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On k-Cost Effective Domination Number in the Join of Graphs

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Abstract. In this paper, we characterized the k-cost effective domination in the join of graphs. Further, we investigate the k-cost effective domination, cost effective domination index, maximal cost effective domination in the join of graphs.

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Key Words and Phrases: k-cost effective set, k-cost effective domination index, maximal cost effective domination.

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1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v in the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v in a graph G, denoted by $deg_G(v)$, is $|N(v)|$. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A subset S of $V(G)$ is an independent set of G if $uv \notin E(G)$ for distinct pairs of vertices u and v in S. An *independent dominating set* in G is an independent set in G which is dominating in G. The minimum cardinality $\gamma_i(G)$ of an independent dominating set in G is called *independence domination number*.

Let $k \geq 0$ be an integer. Consider a vertex v, its neighborhood set, $N(v)$ and the vertex-set of G, $V(G)$. A vertex $v \in S \subseteq V(G)$ is said to be k-cost effective if $|N(v) \cap (V(G) \setminus S)| \geq |N(v) \cap S| + k$. A dominating set S is k-cost effective, if every vertex in S is k-cost effective. The minimum cardinality of a k-cost effective dominating

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set of G is the k-cost effective domination number $\gamma_{ce}^{k}(G)$ of G. In cases where there is no k-cost effective dominating set for G, the k-cost effective domination number of G is infinity. The *k-cost effective domination index* of G, denoted by $\eta(G)$, is the maximum value of k such that k-cost effective domination number is finite. That is,

$$\eta(G) = \max\{k : \gamma_{ce}^{k}(G) \text{ is finite.}\}$$

The *maximal cost effective domination number* of G is equal to $\gamma_{ce}^{\eta(G)}(G)$.

2. Results

Theorem 1. Let G and H be connected graphs, $k \geq max\{|V(G)|, |V(H)|\}$, and $S \subseteq V(G+H)$. Then S is a k-cost effective dominating set in $G+H$ if and only if one of the following holds:

(i) S is $(k-|V(H)|)$-cost effective dominating set in G;

(ii) S is $(k-|V(G)|)$-cost effective dominating set in H;

(iii) $V(G) \cap S$ is $(k-k_1)$-cost effective dominating set in G, where $k_1 = |V(H)| - 2|V(H) \cap S|$ and $V(H) \cap S$ is $(k-k_2)$-cost effective dominating set in H, where $k_2 = |V(G)| - 2|V(G) \cap S|$.

Proof: Let $k \geq max\{|V(G)|, |V(H)|\}$, and $S \subseteq V(G+H)$. Suppose S is a k-cost effective dominating set in $G+H$ and let $x \in S$. Then

$$|N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| \geq k.$$

Suppose $S \subseteq V(G)$. Then S is a dominating set in G. Now,

$$\begin{aligned} |N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| &= |V(H)| + |N_G(x) \setminus S| \\ &\quad - |N_G(x) \cap S| \\ &\geq k. \end{aligned}$$

This implies that,

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - |V(H)|.$$

Hence, S is $(k-|V(H)|)$-cost effective dominating set in G. Similarly, if $S \subseteq V(H)$, then S is $(k-|V(G)|)$-cost effective dominating set in H.

Suppose that $S_1 = V(G) \cap S \neq \varnothing$ and $S_2 = V(H) \cap S \neq \varnothing$. Since S is a k-cost effective dominating set in $G+H$,

$$|N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| \geq k.$$

Let $x \in S_1 \subseteq S$. Then

$$|N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| = |N_G(x) \setminus S_1| + |V(H) \setminus S_2| - |N_G(x) \cap S_1| - |S_2|$$

$$= |N_G(x) \setminus S_1| + |V(H)| - |S_2| - |N_G(x) \cap S_1| - |S_2|$$
$$= |N_G(x) \setminus S_1| - |N_G(x) \cap S_1| + |V(H)| - 2|S_2|.$$

This implies that,

$$\begin{aligned} |N_G(x) \setminus S_1| - |N_G(x) \cap S_1| &\geq k - |V(H)| + 2|V(H) \cap S| \\ &= k - (|V(H)| - 2|V(H) \cap S|) \\ &= k - k_1, \end{aligned}$$

where $k_1 = |V(H)| - 2|V(H) \cap S|$. Thus, $S_1 = V(G) \cap S$ is $(k-k_1)$-cost effective dominating set in G. Similarly, $S_2 = V(H) \cap S$ is $(k - k_2)$-cost effective dominating set in H.

