0.6	.437308400E+00	.437308500E+00	.182211800E+01	.182211900E+01
0.7	.422887900E+00	.422888000E+00	.201375200E+01	.201375300E+01
0.8	.356086300E+00	.356086500E+00	.222554000E+01	.222554100E+01
0.9	.221364000E+00	.221364100E+00	.245960200E+01	.245960300E+01
1.0	-.302392000E-06	-.324044500E-06	.271828000E+01	.271828200E+01

DTM (N=20 and N=24) is **DTM of order 20 and of order 24 respectively ,*

Example 5.3. *Again, following the study of [19], we consider the following linear sixth-*

*order BVP , which is also solved by Adomian decomposition method **(ADM)***

$$y^{(vi)}(x) = y(x) - 6e^x, \qquad 0 < x < 1, \tag{5.9}$$

subject to the boundary conditions

$$y(0) = 1,\ y''(0) = -1,\ \ y^{(iv)}(0) = -3, \quad y(1) = 0,\ y''(1) = -2e, y^{(iv)}(1) = -4e. \tag{5.10}$$

Applying the operations of DT to Eq. (5.11), the following recurrence relation is obtained:

$$Y(k+6) = \frac{k!\left[Y(k) - \frac{6}{k!}\right]}{(k+6)!}. \tag{5.11}$$

By using Eqs. (2.1) and (5.12) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 1, \quad Y(2) = -\frac{1}{2}, \ \ Y(4) = \frac{-1}{8}, \tag{5.12}$$

where, according to Eq. (2.1), $a = Y(1)$, $b = \frac{y'''(0)}{3!} = Y(3)$ and $c = \frac{y^{(v)}(0)}{5!} = Y(5)$. Utilizing the recurrence relation in Eq. (5.13) and the transformed B.C.'s in Eq. (5.14), $Y(k)$ for $k \geq 6$ are easily obtained.

The constants a, b and c are evaluated from the B.C.'s given in Eq. (5.12) for $x = 1$, by taking $N = 13$, to obtain the system:

$$\frac{889750903}{889574400}a + \frac{60481}{60480}b + \frac{332641}{332640}c = -\frac{456655181}{1245404160},$$
$$\frac{332641}{39916800}a + \frac{5041}{840}b + \frac{60481}{3024}c = -2e + \frac{110549143}{39916800},$$
$$\frac{60481}{362880}a + \frac{1}{20}b + \frac{5041}{42}c = -4e + \frac{829261}{120960}.$$

This in turn gives $a = -0.5388992288E^{-6}$, $b = -0.3333328229$ and $c = -0.03333330492$.

For $N = 23$, these values are $a = -0.4667205875E^{-6}$,$b = -0.3333329279$ and $c = -0.03333327191$. Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to $N = 23$:

$$y(x) \ = \ 1 - 0.4667205875E^{-6} - 6x - 0.5x^2 - 0.3333329279x^3 - 0.125x^4$$

$$
\begin{aligned}
&- \quad 0.03333327191x^5 - .6944444444E^{-2} - 2x^6 - .1190476283E^{-2} - 2x^7 \\
&- \quad .1736111111E^{-3} - 3x^8 - .2204584867E^{-4} - 4x^9 - .2480158730E^{-5} - x^{10} \\
&- \quad .2505208992E^{-6}x^{11} - .2296443269E^{-7}x^{12} - .1927085335E^{-8}x^{13} \\
&- \quad .1491196928E^{-9}x^{14} - .1070602736E^{-10}x^{15} - .7169215999E^{-12}x^{16} \\
&- \quad .4498329534E^{-13}x^{17} - .2655265185E^{-14}x^{18} - .1479714382E^{-15}x^{19} \\
&- \quad .7809603484E^{-17}x^{20} - .3914587736E^{-18}x^{21} - .1868326192E^{-19}x^{22} \\
&- \quad .8509971524E^{-21}x^{23}. \qquad (5.13)
\end{aligned}
$$

Example 5.4. *We next consider the following non-linear sixth-order BVP*

$$y^{(vi)}(x) = e^x y^2(x), \qquad 0 < x < 1, \tag{5.14}$$

subject to the boundary conditions

$$y(0) = 1,\ y'(0) = -1,\ \ y''(0) = 1, \quad y(1) = e^{-1},\ y'(1) = -e^{-1},\ y''(1) = e^{-1}. \tag{5.15}$$

