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Fuzzy sets in Hyper UP-algebras

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Abstract. In this paper, we apply the concept of fuzzy set to hyper UP-subalgebras. We introduce the notions of fuzzy hyper UP-subalgebra and fuzzy hyper UP-filter and establish some of their properties. Furthermore, some properties of hyper homomorphism in relation to fuzzy hyper UP-subalgebras and fuzzy hyper UP-filters are also presented

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Key Words and Phrases: Hyper UP-algebra, hyper UP-filter, fuzzy set, fuzzy hyper UP-subalgebra, fuzzy hyper UP-filter

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1. Introduction

The concept of hypergraphs, which is a generalization of the notion of classical algebraic groups, was introduced by F. Marty [14] in 1934. Since then, hyperstructure theory has seen tremendous development. For basic notions and results on hyperstructure theory and some of its applications, see P. Corcini and V. Leoreanu [4].

L. Zadeh [21] defined a fuzzy set as a class of objects with a continuum of grades of membership, as inspired by the process of human perception and recognition. From its inception in 1965, fuzzy set theory became a phenomenon since its logic can deal with information that is imprecise, vague, partially true, or without sharp boundaries. The reader may refer to [19] for a compilation of articles on fuzzy sets, fuzzy logic and their applications.

Fuzzy hyperstructures is an application of the notion of fuzzy sets and their variants to algebra. Many articles and several books are available on fuzzy hyperstructures, such as: In 1997, P. Corcini and I. Tofan [5] introduced and investigated fuzzy hypergroups. In 2001, Y.B. Jun and X.L. Xin [9] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK-ideal, gave relations among them, and investigated some related properties. V. Leoreanu-Fotea and B. Davvaz [11] introduced and investigated

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fuzzy hyperrings in 2009. They analyzed fuzzy substructures and homomorphisms between fuzzy hyperrings. In 2011, R. Ameri and T. Nozari [2] introduced the concept of fuzzy regular (resp., fuzzy strongly regular) relations of hyperalgebras and obtained their basic properties. They also proved that with every fuzzy hyper algebra, a unique hyperalgebra can be associated via a regular (resp., strongly regular) relation. In 2012, F. Nisar et al. [7] introduced distributive hyper BCI-ideals and applied the concept of fuzzy set to investigate the relations between fuzzy distributive hyper BCI-ideals and distributive hyper BCI-ideals of a hyper BCI-algebra. In 2015, B. Davvaz and I. Cristea [6] summarized the research progress of fuzzy hyperstructures. In 2017, P. Corcini [3] gave a brief excursus on some results on hyperstructures, their connections with fuzzy sets, and extensions to weak structures. In 2019, X. Xin et al. [20] introduced the notions of intuitionistic fuzzy soft (resp., weak, s-weak, strong) hyper BCK-ideal and investigated related properties and relations. Moreover, G. Tabaranza and J. Vilela [18] applied the fuzzy set to hyper B-algebras, while A. Macodi-Ringia and G. Petalcorin Jr. [12] introduced the implicative hyper GR-ideal and the fuzzy implicative hyper GR-ideal of type 1 (resp., of type 2), and investigated several properties. Characterizations of fuzzy implicative hyper GR-ideals of type 1 are also given. Ringia and Petalcorin [13] also investigated intuitionistic fuzzy hyper GR-ideals in hyper GR-algebras in 2020.

In 2017, A. Iampan [8] defined a new algebraic structure called UP-algebra and showed that the notion of UP-algebra is a generalization of KU-algebra that was introduced by C. Prabpayak and U. Leerawat [15]. In the same year, S. Mostafa et al. [17] applied the hyper structure theory to KU-algebras. In 2019, D. Romano [16] introduced the concept of hyper UP-algebra, presented some related properties, and considered homomorphisms between hyper UP-algebras. In 2020, R. Amairanto and R. Isla [1] investigated the concept of regular congruence relation on hyper UP-algebras and established some homomorphism theorems on such algebras. They also examined the notion of hyper product of hyper UP-algebras.

In this paper, we apply the concept of fuzzy set to hyper UP-algebras. We introduce the notions of fuzzy hyper UP-subalgebra and fuzzy hyper UP-filter and investigate their basic properties. Some properties of hyper homomorphism in relation to fuzzy hyper UP-subalgebra and fuzzy hyper UP-filter are also provided.

2. Preliminaries

Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a mapping from $H \times H$ into $\mathcal{P}^*(H)$.

Definition 1. [16] A *hyper* UP-*algebra* is a set H with constant 0 and hyperoperation $\circledast$ satisfying the following axioms: for all $x, y, z \in H$,

(HUP1) $y \circledast z \ll [(x \circledast y) \circledast (x \circledast z)]$,

(HUP2) $x \circledast 0 = \{0\}$,

(HUP3) $0 \circledast x = \{x\}$,

(HUP4) $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x \circledast y$ and for every $A, B \subseteq \mathcal{P}^*(H)$, $A \ll B$ is defined by: for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

A hyper UP-algebra H with constant 0 and hyperoperation $\circledast$ is denoted by $(H; \circledast, 0)$. By (HUP2) or (HUP3), $x \circledast y \neq \varnothing$ for all $x, y \in H$.