Conversely, suppose that S satisfies Property (i). Then S is a dominating set in $G+H$ and

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - |V(H)|, \forall x \in S.$$

Now,

$$\begin{aligned} |N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| &= |V(H)| + |N_G(x) \setminus S| \\ &\quad - |N_G(x) \cap S| \\ &\geq |V(H)| + k - |V(H)| \\ &= k, \end{aligned}$$

for all $x \in S$. Since x is arbitrary, S is a k-cost effective dominating set in $G+H$. Similarly, if S satisfies Property (ii), then S is a k-cost effective dominating set in $G + H$. Suppose S satisfies Property (iii) and $x \in V(G) \cap S$. Then

$$|N_G(x) \setminus S| - |N_G(x) \cap S| \geq k - k_1,$$

where $k_1 = |V(H)| - 2|V(H) \cap S|$. Now,

$$\begin{aligned} |N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| &= |N_G(x) \setminus S| + |V(H) \setminus S| - |N_G(x) \cap S| + |V(H) \cap S| \\ &= |N_G(x) \setminus S| - |N_G(x) \cap S| + |V(H) \setminus S| - |V(H) \cap S| \\ &= |N_G(x) \setminus S| - |N_G(x) \cap S| + |V(H)| - 2|V(H) \cap S| \\ &\geq k - k_1 + k_1 \\ &= k. \end{aligned}$$

Similarly, for each $x \in V(H) \cap S$, $|N_{G+H}(x) \setminus S| - |N_{G+H}(x) \cap S| \geq k$. Therefore, S is a k-cost effective dominating set in $G + H$. □

Corollary 1. Let G and H be connected graphs, $k \geq max\{|V(G)|, |V(H)|\}$. If S is a k-cost effective dominating set in $G + H$, then one of the following holds:

(i) $S \subseteq V(G)$ and $k \leq \eta(G) + |V(H)|$;

(ii) $S \subseteq V(H)$ and $k \leq \eta(H) + |V(G)|$;

(iii) $k \leq min\{\eta(G) + |V(H)| - 2|V(H) \cap S|, \eta(H) + |V(G)| - 2|V(G) \cap S|\}$.

Theorem 2. Let G and H be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$ and $0 \leq k \leq |V(H)| + |V(G)| - 1$. Then $S \subseteq V(G+H)$ is a γ_{ce}^k-set in $G+H$ if and only if S is a γ-set in G or S is a γ-set in H.

Corollary 2. Let G and H be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$. Then

$$\gamma_{ce}^k(G+H) = \begin{cases} 1, & if\ 0 \leq k \leq |V(H)| + |V(G)| - 1 \\ \infty, & if\ k > |V(H)| + |V(G)| - 1. \end{cases}$$

Corollary 3. Let G and H be connected graphs such that $\gamma(G) = 1$ or $\gamma(H) = 1$. Then $\eta(G+H) = |V(H)| + |V(G)| - 1$ and $\gamma_{ce}^{\eta(G+H)}(G+H) = 1$.

In the succeeding theorems, $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$ and assume that $\Delta(G) + |V(H)| \leq \Delta(H) + |V(G)|$.

Theorem 3. Let G and H be connected graphs such that $min\{\gamma(G), \gamma(H)\} \geq 2$ and $0 \leq k \leq \Delta(G) + |V(H)| - 2$. Then S is a γ_{ce}^k-set in $G+H$ if and only if $|S| = 2$ and one of the following holds:

(i) $|V(G) \cap S| = 1$ and $|V(H) \cap S| = 1$;

(ii) S is a γ-set in G such that $k - |V(H)| + 2 \leq \delta(S : G)$;

(iii) S is a γ-set in H such that $k - |V(G)| + 2 \leq \delta(S : H)$.