Applying the operations of DT to Eq. (5.16), the following recurrence relation is obtained:

$$Y(k+6) = \tfrac{k!}{(k+6)!} \sum_{l=0}^{k} \sum_{s=0}^{l} \tfrac{1}{s!} Y(l-s) Y(k-l). \tag{5.16}$$

By using Eqs. (2.1) and (5.17) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 1, \quad Y(1) = -1, \quad Y(2) = \tfrac{1}{2}, \tag{5.17}$$

where, according to Eq. (2.1), $a_1 = \frac{y'''(0)}{3!} = Y(3), a_2 = \frac{y^{(iv)}(0)}{4!} = Y(4)$ and $a_3 = \frac{y^{(v)}(0)}{5!} = Y(5)$. Utilizing the recurrence relation in Eq. (5.18) and the transformed B.C.'s in Eq. (5.19), $Y(k)$ for $k \geq 6$ are given.

$$
\begin{gathered}
Y(6) = \frac{1}{720}, \quad Y(7) = -\frac{1}{5040}, \quad Y(8) = \frac{1}{40320}, \quad Y(9) = \frac{1}{30240}a_1 + \frac{1}{362880}, \\
Y(10) = \frac{1}{75600}a_2 - \frac{1}{3628800}, \quad Y(11) = \frac{1}{166320}a_3 + \frac{1}{39916800},
\end{gathered}
$$

$$\vdots$$

and so on.

The constants a_1*,* a_2 *and* a_3 *are evaluated from the B.C.'s given in Eq. (5.17) for* $x = 1$*, by taking* $N = 11$*, to obtain the system:*

$$\begin{aligned} e^{-1} - \frac{2000701}{3991680} &= \frac{30241}{30240}a_1 + \frac{75601}{75600}a_2 + \frac{166321}{166320}a_3, \\ -e^{-1} - \frac{321}{44800} &= \frac{10081}{3360}a_1 + \frac{30241}{7560}a_2 + \frac{75601}{15120}a_3, \\ e^{-1} - \frac{46943}{45630} &= \frac{2521}{420}a_1 + \frac{10081}{840}a_2 + \frac{30241}{1512}a_3. \end{aligned}$$

This in turn gives $a_1 = -.1666630261$*,* $a_2 = .04166085689$ *and* $a_3 = .008330914797$. *For* $N = 17$*, these values are* $a_1 = .1666633333$*,* $a_2 = .04166152737$ *and* $a_3 = -.008331283835$*. Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to* $N = 17$*:*

$$\begin{aligned} y(x) = {} & 1 - x + 0.5x^2 + 0.1666633333x^3 + 0.4166152737E - 1x^4 - 0.8331283835E - 2x^5 \\ & +0.1388888889E - 2\,x^6 - 0.1984126984E - 3\,x^7 + 0.2480158730E - 4\,x^8 \\ & -0.2755621693E - 5x^9 + 0.2755052123E - 6x^{10} - 0.2503978576E - 7x^{11} \\ & +0.2087675720E - 8x^{12} - 0.1605904400E - 9x^{13} + 0.1147075100E - 10x^{14} \\ & -0.7646535000E - 12x^{15} + 0.3944023390E - 8x^{16} - 0.7792679132E - 10x^{17}. \end{aligned}$$

Numerical results for linear and non-linear of sixth- order BVP, the differential transformation method DTM with comparison to the exact solution are given in Table 2.

Table 2:
Comparison of numerical result of **BVP's** (5.11)-(5.12) and (5.16)-(5.17) see Fig. 2.