Example 1. Let $H = \{0, a, b, c, d\}$ be a set with a binary operation $\circledast$ defined by the following Cayley table:

$\circledast$	0	a	b	c	d
0	{0}	{a}	{b}	{c}	{d}
a	{0}	{0,a}	{0,b}	{c}	{d}
b	{0}	{a}	{0,b}	{c}	{d}
c	{0}	{0,a}	{0,b}	{0,a,c}	{d}
d	{0}	{0,a}	{0,b}	{0,a,c}	{0,d}

By routine calculations, $(H; \circledast, 0)$ is a hyper UP-algebra.

Definition 2. [16] Let $(H, \circledast, 0)$ be a hyper UP-algebra and let I be a subset of H containing 0. If I is a hyper UP-algebra with respect to the hyper operation " $\circledast$ " on H, we say that I is a *hyper* UP-*subalgebra* of H.

Example 2. In Exampe 1, it can be verified that the set $\{0, a, b, c\}$ is a hyper UP-algebra. Thus, $\{0, a, b, c\}$ is a hyper UP-subalgebra of H.

Example 3. Let $K = \{0, 1, 2\}$ be a set with a binary operation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0,2}	{0,2}
2	{0}	{1}	{0,2}

By routine calculations, $(K; \circledast, 0)$ is a hyper UP-algebra. Note that if $I = \{0, 1\}, I$ is not a hyper UP-subalgebra since $1 \circledast 1 = \{0, 2\} \nsubseteq I$.

Proposition 1. [16] *Let* $(H; \circledast, 0)$ *be a hyper* UP-*algebra. Then the following hold for all* $x, y \in H$ *and for every nonempty subsets* $A, B \subseteq H$:

(*i*) $0 \circledast 0 = \{0\}$

(*ii*) $x \ll 0$

(*iii*) $x \ll x$

(*iv*) $y \ll x \circledast y$

(*v*) $A \subseteq B$ *implies* $A \ll B$

(*vi*) $0 \circledast A = A$

(*vii*) $A \circledast 0 = \{0\}$

Proposition 2. [16] Let I be a non-empty subset of a hyper UP-algebra $(H, \circledast, 0)$. Then I is a hyper UP-subalgebra of H if and only if for all $x, y \in I, x \circledast y \subseteq I$ holds.

Definition 3. [16] Let $(H; \circledast, 0)$ and $(H'; \circledast', 0')$ be hyper UP-algebras. A mapping $f : H \to H'$ is called a *hyper homomorphism* if

(HH1) $f(0) = 0'$, and

(HH2) $f(x \circledast y) = f(x) \circledast' f(y)$ for all $x, y \in H$.

Definition 4. [21] A *fuzzy set* in a nonempty set H (or a *fuzzy subset* of H) is an arbitrary function $\mu : H \longrightarrow [0, 1]$, where $[0, 1]$ is the unit segment of the real line.

If $I \subseteq H$, the *characteristic function* μ_I of H is a function of H into $\{0, 1\}$ defined as follows:

$$\mu_I(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{if } x \notin I. \end{cases}$$

By the definition of characteristic function, μ_I is a function of H into $\{0, 1\} \subset [0, 1]$. Then, μ_I is a fuzzy set in H.

Definition 5. [10] Let X and Y be two nonempty sets, μ a fuzzy set of Y and $f : X \longrightarrow Y$ a mapping. The *preimage* of μ under f, denoted by μ^f, is the fuzzy set of X defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$, that is, $\mu^f = \mu \circ f$.

Definition 6. [10] Let μ be a fuzzy set of X and $f : X \longrightarrow Y$ a mapping. The mapping $f(\mu) : Y \longrightarrow [0, 1]$ defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)}\{\mu(x)\}, & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{if } f^{-1}(y) = \varnothing, \end{cases}$$

is called the *image of μ under f*, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Definition 7. [10] Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy sets of X, where $\mathcal{A}$ is an arbitrary index set. The *intersection* of μ_α, denoted by $\wedge_{\alpha \in \mathcal{A}} \mu_\alpha$, is defined by $\wedge_{\alpha \in \mathcal{A}} \mu_\alpha(x) = \inf_{\alpha \in \mathcal{A}}\{\mu_\alpha(x)\}$ for all $x \in X$.

3. Fuzzy hyper UP-subalgebras

In this section, we introduce the notion of fuzzy hyper UP-subalgebra and study some of its basic properties. For brevity, we denote a hyper UP-algebra $(H; \circledast, 0)$ by H, from here onwards.