Proof: Suppose that $A = \{a, b\}$ such that $deg_G(a) = \Delta(G)$ and $deg_H(b) = \Delta(H)$. Clearly, A is a dominating set in $G+H$. Moreover,

$$\begin{aligned} |N_{G+H}(a) \setminus A| - |N_{G+H}(a) \cap A| &= deg_G(a) + |V(H)| \\ &= \Delta(G) + |V(H)| \\ &> \Delta(G) + |V(H)| - 2 \\ &\geq k. \end{aligned}$$

and

$$\begin{aligned} |N_{G+H}(b) \setminus A| - |N_{G+H}(b) \cap A| &= deg_H(b) + |V(G)| \\ &= \Delta(H) + |V(G)| \\ &\geq \Delta(G) + |V(H)| \\ &> \Delta(G) + |V(H)| - 2 \\ &\geq k. \end{aligned}$$

Thus, A is a k-cost effective dominating set in $G+H$. Accordingly, $\gamma_{ce}^k(G+H) = |S| \leq 2$. Suppose that $|S| = 1$. Then $\gamma(G) = 1$ or $\gamma(H) = 1$, which is a contradiction to that fact that $min\{\gamma(G), \gamma(H)\} \geq 2$. Therefore, $\gamma_{ce}^k(G+H) = 2$. Since S is a γ_{ce}^k-set in $G+H$, $|S| = 2$.

Clearly, $|V(G) \cap S| = 1$ and $|V(H) \cap S| = 1$. Thus, Property (i) holds.

Suppose that $S \subseteq V(G)$. Since S is a dominating set in $G+H$, S is a dominating set in G. Now, $\gamma(G) \geq 2$, so S is a minimum dominating set in G, that is, S is a γ-set in G. Let $S = \{a_1, a_2\} \subseteq V(G)$. Suppose a_1 and a_2 are adjacent in S. Then

$$\begin{aligned}|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| &= (|V(H)| + deg_G(a_i) - 1) - 1\\ &= |V(H)| + deg_G(a_i) - 2\\ &\geq |V(H)| + \delta(S : G) - 2\\ &\geq k,\ i = 1, 2.\end{aligned}$$

Thus, $k - |V(H)| + 2 \leq \delta(S : G)$. Suppose a_1 and a_2 are not adjacent in S. Then

$$\begin{aligned}|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| &= |V(H)| + deg_G(a_i)\\ &> |V(H)| + deg_G(a_i) - 2\\ &\geq |V(H)| + \delta(S : G) - 2\\ &= k,\ i = 1, 2.\end{aligned}$$

Thus, $k - |V(H)| + 2 \leq \delta(S : G)$. Similarly, $k - |V(G)| + 2 \leq \delta(S : H)$.

Conversely, suppose that S satisfies Property (i). Then S is a γ-set in $G+H$. Moreover,

$$\begin{aligned}|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= |V(H)| - 1 + deg_G(a) - 1\\ &= |V(H)| + \Delta(G) - 2\\ &\geq k.\end{aligned}$$

and

$$\begin{aligned}|N_{G+H}(b) \setminus S| - |N_{G+H}(b) \cap S| &= |V(G)| - 1 + deg_H(b) - 1\\ &= |V(G)| + \Delta(H) - 2\\ &= |V(H)| + \Delta(G) - 2\\ &\geq k.\end{aligned}$$

Thus, S is a k-cost effective dominating set in $G+H$. Hence, S is a γ_{ce}^k-set in $G+H$. Suppose that S satisfies Property (ii). Then S is a γ-set in $G+H$. Suppose a_1 and a_2 are adjacent in S. Then

$$\begin{aligned}|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| &= (|V(H)| + deg_G(a_i) - 1) - 1\\ &= |V(H)| + deg_G(a_i) - 2\\ &\geq |V(H)| + \delta(S : G) - 2\\ &\geq k.\end{aligned}$$