	LN-6^{th} order bvp		**NLN-6^{th} order-bvp**	
x	$DTM(N=23)^*$	$y_{exact} = (1-x)e^x$	$DTM(N=17)^*$	$y_{exact} = e^{-x}$
0.0	.100000000E+01	.100000000E+01	.100000000E+01	.100000000E+01
0.1	.994653800E+00	.994653800E+00	.904837400E+00	.904837400E+00
0.2	.977122100E+00	.977122200E+00	.818730800E+00	.818730800E+00
0.3	.944901000E+00	.944901200E+00	.740818300E+00	.740818200E+00
0.4	.895094600E+00	.895094800E+00	.670320200E+00	.670320000E+00
0.5	.824360400E+00	.824360700E+00	.606530800E+00	.606530700E+00
0.6	.728847300E+00	.728847500E+00	.548811800E+00	.548811600E+00
0.7	.604125600E+00	.604125700E+00	.496585500E+00	.496585300E+00
0.8	.445107900E+00	.445108100E+00	.449329200E+00	.449328900E+00
0.9	.245960000E+00	.245960100E+00	.406569900E+00	.406569600E+00
1.0	-.299616900E-06	-.324044500E-06	.367879600E+00	.367879400E+00

*∗DTM (N=17 and N=23) is **DTM** of order 17 and of order 23 respectively ,*

Example 5.5. *We consider the following linear ninth-order BVP [20],*

$$y^{(ix)}(x) = y(x) - 9e^x, \qquad 0 < x < 1, \tag{5.18}$$

subject to the boundary conditions

$$\begin{aligned} y^{(j)}(0) &= (1-j),\ j = 0,1,2,3,4, \\ y^{(j)}(1) &= -je,\ j = 0,1,2,3. \end{aligned} \tag{5.19}$$

Applying the operations of DT to Eq. (5.21), the following recurrence relation is obtained:

$$Y(k+9) = \frac{k!\left[Y(k) - \frac{9}{k!}\right]}{(k+9)!}. \tag{5.20}$$

By using Eqs. (2.1) and (5.22) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 1, \quad Y(1) = 0, \ \ Y(2) = \frac{-1}{2},\ Y(3) = \frac{-1}{3},\ Y(4) = \frac{-1}{8}, \tag{5.21}$$

where, according to Eq. (2.1), we have

$$\begin{aligned} a &= \frac{y^{(v)}(0)}{5!} = Y(5), \quad b = \frac{y^{(vi)}(0)}{6!} = Y(6), \\ c &= \frac{y^{(vii)}(0)}{7!} = Y(7), \quad d = \frac{y^{(viii)}(0)}{8!} = Y(8). \end{aligned}$$

Utilizing the recurrence relation in Eq. (5.23) and the transformed B.C.'s in Eq. (5.24), $Y(k)$ for $k \geq 9$ are easily obtained:

$$\begin{aligned} Y(9) &= -\frac{1}{45360}, \\ Y(10) &= -\frac{1}{403200}, \\ Y(11) &= -\frac{1}{3991680}, \\ Y(12) &= -\frac{1}{43545600}, \\ Y(13) &= -\frac{1}{518918400}, \\ Y(14) &= \frac{1}{726485760}a - \frac{1}{726485760}, \end{aligned}$$

$$
\begin{aligned}
Y(15) &= \frac{1}{726485760}b - \frac{1}{145297152000},\\
Y(16) &= \frac{1}{4151347200}c - \frac{1}{2324754432000},\\
Y(17) &= \frac{1}{8821612800}d - \frac{1}{39520825344000},\\
Y(18) &= -\frac{1}{376610217984000},\\
Y(19) &= -\frac{1}{6758061133824000},\\
Y(20) &= -\frac{1}{128047474114560000},
\end{aligned}
$$

and so on.

The constants a, b, c and d are evaluated from the B.C.'s given in Eq. (5.22) for $x = 1$, by taking $N = 17$, to obtain the system:

$$
\begin{aligned}
\frac{726485761}{726485760}a + \frac{1816214401}{1816214400}b + \frac{4151347201}{4151347200}c + \frac{8821612801}{8821612800}d &= \frac{-493716263633}{11856247603200},\\
\frac{259459201}{51891840}a + \frac{726485761}{121080960}b + \frac{1816214401}{259459200}c + \frac{4151347201}{518918400}d &= -e + \frac{5812412107297}{2324754432000},\\
\frac{79833601}{3991680}a + \frac{259459201}{8648640}b + \frac{726485761}{17297280}c + \frac{1816214401}{32432400}d &= -2e + \frac{163526184469}{36324288000},\\
\frac{19958401}{332640}a + \frac{79833601}{665280}b + \frac{259459201}{1235520}c + \frac{726485761}{2162160}d &= -3e + \frac{145680117719}{29059430400}.
\end{aligned}
$$

This in turn gives $a = -.3336167167E^{-1}$, $b = -.6870399019E^{-2}$ and $c = -.1255380280E^{-2}$ and $d = -.1544141065E^{-3}$.