Definition 8. A fuzzy set μ in a hyper UP-algebra H is called a *fuzzy hyper* UP-*subalgebra* of H if for any $x, y \in H$,

$$\inf_{a \in x \circledast y} \{\mu(a)\} \geq \min\{\mu(x), \mu(y)\}.$$

Example 4. 1. By Examples 1 and 2, $H = \{0, a, b, c, d\}$ is a hyper UP-algebra and the set $\{0, a, b, c\}$ is a hyper UP-subalgebra. It can be easily verified that

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \{0, a, b, c\} \\ 0, & \text{if } x \in \{d\} \end{cases}$$

is a fuzzy hyper UP-subalgebra of H.

2. Let $H = \{0, 1, 2\}$ be a set with a binary operation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0,2}	{0,2}
2	{0}	{1}	{0,2}

Define a fuzzy subset $\mu : H \to [0, 1]$ by $\mu(0) = 0.9, \mu(1) = 0.5, \mu(2) = 0.3$. Then μ is not a fuzzy hyper UP-subalgebra of H since $0.3 = \mu(2) = \inf_{a \in 1 \circledast 1} \{\mu(a)\} < \min\{\mu(1), \mu(1)\} = 0.5$.

Lemma 1. *Let μ be a fuzzy hyper* UP*-subalgebra of a hyper* UP*-algebra H. Then $\mu(0) \geq \mu(x)$ for all $x \in H$. Moreover, if μ is onto, then $\mu(0) = 1$.*

Proof. Let $x \in H$. By Proposition 1(iii), $x \ll x$, that is, $0 \in x \circledast x$. Then $\mu(0) \geq \inf_{a \in x \circledast x} \{\mu(a)\} \geq \min\{\mu(x), \mu(x)\} = \mu(x)$. If μ is onto, then $\mu(y) = 1$, for some $y \in H$. Thus, $1 = \mu(y) \leq \mu(0) \leq 1$. Hence, $\mu(0) = 1$. □

Theorem 1. *Let μ be a fuzzy hyper* UP*-subalgebra of a hyper* UP*-algebra H. Then there exists a sequence $\langle x_n \rangle \subseteq H$ such that $\lim_{n \to \infty} \mu(x_n) = 1$ if and only if $\mu(0) = 1$.*

Proof. Suppose that there exists $\langle x_n \rangle \subseteq H$ such that $\lim_{n \to \infty} \mu(x_n) = 1$. By Lemma 1, it will follow that $\mu(0) \geq \mu(x_n)$ for all n. This implies that $1 \geq \mu(0) = \lim_{n \to \infty} \mu(0) \geq \lim_{n \to \infty} \mu(x_n) = 1$, that is, $\mu(0) = 1$. Conversely, assume that $\mu(0) = 1$. Consider the sequence $\langle x_n \rangle = \langle 0, 0, 0, \cdots, 0, \cdots \rangle$ in H. Thus, $\langle \mu(x_n) \rangle = \langle \mu(0), \mu(0), \mu(0), \cdots, \mu(0), \cdots \rangle = \langle 1, 1, 1, \cdots, 1, \cdots \rangle$. Hence, $\lim_{n \to \infty} \mu(x_n) = 1$. □

Definition 9. Let μ be a fuzzy subset of a hyper UP-algebra H and $t \in [0, 1]$. Then the *upper level set* μ_t is the set $\mu_t = \{x \in H : \mu(x) \geq t\}$.

Theorem 2. *Let μ be a fuzzy subset of a hyper* UP*-algebra H. Then μ is a fuzzy hyper* UP*-subalgebra of H if and only if for all $t \in [0, 1]$, $\varnothing \neq \mu_t$ is a hyper* UP*-subalgebra of H.*

Proof. Suppose that μ is a fuzzy hyper UP-subalgebra of H and $\mu_t \neq \varnothing$. Let $x, y \in \mu_t$ and let $a \in x \circledast y$. Then $\mu(x), \mu(y) \geq t$ and so $\min\{\mu(x), \mu(y)\} \geq t$. Since μ is a fuzzy hyper UP-subalgebra of H, by Definition 8, $\mu(a) \geq \inf_{a' \in x \circledast y} \{\mu(a')\} \geq t$. It follows that $a \in \mu_t$. Since a is arbitrary, $x \circledast y \subseteq \mu_t$. By Proposition 2, μ_t is a hyper UP-subalgebra of H. Conversely, suppose μ_t is a hyper UP-subalgebra of H for each $t \in [0, 1]$. Let $x, y \in H$.