Suppose a_1 and a_2 are not adjacent in S. Then

$$\begin{aligned}|N_{G+H}(a_i) \setminus S| - |N_{G+H}(a_i) \cap S| &= |V(H)| + deg_G(a_i)\\ &> |V(H)| + deg_G(a_i) - 2\\ &\geq |V(H)| + \delta(S:G) - 2\\ &= k.\end{aligned}$$

Thus, S is a k-cost effective dominating set in $G+H$. Suppose that a singleton set is a dominating set in $G+H$. Then $\gamma(G) = 1$ or $\gamma(H) = 1$, which a contradiction to the fact that $min\{\gamma(G), \gamma(H)\} \geq 2$. Hence, S is a γ_{ce}^k-set in $G+H$. Similarly, if S satisfies Property (iii), then S is a γ_{ce}^k-set in $G+H$.

Therefore, S is a γ_{ce}^k-set in $G+H$. □

Theorem 4. Let G and H be connected graphs such that $min\{\gamma(G), \gamma(H)\} \geq 2$ and $k = \Delta(G) + |V(H)| - 1$. Then S is a k-cost effective dominating set in $G+H$ if and only if one of the following holds:

(i) S is an independent dominating set in G such that $\delta(S:G) \geq \Delta(G) - 1$;

(ii) S is a dominating set in H such that $0 \leq r_H(a) + 2|N_H(a) \cap S| - t \leq 1$, where $r_H(a) = \Delta(H) - deg_H(a)$ and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$, and $deg_H(a) + |V(G)| - 2|N_H(a) \cap S| = \Delta(G) + |V(H)| - 1$.

Proof: Suppose that S is a k-cost effective dominating set in $G+H$. Consider the following cases:

Case 1: $V(G) \cap S \neq \varnothing$ and $V(H) \cap S \neq \varnothing$.

Let $a \in V(G) \cap S$. Then

$$\begin{aligned}|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &\leq \Delta(G) - 1 + |V(H)| - 1\\ &< \Delta(G) + |V(H)| - 1\\ &= k,\end{aligned}$$

a contradiction. Thus, this case is not possible.

Case 2: $S \subseteq V(G)$.

Suppose S is not an independent dominating set G. Let $a \in S$. Then there exists $a' \in S$ such that $d_G(a, a') = 1$. Now

$$\begin{aligned}|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &\leq \Delta(G) - 1 + |V(H)| - 1\\ &< \Delta(G) + |V(H)| - 1\\ &= k,\end{aligned}$$

a contradiction. Thus, in this case S is an independent dominating set in G. Let $r_G(a) = \Delta(H) - deg_G(a)$. Now, S is a k-cost effective dominating set in $G+H$, so

$$|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| = deg_G(a) + |V(H)|$$

$$= \Delta(G) - r_G(a) + |V(H)|$$
$$\geq \Delta(G) + |V(H)| - 1.$$

Thus, $r_G(a) \leq 1$ and $deg_G(a) \geq \Delta(G) - 1$ for all $a \in S$. Hence, $\delta(S : G) \geq \Delta(G) - 1$.
Case 3: $S \subseteq V(H)$.

Since S is a k-cost effective dominating set in $G + H$, S is a dominating set in H. Let $a \in S$ and $r_H(a) = \Delta(H) - deg_H(a)$, and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$. Then

$$\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
&= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t).
\end{aligned}$$

Thus, $0 \leq r_H(a) + 2|N_H(a) \cap S| - t \leq 1$. Hence, $deg_H(a) + |V(G)| - 2|N_H(a) \cap S| = \Delta(G) + |V(H)| - 1$.

Conversely, suppose that S satisfies Property (i). Then S is a dominating set in $G+H$. Let $a \in S$. Then

$$\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_G(a) + |V(H)| \\
&= \Delta(G) - 1 + |V(H)| \\
&= k.
\end{aligned}$$

Hence, S is a k-cost effective dominating set in $G + H$.