For $N = 26$, these values are $a = -.3336167167E^{-1}$,$b = -.6870399019E^{-2}$ and $c = -.1255380280E^{-2}$ and $d = -.1544141066E^{-3}$. Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to $N = 26$:

$$
\begin{aligned}
y(x) = {} & 1 - 0.5x^2 - 0.3333333333x^3 - 0.125x^4 - 0.03336167167x^5 - 0.6870399019E - 2x^6\\
& -0.1255380280E - 2x^7 - 0.1544141066E - 3x^8 - 0.2204585538E - 4x^9\\
& -0.2480158730E - 5x^{10} - 0.2505210839E - 6x^{11} - 0.2296443269E - 7x^{12}\\
& -0.1927085260E - 8x^{13} - 0.1491587002E - 9 * x^{14} - 0.1066526013E - 10x^{15}\\
& -0.7325560642E - 12x^{16} - 0.4280718287E - 13x^{17} - 0.2655265185E - 14x^{18}\\
& -0.1479714344E - 15x^{19} - 0.7809603484E - 17x^{20} - 0.3914588213E - 18x^{21}\\
& -0.1868326192E - 19x^{22} - 0.8511289785E - 21x^{23} - 0.8511289785E - 21x^{23}\\
& -0.3698403804E - 22x^{24} - 0.1568357113E - 23x^{25} - 0.6007064149E - 25x^{26}
\end{aligned}
$$

Numerical results for linear ninth- order BVP, the differential transformation method (DTM) with comparison to the exact solution are given in Table 3.

Table 3: Comparison of numerical result of BVP's (5.21)-(5.22) see Fig.3.

	linear ninth-order BVP	
x	$DTM(N=26)^*$	$y_{exact}=(1-x)e^x$
0.0	.100000000E+01	.100000000E+01
0.1	.994653800E+00	.994653800E+00
0.2	.977122200E+00	.977122200E+00
0.3	.944901100E+00	.944901200E+00
0.4	.895094800E+00	.895094800E+00
0.5	.824360500E+00	.824360700E+00
0.6	.728847300E+00	.728847500E+00
0.7	.604125400E+00	.604125700E+00
0.8	.445107800E+00	.445108100E+00
0.9	.245959900E+00	.245960100E+00
1.0	-.332089400E-06	-.324044500E-06

*∗DTM (N=26) is **DTM** of order 26.,*

Example 5.6. *We consider the following non-linear tenth-order BVP*

$$y^{(x)}(x)=e^{-x}y^2(x), \qquad 0<x<1, \tag{5.22}$$

subject to the boundary conditions

$$\begin{aligned} &y^{(2j)}(0)=1,\ j=0,1,2,3,4,\\ &.\\ &y^{(2j)}(1)=e,\ j=0,1,2,3,4. \end{aligned} \tag{5.23}$$

Applying the operations of DT to Eq. (5.26), the following recurrence relation is obtained

$$Y(k+10)=\frac{k!}{(k+10)!}\sum_{l=0}^{k}\sum_{s=0}^{l}\frac{(-1)^s}{s!}Y(l-s)Y(k-l). \tag{5.24}$$

By using Eqs. (2.1) and (5.27) the following transformed B.C.'s at $x=0$ can be obtained:

$$Y(0)=1, \quad Y(2)=\frac{1}{2!},\ Y(4)=\frac{1}{4!},\ Y(6)=\frac{1}{6!},\ Y(8)=\frac{1}{8!}, \tag{5.25}$$

where, according to Eq. (2.1), we have

$$a_1=y'(0)=Y(1),\ \ Y(5),\quad a_2=\frac{y^{(iii)}(0)}{3!}=Y(3),\quad a_3=\frac{y^{(v)}(0)}{5!}=Y(5),$$

$$a_4 = \frac{y^{(vii)}(0)}{7!} = Y(7), \quad a_5 = \frac{y^{(ix)}(0)}{9!} = Y(9),$$