Then $\mu(x) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(y) \geq \min\{\mu(x), \mu(y)\}$. Take $t_0 = \min\{\mu(x), \mu(y)\}$. Then $x, y \in \mu_{t_0}$ and so $x \circledast y \subseteq \mu_{t_0}$. Let $a \in x \circledast y$. Then $\mu(a) \geq t_0$ which implies that $\inf_{a \in x \circledast y} \{\mu(a)\} \geq t_0 = \min\{\mu(x), \mu(y)\}$. Thus, μ is a fuzzy hyper UP-subalgebra of H. □

Theorem 3. *Let μ be a fuzzy hyper* UP*-subalgebra of a hyper* UP*-algebra H. Then the set $H_\mu = \{x \in H : \mu(x) = \mu(0)\}$ is a hyper* UP*-subalgebra of H.*

Proof. Note that $0 \in H_\mu$ so that $H_\mu \neq \varnothing$. Let $x, y \in H_\mu$ and let $a \in x \circledast y$. Then $\mu(x) = \mu(y) = \mu(0)$. Since μ is a fuzzy hyper UP-subalgebra, $\mu(a) \geq \mu(0)$ by Definition 8. Thus, by Lemma 1, $\mu(a) = \mu(0)$ for all $a \in x \circledast y$, that is, $a \in H_\mu$. Since a is an arbitrary element of $x \circledast y$, it follows that $x \circledast y \subseteq H_\mu$. Thus, H_μ is a hyper UP-subalgebra of H. □

Lemma 2. *Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy hyper* UP*-algebras of a hyper* UP*-algebra H. Then*

(i) $\inf\limits_{a \in x \circledast y} \{\inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(a)\}\} \geq \inf\limits_{\alpha \in \mathcal{A}} \{\inf\limits_{a \in x \circledast y} \{\mu_\alpha(a)\}\}$.

(ii) $\inf_{\alpha \in \mathcal{A}} \{\min\{\mu_\alpha(x), \mu_\alpha(y)\}\} \geq \min\{\inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(y)\}\}$.

Proof. Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy hyper UP-subalgebras of H and let $x, y \in H$.

(i) For all $a \in x \circledast y$, we have $\mu_\alpha(a) \geq \inf_{a \in x \circledast y} \{\mu_\alpha(a)\}$ for all $\alpha \in \mathcal{A}$. Thus, for all $a \in x \circledast y, \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(a)\} \geq \inf_{a \in x \circledast y} \{\mu_\alpha(a)\} \geq \inf_{\alpha \in \mathcal{A}} \{\inf_{a \in x \circledast y} \{\mu_\alpha(a)\}\}$. Hence, $\inf_{a \in x \circledast y} \{\inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(a)\}\} \geq \inf_{\alpha \in \mathcal{A}} \{\inf_{a \in x \circledast y} \{\mu_\alpha(a)\}\}$.

(ii) Note that $\mu_\alpha(x) \geq \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}$ and $\mu_\alpha(y) \geq \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(y)\}$. Then, $\{\min\{\mu_\alpha(x), \mu_\alpha(y)\}\} \geq \min\{\inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(y)\}$. Thus, $\inf_{\alpha \in \mathcal{A}} \{\min\{\mu_\alpha(x), \mu_\alpha(y)\}\} \geq \min\{\inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf\limits_{\alpha \in \mathcal{A}} \{\mu_\alpha(y)\}$.

Hence, the conclusion follows. □

Example 5. Let $H = \{0, 1, 2, 3\}$ be a set. Define the hyperoperation $\circledast$ by the following Cayley table:

$\circledast$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0, 1}	{1, 2, 3}	{0, 3}
2	{0}	{0, 1}	{0, 2}	{0, 3}
3	{0}	{1}	{1, 2, 3}	{0, 3}

By routine calculations, $(H; \circledast, 0)$ is a hyper UP-algebra. Define a fuzzy subset $\mu : H \to [0, 1]$ by $\mu(0) = 0.9, \mu(1) = 0.7, \mu(2) = 0.4, \mu(3) = 0.2$. Let $A = \{0, 1\}$ and $B = \{0, 2\}$. By routine calculations, μ_A and μ_B are fuzzy hyper UP-subalgebras of H. But $\mu_{A \cup B}$ is not a fuzzy hyper UP-subalgebra of H since $0.2 = \inf_{a \in 1 \circledast 2} \{\mu(a)\} < \min\{\mu(1), \mu(2)\} = 0.4$.

Remark 1. *The union of two fuzzy hyper UP-subalgebras of a hyper* UP*-algebra H is not necessarily a fuzzy hyper UP-subalgebra of H.*

Theorem 4. *The intersection of any nonempty family of fuzzy hyper* UP*-subalgebras of a hyper* UP*-algebra H is also a fuzzy hyper* UP*-subalgebra of H.*

Proof. Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy hyper UP-subalgebras of H. Let $x, y \in H$. Then by Definitions 7 and 8 and Lemma 2(i) and (ii),

$$\begin{aligned}\inf_{a\in x\circledast y}\{\wedge_{\alpha\in\mathcal{A}}\mu_\alpha(a)\} &= \inf_{a\in x\circledast y}\{\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(a)\}\} \\ &\geq \inf_{\alpha\in\mathcal{A}}\{\inf_{a\in x\circledast y}\{\mu_\alpha(a)\}\} \\ &\geq \inf_{\alpha\in\mathcal{A}}\{\min\{\mu_\alpha(x), \mu_\alpha(y)\}\} \\ &\geq \min\{\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(x)\}, \inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(y)\}\} \\ &= \min\{\wedge_{\alpha\in\mathcal{A}}\mu_\alpha(x), \wedge_{\alpha\in\mathcal{A}}\mu_\alpha(y)\}.\end{aligned}$$