Suppose that S satisfies Property (ii). Then S is a dominating set in $G + H$. Now,

$$\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
&= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t).
\end{aligned}$$

If $r_H(a) + 2|N_H(a) \cap S| - t = 0$, then

$$\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
&= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t \\
&= \Delta(G) + |V(H)| \\
&> \Delta(G) + |V(H)| - 1 \\
&= k.
\end{aligned}$$

Hence, S is a k-cost effective dominating set in $G + H$. If

$$r_H(a) + 2|N_H(a) \cap S| - t = 1$$

then

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &= deg_H(a)-|N_H(a)\cap S|+|V(G)|-|N_H(a)\cap S|\\ &= \Delta(H)-r_H(a)-2|N_H(a)\cap S|+|V(G)|\\ &= \Delta(G)+|V(H)|+t-r_H(a)-2|N_H(a)\cap S|\\ &= \Delta(G)+|V(H)|-(r_H(a)+2|N_H(a)\cap S|-t\\ &= \Delta(G)+|V(H)|-1\\ &= k.\end{aligned}$$

Hence, S is a k-cost effective dominating set in $G+H$.

Therefore, S is a k-cost effective dominating set in $G+H$. □

Theorem 5. Let G and H be connected graphs such that $min\{\gamma(G),\gamma(H)\}\geq 2$ and $k=\Delta(G)+|V(H)|$. Then S is a k-cost effective dominating set in $G+H$ if and only if one of the following holds:

(i) S is an independent dominating set in G such that $\delta(S:G)=\Delta(G)$;

(ii) S is a dominating set in H such that $deg_H(a)+|V(G)|=2|N_H(a)\cap S|+\Delta(G)+|V(H)|$ and $r_H(a)+2|N_H(a)\cap S|-t=0$, where $r_H(a)=\Delta(H)-deg_H(a)$, $t=\Delta(H)+|V(G)|-\Delta(G)-|V(H)|$.

Proof: Suppose that S is a k-cost effective dominating set in $G+H$. Consider the following cases:

Case 1: $V(G)\cap S\neq\varnothing$ and $V(H)\cap S\neq\varnothing$.

Let $a\in V(G)\cap S$. Then

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &\leq \Delta(G)-1+|V(H)|-1\\ &< \Delta(G)+|V(H)|\\ &= k,\end{aligned}$$

a contradiction. Thus, this case is not possible.

Case 2: $S\subseteq V(G)$.

Suppose S is not an independent dominating set G. Let $a\in S$. Then there exists $a^{'}\in S$ such that $d_G(a,a^{'})=1$. Now

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &\leq \Delta(G)-1+|V(H)|-1\\ &< \Delta(G)+|V(H)|\\ &= k,\end{aligned}$$

a contradiction. Thus, in this case S is an independent dominating set in G. Now,

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &= deg_G(a)+|V(H)|\\ &= \Delta(G)+|V(H)|\end{aligned}$$

$$= k,$$

Thus, $deg_G(a) = \Delta(G)\ \forall\ a \in S$. Hence, $\delta(S : G) = \Delta(G)$.
Case 3: $S \subseteq V(H)$.

Since S is a k-cost effective dominating set in $G + H$, S is a dominating set in H. Let $a \in S$ and $r_H(a) = \Delta(H) - deg_H(a)$, and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$. Then

$$\begin{aligned} |N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\ &= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\ &= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\ &= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t). \end{aligned}$$

Thus, $r_H(a) + 2|N_H(a) \cap S| - t = 0$. Hence, $deg_H(a) + |V(G)| = 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$.

Conversely, suppose that S satisfies Property (i). Then S is a dominating set in $G+H$. Let $a \in S$. Then

$$\begin{aligned} |N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_G(a) + |V(H)| \\ &= \delta(S : G) + |V(H)| \\ &= \Delta(G) + |V(H)| \\ &= k. \end{aligned}$$

Hence, S is a k-cost effective dominating set in $G + H$.

Suppose that S satisfies Property (ii). Then S is a dominating set in $G + H$. Now,

$$\begin{aligned} |N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\ &= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\ &= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\ &= \Delta(G) + |V(H)| + \Delta(H) + |V(G)| - \Delta(G) \\ &\quad - |V(H)| - \Delta(H) + deg_H(a) - 2|N_H(a) \cap S| \\ &= |V(G)| + deg_H(a) - 2|N_H(a) \cap S| \\ &= \Delta(G) + |V(H)| \\ &= k. \end{aligned}$$

Thus, S is a k-cost effective dominating set in $G + H$.