Utilizing the recurrence relation in Eq. (5.28) and the transformed B.C.'s in Eq. (5.29), $Y(k)$ for $k \geq 10$ are easily obtained:

1;

$$Y(10) = \frac{1}{3628800},$$
$$Y(11) = \frac{1}{19958400}a_1 - \frac{1}{39916800},$$
$$Y(12) = \frac{1}{159667200} + \frac{1}{239500800}a_1^2 - \frac{1}{119750400}a_{1,}$$
$$Y(13) = \frac{1}{518918400}a_2 + \frac{1}{518918400}a_1 - \frac{1}{889574400} - \frac{1}{1037836800}a_1^2,$$
$$Y(14) = \frac{1}{1037836800} + \frac{1}{1816214400}a_1a_2 - \frac{1}{1816214400}a_2 - \frac{1}{2724321600}a_1 + \frac{1}{7264857600}a_1^2$$
$$\vdots$$

and so on.

The constants a_1,a_2 , a_3, a_4 and a_5 are evaluated from the B.C.'s given in Eq. (5.27) for $x = 1$, by taking $N = 12$, to obtain :

$$a_1 = 1.000000124,$$
$$a_2 = .9999819650,$$
$$a_3 = 1.000157229,$$
$$a_4 = .9985666714,$$
$$a_5 = 1.009946626.$$

For $N = 17$, these values are $a_1 = .9999698990$, $a_2 = 1.000278188$, $a_3 = .9973109664$, $a_4 = 1.023991491$ and $a_5 = .8383579606$.

Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to $N = 17$:

$$\begin{aligned} y(x) &= 1 + 0.9999698990 * x - 0.5x^2 + 0.1667130314x^3 + 0.04166666667x^4 \\ &+0.008310924720x^5 + 0.001388888889x^6 + 0.0002031729149x^7 \\ &+0.00002480158730x^8 + 0.2310289794E - 5x^9 + 0.2755731922E - 6x^{10} \\ &+0.2505060019E - 7x^{11} + 0.2087675700E - 8x^{12} + .1767030912E - 8x^{13} \\ &+0.1145693000E - 10x^{14} + 0.1822782951E - 9x^{15} + 0.2395822685E - 10x^{16} \\ &+.1917087518E - 10x^{17}. \end{aligned} \tag{5.30}$$

Numerical results for non-linear tenth- order BVP, the differential transformation method (DTM) with comparison to the exact solution are given in Table 4, and which have been indicated in Fig. 4.

Table 4: Comparison of numerical result of BVP's (5.26)-(5.27), see Fig.4.

	nonlinear tenth-order BVP	
x	$DTM(N = 17)$	$y_{exact} = e^x$
0.0	.100000000E+01	.100000000E+01
0.1	.110516800E+01	.110517100E+01
0.2	.122139700E+01	.122140300E+01
0.3	.134985100E+01	.134985900E+01
0.4	.149181500E+01	.149182500E+01
0.5	.164871100E+01	.164872100E+01
0.6	.182210900E+01	.182211900E+01
0.7	.201374400E+01	.201375300E+01
0.8	.222553400E+01	.222554100E+01
0.9	.245959900E+01	.245960300E+01
1.0	.271828000E+01	.271828200E+01

∗DTM (N=17) is differential transformation method of order 17.

Example 5.7. *Finally, we consider the non-linear twelfth-order BVP [20]*

$$y^{(xii)}(x) = 2e^x y^2(x) + y^{(iii)}(x), \qquad 0 < x < 1, \tag{5.26}$$

subject to the boundary conditions

$$\begin{aligned} y^{(2j)}(0) &= 1,\ j = 0,1,2,3,4,5, \\ y^{(2j)}(1) &= e^{-1},\ j = 0,1,2,3,4,5. \end{aligned} \tag{5.27}$$

Applying the operations of (DT) to Eq. (5.31), the following recurrence relation is *obtained:*

$$Y(k+12) = \frac{k!}{(k+12)!}\left[2\sum_{l=0}^{k}\sum_{s=0}^{l}\frac{1}{s!}Y(l-s)Y(k-l) + (k+1)(k+2)(k+3)Y(k+3)\right]. \tag{5.28}$$