Hence, the conclusion follows. □

Theorem 5. *Let K be a nonempty subset of a hyper* UP*-algebra H and let $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Let μ_K be a fuzzy subset of H defined by*

$$\mu_K(x) = \begin{cases} \alpha, & \text{if } x \in K \\ \beta, & \text{if } x \notin K \end{cases}$$

for all $x \in H$. Then μ_K is a fuzzy hyper UP*-subalgebra of H if and only if K is a hyper* UP*-subalgebra of H.*

Proof. Suppose that μ_K is a fuzzy hyper UP-subalgebra of H and let $x, y \in K$. Then $\mu_K(x) = \alpha = \mu_K(y)$. By Definition 8, $\inf_{a\in x\circledast y}\{\mu_K(a)\} \geq \alpha$. Since $\beta < \alpha, \mu_K(a) = \alpha$ for all $a \in x \circledast y$. This implies that $x \circledast y \subseteq K$. Thus, K is a hyper UP-subalgebra of H.

Conversely, suppose K is a hyper UP-subalgebra of H. Let $x, y \in H$. If $x, y \in K$, then $x \circledast y \in K$ since K is a hyper UP-subalgebra of H. Hence, $\inf_{a\in x\circledast y}(\mu_K(a)) = \alpha = \min\{\mu_K(x), \mu_K(y)\}$. Suppose $x \notin K$ or $y \notin K$. Then $\min\{\mu_K(x), \mu_K(y)\} = \beta$. Thus, $\inf_{a\in x\circledast y} \mu_K(a) \geq \min\{\mu_K(x), \mu_K(y)\}$. This shows that μ_K is a fuzzy hyper UP-subalgebra of H. □

Theorem 6. *Let H be a hyper* UP*-algebra. Then every hyper* UP*-subalgebra of H is an upper level hyper* UP*-subalgebra of a fuzzy hyper* UP*-subalgebra of H.*

Proof. Let K be a hyper UP-subalgebra of H. For a fixed $t \in (0, 1]$, we consider the fuzzy subset μ defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in K \\ 0, & \text{if } x \notin K. \end{cases}$$

By Theorem 5, μ is a fuzzy hyper UP-subalgebra of H. Since $0 \in K, \mu(0) = t$. Hence, $0 \in \mu_t = \{x \in H : \mu(x) = t\}$. By Theorem 2, μ_t is a hyper UP-subalgebra of H. Let

$x \in K$. Then $\mu(x) = t$ which implies that $x \in \mu_t$. Thus, $K \subseteq \mu_t$. On the other hand, suppose that $x \in \mu_t$. Then $\mu(x) = t$ which means that $x \in K$. Thus, $\mu_t \subseteq K$. Hence, $K = \mu_t$. □

Recall that each hyper UP-subalgebra of a hyper UP-algebra H contains the element 0. Thus, for any family $\{K_n\}$ of hyper UP-subalgebras of H, $0 \in \bigcap_{n=1}^{\infty} K_n$ and so $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Theorem 7. *Let H be a hyper* UP*-algebra and let $\{K_n : n = 1, 2, \cdots\}$ be a family of hyper* UP*-subalgebras of H such that $H = K_1 \supseteq K_2 \supseteq \cdots$. Let μ be a fuzzy set in H defined by*

$$\mu(x) = \begin{cases} \frac{n}{n+1}, & \text{if } x \in K_n \backslash K_{n+1}, \\ 1, & \text{if } x \in \bigcap_{n=1}^{\infty} K_n. \end{cases}$$

Then μ is a fuzzy hyper UP*-subalgebra of H.*

Proof. Let $x, y \in H$. Consider the following cases.

Case 1: $x, y \in K_n \backslash K_{n+1}$. Then $\mu(x) = \frac{n}{n+1} = \mu(y)$. Since K_n is a hyper UP-subalgebra, $x \circledast y \subseteq K_n$. Then $x \circledast y \subseteq K_{n+1}$ or $x \circledast y \nsubseteq K_{n+1}$. If $x \circledast y \nsubseteq K_{n+1}$, then $x \circledast y \subseteq K_n \backslash K_{n+1}$ and for all $a \in x \circledast y$, $\mu(a) = \frac{n}{n+1}$. Thus, $\inf_{a \in x \circledast y}\{\mu(a)\} = \frac{n}{n+1} = \min\{\mu(x), \mu(y)\}$. Suppose $x \circledast y \nsubseteq K_n \backslash K_{n+1}$. If $x \circledast y \subseteq \cap_{m=1}^{\infty} K_m$, then $\inf_{a \in x \circledast y}\{\mu(a)\} = 1 > \frac{n}{n+1} = \min\{\mu(x), \mu(y)\}$. Suppose $x \circledast y \nsubseteq K_m$. Then there exists $a \in x \circledast y$ such that $a \in K_m \backslash K_{m+1}$ for some $m \geq n$. By the Well-Ordering Principle, there exists a smallest positive integer $p \geq n$ such that $z \in K_p \backslash K_{p+1}$ for some $z \in x \circledast y$. Hence, $\inf_{a \in x \circledast y}\{\mu(a)\} = \frac{p}{p+1} \geq \frac{n}{n+1}$.