Therefore, S is a k-cost effective dominating set in $G + H$. □

Theorem 6. Let G and H be connected graphs such that $min\{\gamma(G), \gamma(H)\} \geq 2$ and $\Delta(G) + |V(H)| + 1 \leq k \leq \Delta(H) + |V(G)|$. Then S is a k-cost effective dominating set in $G + H$ if and only if S is a dominating set in H such that $t - r_H(a) - 2|N_H(a) \cap S| \geq p$, where $1 \leq p \leq t$ and $t = \Delta(H) + |V(G)| - \Delta(G) - |V(H)|$, and $r_H(a) = \Delta(H) - deg_H(a)$ and $deg_H(a) + |V(G)| \geq p + 2|N_H(a) \cap S| + \Delta(G) + |V(H)|$.

Proof: Suppose that S is a k-cost effective dominating set in $G+H$. Consider the following cases:
Case 1: $V(G)\cap S\neq\varnothing$ and $V(H)\cap S\neq\varnothing$.
Let $a\in V(G)\cap S$. Then

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &\leq \Delta(G)-1+|V(H)|-1\\ &< \Delta(G)+|V(H)|+1\\ &\leq k,\end{aligned}$$

a contradiction. Thus, in this case is not possible.
Case 2: $S\subseteq V(G)$.
Suppose S is not an independent dominating set G. Let $a\in S$. Then there exists $a^{'}\in S$ such that $d_G(a,a^{'})=1$. Now

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &\leq \Delta(G)-1+|V(H)|-1\\ &< \Delta(G)+|V(H)|+1\\ &\leq k,\end{aligned}$$

a contradiction. Thus, in this case S is an independent dominating set in G. Thus,

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &= deg_G(a)+|V(H)|\\ &= \Delta(G)-r_G(a)+|V(H)|\\ &\leq \Delta(G)+|V(H)|+1\\ &\geq k,\end{aligned}$$

a contradiction. Thus, in this case is not possible.
Case 3: $S\subseteq V(H)$.
Since S is a k-cost effective dominating set in $G+H$, S is a dominating set in H. Let $a\in S$, $r_H(a)=\Delta(H)-deg_H(a)$ and $1\leq p\leq t$, where $t=\Delta(H)+|V(G)|-\Delta(G)-|V(H)|$. Then

$$\begin{aligned}|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| &= deg_H(a)-|N_H(a)\cap S|+|V(G)|-|N_H(a)\cap S|\\ &= \Delta(H)-r_H(a)-2|N_H(a)\cap S|+|V(G)|\\ &= \Delta(G)+|V(H)|+t-r_H(a)-2|N_H(a)\cap S|.\end{aligned}$$

Thus, $t-r_H(a)-2|N_H(a)\cap S|\geq p$. Hence, $deg_H(a)+|V(G)|\geq p+2|N_H(a)\cap S|+\Delta(G)+|V(H)|$.

Conversely, suppose that S is a dominating set in H such that $t-r_H(a)-2|N_H(a)\cap S|\geq p$, where $1\leq p\leq t$ and $t=\Delta(H)+|V(G)|-\Delta(G)-|V(H)|$, and $r_H(a)=\Delta(H)-deg_H(a)$ and $deg_H(a)+|V(G)|\geq p+2|N_H(a)\cap S|+\Delta(G)+|V(H)|$. Then

$$|N_{G+H}(a)\setminus S|-|N_{G+H}(a)\cap S| = deg_H(a)-|N_H(a)\cap S|+|V(G)|-|N_H(a)\cap S|$$

$$
\begin{aligned}
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| \\
&= k.
\end{aligned}
$$

Hence, S is a k-cost effective dominating set in $G + H$. □

Theorem 7. Let G and H be connected graphs such that $min\{\gamma(G), \gamma(H)\} \geq 2$ and $k \geq \Delta(H) + |V(G)| + 1$. Then $\gamma_{ce}^{k}(G + H) = \infty$.