By using Eqs. (2.1) and (5.32) the following transformed B.C.'s at $x = 0$ can be obtained:

$$Y(0) = 1, \quad Y(2) = \frac{1}{2!}, \quad Y(4) = \frac{1}{4!}, \; Y(6) = \frac{1}{6!}, \; Y(8) = \frac{1}{8!}, \quad Y(10) = \frac{1}{10!}, \tag{5.29}$$

where, according to Eq. (2.1), we have

$$\begin{aligned}
Y(1) &= a_1 = y'(0),\\
Y(3) &= a_2 = \frac{y^{(iii)}(0)}{3!},\\
Y(5) &= a_3 = \frac{y^{(v)}(0)}{5!},\\
Y(7) &= a_4 = \frac{y^{(vii)}(0)}{7!},\\
Y(9) &= a_5 = \frac{y^{(ix)}(0)}{9!},\\
Y(11) &= a_6 = \frac{y^{(xi)}(0)}{11!}.
\end{aligned}$$

Utilizing the recurrence relation in Eq. (5.33) and the transformed B.C.'s in Eq. (5.34), $Y(k)$ for $k \geq 12$ are easily obtained:

$$\begin{aligned}
Y(12) &= \frac{1}{239500800} + \frac{1}{79833600}a_2,\\
Y(13) &= \frac{1}{2075673600} + \frac{1}{1556755200}a_1,\\
Y(14) &= \frac{1}{14529715200} + \frac{1}{10897286400}a_1 + \frac{1}{21794572800}a_1^2 + \frac{1}{726485760}a_3,\\
Y(15) &= \frac{1}{87178291200} + \frac{1}{54486432000}a_1 + \frac{1}{108972864000}a_1^2 + \frac{1}{54486432000}a_2,\\
Y(16) &= \frac{1}{498161664000} + \frac{1}{326918592000}a_1 + \frac{1}{871782912000}a_1^2 + \frac{1}{217945728000}a_2 +\\
&\quad \frac{1}{217945728000}a_1a_2 + \frac{1}{4151347200}a_4,\\
&\vdots
\end{aligned}$$

and so on.

The constants a_1,a_2 , a_3, a_4 , a_5 and a_6 are evaluated from the B.C.'s given in Eq. (5.32) for x =1, by taking $N = 16$, to obtain:

$$a_1 = -0.9999999967,\ a_2 = -0.1666666720,$$

$$a_3 = -0.0083333307, a_4 = -0.0001984133,$$

$$a_5 = -0.0000027557, a_5 = -0.0000000251$$

Then, by using the inverse transformation rule in Eq. (4.2), we get the following series solution is evaluated up to $N = 16$:

$$\begin{aligned} y(x) = {} & 1 - x + 0.5x^2 - 0.166667x^3 + 0.0416667x^4 - 0.00833333x^5 + 0.00138889x^6 \\ & -0.000198413x^7 + 0.0000248016x^8 - 2.75566E - 6x^9 + 2.75573E - 7x^{10} \\ & -250565E - 8x^{11} + 2.08768E - 9x^{12} - 1.6059E - 10x^{13} + 1.14707E - 11x^{14} \\ & -7.64716E - 13x^{15} + 4.77946E - 14x^{16} + O(x^{17}). \end{aligned} \qquad (5.35)$$

Numerical results for non-linear twelfth- order BVP, the differential transformation method DTM with comparison to the exact solution are given in Table 5, and which have been indicated in Fig. 5.

Table: 5 Comparison of numerical result of BVP's (5.31)-(5.32), see Fig.5.

	non-linear twelfth-order BVP	
x	$DTM(N = 16)$	$y_{exact} = e^{-x}$
0.0	10.00000000E+01	10.00000000E+01
0.1	9.048374184E-01	9.048374180E-01
0.2	8.187307537E-01	8.187307531E-01
0.3	7.408182215E-01	7.408182207E-01
0.4	6.703200470E-01	6.703200460E-01
0.5	6.065306608E-01	6.065306597E-01
0.6	5.488116371E-01	5.488116361E-01
0.7	4.965853046E-01	4.965853038E-01
0.8	4.493289647E-01	4.493289641E-01
0.9	4.065696601E-01	4.065696597E-01
1.0	3.678794412E-01	3.678794412E-01