Case 2: $x \in K_s \backslash K_{s+1}, y \in K_r \backslash K_{r+1}$. Without loss of generality, assume that $s < r$. The $\frac{s}{s+1} < \frac{r}{r+1}$ and $K_s \supseteq K_r$. Since K_s is a hyper UP-subalgebra, $x \circledast y \subseteq K_s$. As in the case of a portion of Case 1, $\inf_{a \in x \circledast y}\{\mu(a)\} = \frac{s+j}{s+j+1} > \frac{s}{s+1} = \min\{\mu(x), \mu(y)\}$.

Case 3: $x, y \in \bigcap_{n=1}^{\infty} K_n$. Then $x \circledast y \subseteq \bigcap_{n=1}^{\infty} K_n$ so that for all $a \in x \circledast y, \mu(a) = 1$. Thus, $\inf_{a \in x \circledast y}\{\mu(a)\} = 1 = \min\{\mu(x), \mu(y)\}$.

Case 4: $x \in \bigcap_{n=1}^{\infty} K_n$, and $y \notin \bigcap_{n=1}^{\infty} K_n$ or ($x \notin \bigcap_{n=1}^{\infty} K_n$ and $y \in \bigcap_{n=1}^{\infty} K_n$.) Without loss of generality, assume that $x \in \bigcap_{n=1}^{\infty} K_n$ and $y \notin \bigcap_{n=1}^{\infty} K_n$. This means that there exists r such that $y \notin K_r$. Thus, the set $S = \{q : y \notin K_q\} \neq \varnothing$. By Well-Ordering Principle, there exists a smallest element $t \in S$. This means that $y \in K_{t-1} \backslash K_t$ so that $\mu(y) = \frac{t-1}{t}$. So, $\min\{\mu(x), \mu(y)\} = \min\{1, \frac{t-1}{t}\} = \frac{t-1}{t}$. As in the case of a part of Case 1, $\inf_{a \in x \circledast y}\{\mu(a)\} = \frac{t+j}{t+j+1} > \frac{t-1}{t} = \min\{\mu(x), \mu(y)\}$. □

Corollary 1. *Let H be a hyper* UP*-algebra. Then for any family of hyper* UP*-subalgebras $\{K_n : n = 1, 2, \cdots\}$ of H such that $H = K_1 \supseteq K_2 \supseteq \cdots$, there exists fuzzy hyper* UP*-subalgebras of H whose upper level sets are exactly the hyper* UP*-subalgebras in the family.*

Proof. Define a fuzzy set μ of H by

$$\mu(x) = \begin{cases} \frac{n}{n+1}, & \text{if } x \in K_n \backslash K_{n+1}, \\ 1, & \text{if } x \in \bigcap_{n=1}^{\infty} K_n. \end{cases}$$

Then by Theorem 7, μ is a fuzzy hyper UP-subalgebra of H. Let $x \in H$. For each n, consider $\mu_{\frac{n}{n+1}} = \{x \in H : \mu(x) \geq \frac{n}{n+1}\}$. Let $x \in K_n$. If $x \in K_n \backslash K_{n+1}$, then $\mu(x) = \frac{n}{n+1}$. If $x \in \cap_{m=1}^{\infty} K_m$, then $\mu(x) = 1 > \frac{n}{n+1}$. Hence, $x \in \mu_{\frac{n}{n+1}}$, showing that $K_n \subseteq \mu_{\frac{n}{n+1}}$. If $y \in \mu_{\frac{n}{n+1}}$, then $\mu(y) \geq \frac{n}{n+1}$. If $\mu(y) = 1$, then $y \in \cap_{m=1}^{\infty} K_m$. Hence, $y \in K_n$. Suppose that $y \notin \cap_{m=1}^{\infty} K_m$. Choose the smallest integer p such that $y \notin K_p$. Then $y \in K_{p-1} \backslash K_p$ and $\mu(y) = \frac{p-1}{p} \geq \frac{n}{n+1}$. Hence, $p - 1 \geq n$. This implies that $y \in K_n$ and $\mu_{\frac{n}{n+1}} \subseteq K_n$. Thus, $\frac{n}{n+1} = K_n$. □

4. Fuzzy hyper UP-filter

In this section, we introduce the notion of fuzzy hyper UP-filter of a hyper UP-algebra and study some of its basic properties.

Definition 10. A subset G of a hyper UP-algebra H is called a *hyper* UP-*filter* of H if it satisfies the following properties:

(i) $0 \in G$

(ii) for any $x, y \in H, x \in G$ and $x \circledast y \subseteq G$ imply $y \in G$.