Proof: Let $k \geq \Delta(H) + |V(G)| + 1$. Suppose that there exists a k-cost effective dominating set S in $G + H$. Consider the following cases:
Case 1: $V(G) \cap S \neq \varnothing$ and $V(H) \cap S \neq \varnothing$.
Let $a \in V(H) \cap S$. Then

$$
\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &\leq \Delta(H) - 1 + |V(G)| - 1 \\
&< \Delta(H) + |V(G)| + 1 \\
&= k,
\end{aligned}
$$

a contradiction.
Case 2: $S \subseteq V(G)$.
Suppose S is not an independent dominating set G. Let $a \in S$. Then there exists $a' \in S$ such that $d_G(a, a') = 1$. Now

$$
\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &\leq \Delta(G) - 1 + |V(H)| - 1 \\
&\leq \Delta(H) - 1 + |V(G)| - 1 \\
&< \Delta(H) + |V(G)| + 1 \\
&= k,
\end{aligned}
$$

a contradiction. Thus, in this case S is an independent dominating set in G. Thus,

$$
\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_G(a) + |V(H)| \\
&= \Delta(G) - r_G(a) + |V(H)| \\
&= \Delta(H) - r_G(a) + |V(G)| \\
&< \Delta(H) + |V(G)| + 1 \\
&= k,
\end{aligned}
$$

a contradiction.
Case 3: $S \subseteq V(H)$. . Let $a \in S$. Then

$$
\begin{aligned}
|N_{G+H}(a) \setminus S| - |N_{G+H}(a) \cap S| &= deg_H(a) - |N_H(a) \cap S| + |V(G)| - |N_H(a) \cap S| \\
&= \Delta(H) - r_H(a) - 2|N_H(a) \cap S| + |V(G)| \\
&= \Delta(G) + |V(H)| + t - r_H(a) - 2|N_H(a) \cap S| \\
&= \Delta(G) + |V(H)| - (r_H(a) + 2|N_H(a) \cap S| - t)
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta(H) + |V(G)| - (r_H(a) + 2|N_H(a) \cap S| - t) \\
&< \Delta(H) + |V(G)| + 1 \\
&= k,
\end{aligned}
$$

a contradiction. Hence, $\gamma_{ce}^k(G+H) = \infty$. □

The next result follows from Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7.

Corollary 4. *Let G and H be connected graphs such that $\gamma(G) \geq 2$, $\gamma(H) \geq 2$ and $|V(H)| + \Delta(G) \leq |V(G)| + \Delta(H)$. Then*

$$
\gamma_{ce}^k(G+H) = \begin{cases} 2, & if\ 0 \leq k \leq |V(G)| + \Delta(H) - 2 \\ min\{\gamma_i^*(G), \gamma^*(H)\}, & if\ |V(G)| + \Delta(H) - 1 \leq k \leq \Delta(G) + |V(H)| \\ \gamma(H), & if\ |V(H)| + \Delta(G) + 1 \leq k \leq |V(G)| + \Delta(H) \\ \infty & if\ k \geq |V(G)| + \Delta(H) + 1 \end{cases},
$$

where
$\gamma_i^*(G) = min\{|S| : S\ is\ a\ \gamma_i\text{-}set\ in\ G\ and\ \delta(S : G) \geq \Delta(G) - 1\}$,
$\gamma^*(H) = min\{|S| : S\ is\ a\ \gamma\text{-}set\ in\ G\ and\ 0 \leq \Delta(G) + |V(H)| - |V(G)| - deg_H(a) + 2|N_H(a) \cap S| \leq 1\}$, *and*
$\gamma(H) = min\{|S| : S\ is\ a\ \gamma\text{-}set\ in\ G\ and\ deg_H(a) + |V(G)| - |V(H)| - 2|N_H(a) \cap S| \geq p\}$

Corollary 5. Let G and H be connected graphs such that $\gamma(G) \geq 2$, $\gamma(H) \geq 2$ and $|V(H)| + \Delta(G) \leq |V(G)| + \Delta(H)$. Then $\eta(G+H) = |V(G)| + \Delta(H)$ and $\gamma_{ce}^{\eta(G+H)}(G+H) = \gamma(H)$.

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