∗DTM (N=16) is differential transformation method of order 16,

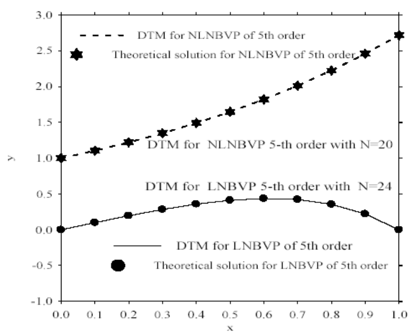

Fig. 1 Comparison differential transformation method (DTM) solution of linear and nonlinear BVP of 5-th order with exact solution

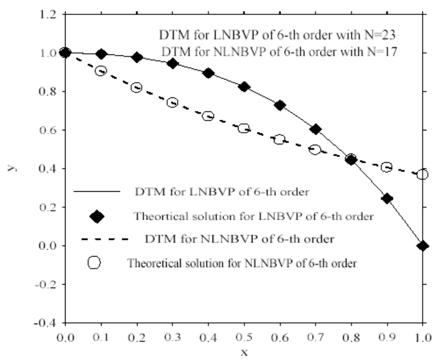

Fig. 2 Comparison differential transformation method (DTM) solution of linear and nonlinear BVP of 6-th order with exact solution

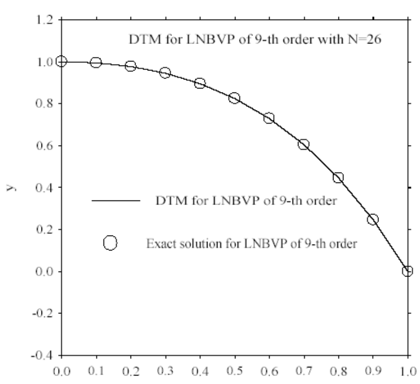

Fig. 3 Comparison differential transformation method (DTM) solution of linear BVP of 9-th order with exact solution y(x)=(1-x)exp(x)

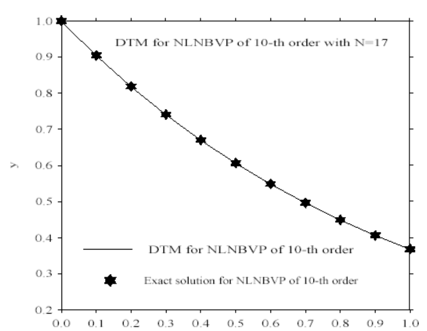

Fig. 4 Comparison differential transformation method (DTM) solution of nonlinear BVP of 10-th order with exact solution y(x)=exp(x)

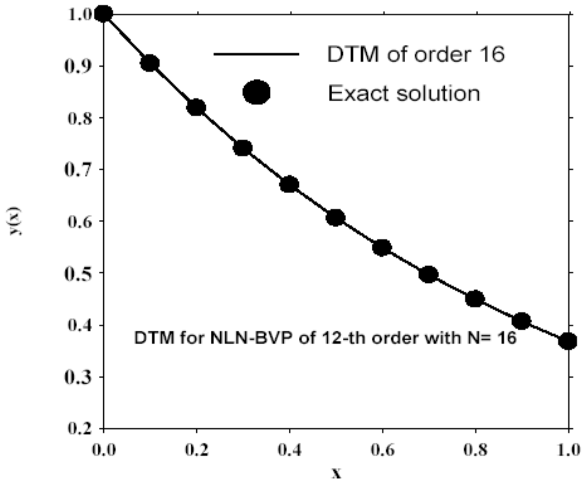

Fig. 5: Comparison differential transformation method (DTM) solutipn on nonlinear BVP of 12-th order with exact solution y(x)=exp(- x)

6. Conclusion

The computations associated with the examples discussed above were performed by using Maple 4. The existence and uniqueness of the solution is guaranteed by Agarwal's [1]. We first gave the definition of HOBVPs, second gave the analysis of HOBVPs by DTM. In above problems, we gave higher-order series solution. It is shown that DTM is a very fast convergent, precise and cost efficient tool for solving HOBVPs in the bounded domains.

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