In Example 1, it can be verified by routine calculations that the set $\{0, a, c\}$ is a hyper UP-filter of H.

Definition 11. A fuzzy set μ in a hyper UP-algebra H is called a *fuzzy hyper* UP-*filter* of H if it satifies the following properties: for any $x, y \in H$,

(i) $\mu(0) \geq \mu(x)$,

(ii) $\mu(y) \geq \min\{\mu(x), \inf_{a \in x \circledast y}\{\mu(a)\}\}$.

Remark 2. *A fuzzy hyper UP-filter need not be a fuzzy hyper UP-subalgebra.*

Example 6. Consider the hyper UP-filter $\{0, a, c\}$ of H in Example 1. It can be easily verified that

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \{0, a, c\} \\ 0, & \text{if } x \in \{b, d\} \end{cases}$$

is a fuzzy hyper UP-filter of H.

Example 7. Consider the fuzzy set μ in Example 4(2). Note that μ is not a fuzzy hyper UP-subalgebra of H but by routine calculations, μ is a fuzzy hyper UP-filter of H.

Proposition 3. *Let μ be a fuzzy subset of a hyper* UP*-algebra H. If μ is a fuzzy hyper* UP*-filter of H, then μ_t is a hyper* UP*-filter of H for each $t \in [0,1]$ with $\mu_t \neq \varnothing$.*

Proof. Suppose that μ is a fuzzy hyper UP-filter of H and $\mu_t \neq \varnothing$, where $t \in [0,1]$. Then there exists $a \in \mu_t$ and so $\mu(a) \geq t$. By Definition 11, $\mu(0) \geq \mu(a) \geq t$, that is, $0 \in \mu_t$. Let $x, y \in H$ with $x \in \mu_t$ and $x \circledast y \subseteq \mu_t$. Then $\mu(x) \geq t$ and for all $a \in x \circledast y, \mu(a) \geq t$. Hence, $\inf_{a \in x \circledast y}\{\mu(a)\} \geq t$. Since μ is a fuzzy hyper UP-filter, $\mu(y) \geq \min\{\mu(x), \inf_{a \in x \circledast y}\{\mu(a)\}\} \geq t$. Hence, $y \in \mu_t$ and so μ_t is a hyper UP-filter of H. □

For any nonempty subset K of a hyper UP-algebra H, we define a fuzzy set μ_K in H by

$$\mu_K(x) = \begin{cases} \alpha, & \text{if } x \in K \\ \beta, & \text{if } x \notin K \end{cases}$$

for all $x \in H$ and $\alpha, \beta \in [0,1]$ with $\alpha > \beta$.

Lemma 3. *Let K be a nonempty subset of a hyper* UP*-algebra H. If μ_K is a fuzzy hyper* UP*-filter of H, then $0 \in K$.*

Proof. Let μ_K be a fuzzy hyper UP-filter of H. Since $K \neq \varnothing$, it follows that there exists $a \in K$ with $\mu_K(a) = \alpha$. By Definition 11(i), $\mu_K(0) \geq \mu_K(a) = \alpha$. Since $\alpha > \beta$, we have $\mu_K(0) = \alpha$. Hence, $0 \in K$. □

Theorem 8. *Let K be a nonempty subset of a hyper* UP*-algebra H. If μ_K is a fuzzy hyper* UP*-filter of H, then K is a hyper* UP*-filter of H.*

Proof. Suppose that μ_K is a fuzzy hyper UP-filter of H. Let $x, y \in H$ with $x \in K$ and $x \circledast y \subseteq K$. Then for all $a \in x \circledast y$, we have $a \in K$, that is, $\mu_K(a) = \alpha$. It follows that $\inf_{a \in x \circledast y}\{\mu(a)\} = \alpha$. By Definition 11(ii), $\mu(y) \geq \min\{\mu(x), \inf_{a \in x \circledast y}\{\mu(a)\}\} = \alpha$. Since $\alpha > \beta$, it follows that $\mu_K(y) = \alpha$. Hence, $y \in \mu_K$ and since $0 \in K$ by Lemma 3, μ_K is a hyper UP-filter of H. □

Lemma 4. *Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy subsets of a hyper* UP*-algebra H. Then*

$$\inf_{\alpha \in \mathcal{A}} \left\{ \min \left\{ \mu_\alpha(x), \inf_{a \in x \circledast y} \{\mu_\alpha(a)\} \right\} \right\} \geq \min \left\{ \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf_{\alpha \in \mathcal{A}} \{ \inf_{a \in x \circledast y} \{\mu_\alpha(a)\}\} \right\}.$$

Proof. Let $x, y \in H$. Note that $\forall \alpha \in \mathcal{A}, \mu_\alpha(x) \geq \inf_{\alpha \in \mathcal{A}}\{\mu_\alpha(x)\}$ and $\inf_{a \in x \circledast y}\{\mu_\alpha(a)\} \geq \inf_{\alpha \in \mathcal{A}}\{\inf_{a \in x \circledast y}\{\mu_\alpha(a)\}\}$. Thus, $\forall\ \alpha \in \mathcal{A}$, we have

$$\min\{\mu_\alpha(x), \inf_{a \in x \circledast y}\{\mu_\alpha(a)\}\} \geq \min \left\{ \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf_{\alpha \in \mathcal{A}} \{ \inf_{a \in x \circledast y} \{\mu_\alpha(a)\}\} \right\}.$$

Hence,

$$\inf_{\alpha \in \mathcal{A}} \left\{ \min \left\{ \mu_\alpha(x), \inf_{a \in x \circledast y} \{\mu_\alpha(a)\} \right\} \right\} \geq \min \left\{ \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}, \inf_{\alpha \in \mathcal{A}} \{ \inf_{a \in x \circledast y} \{\mu_\alpha(a)\}\} \right\}. \quad \square$$

Theorem 9. *Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy subsets of a hyper* UP-*algebra H. If μ_α is a fuzzy hyper* UP-*filter of H for all $\alpha \in \mathcal{A}$, then so is $\wedge_{\alpha\in\mathcal{A}}\mu_\alpha$.*

Proof. Let $x, y \in H$. Since each μ_α is a fuzzy hyper UP-filter of $H, \mu_\alpha(0) \geq \mu_\alpha(x)$ for all $\alpha \in \mathcal{A}$. Thus for all $\alpha \in \mathcal{A}, \inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(0)\} \geq \mu_\alpha(x)$. Hence, $\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(0)\} \geq \inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(x)\}$, or equivalently, $\wedge_{\alpha\in\mathcal{A}}\mu_\alpha(0) \geq \wedge_{\alpha\in\mathcal{A}}\mu_\alpha(x)$. Now, by Definitions 11 and 7 and Lemma 4,

$$\begin{aligned}\wedge_{\alpha\in\mathcal{A}}\mu_\alpha(y) &= \inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(y)\}\\ &\geq \inf_{\alpha\in\mathcal{A}}\left\{\min\{\mu_\alpha(x), \inf_{a\in x\circledast y}\{\mu_\alpha(a)\}\}\right\}\\ &\geq \min\left\{\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(x), \inf_{\alpha\in\mathcal{A}}\{\inf_{a\in x\circledast y}\{\mu_\alpha(a)\}\}\right\}\\ &\geq \min\left\{\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(x), \inf_{a\in x\circledast y}\{\inf_{\alpha\in\mathcal{A}}\{\mu_\alpha(a)\}\}\right\}\\ &= \min\left\{\wedge_{\alpha\in\mathcal{A}}\mu_\alpha(x), \inf_{a\in x\circledast y}\{\wedge_{\alpha\in\mathcal{A}}\{\mu_\alpha(a)\}\}\right\}.\end{aligned}$$

Hence, the conclusion follows. □

5. Hyper Homomorphism of Fuzzy Hyper UP-algebras

This section provides some properties of hyper homomorphism in relation to the concepts of fuzzy hyper UP-subalgebra and fuzzy hyper UP-filter. By Definitions 3, 5, 7 and Lemma 1, we deduce the following proposition:

Proposition 4. *Let $f : G \to H$ be a hyper homomorphism of hyper* UP-*algebras G and H.*

(i) *If μ is a fuzzy hyper* UP-*subalgebra of G, then $f(\mu)(0_H) = \mu(0_G)$.*

(ii) *If μ is a fuzzy hyper* UP-*subalgebra of H, then $\mu^f(0_G) = 0_H$.*

Proof. Let $f : G \to H$ be a hyper homomorphism.

(i) By Definition 3, $0_G \in f^{-1}(0_H)$ and so $f^{-1}(0_H) \neq \varnothing$. By Definition 6 and Lemma 1,

$$f(\mu)(0_H) = \sup_{x\in f^{-1}(0_H)}\{\mu(x)\} = \mu(0_G).$$

(ii) By Definitions 3 and 5, $\mu^f(0_G) = \mu(f(0_G)) = \mu(0_H)$. □

Lemma 5. *Let $f : G \to H$ be a hyper homomorphism of hyper* UP-*algebras G and H, μ be a fuzzy subset of H and μ^f be a fuzzy hyper* UP-*subalgebra of G. If $A = \{\mu(f(a)) = \mu^f(a) : a \in x \circledast_G y\}$ and $B = \{\mu(f(a)) = \mu^f(a) : f(a) \in f(x) \circledast_H f(y)\}$, then A=B.*

Proof. Let $z \in A$. Then $z = \mu(f(a))$ for some $a \in x \circledast_G y$. Since f is a hyper homomorphism, we have $f(a) \in f(x \circledast_G y) = f(x) \circledast_H f(y)$. Thus $z \in B$ and so $A \subseteq B$. Conversely, suppose that $z \in B$. Then $z = \mu(f(a))$ for some $f(a) \in f(x) \circledast_H f(y)$. Since f is a hyper homomorphism, $f(a) \in f(x) \circledast_H f(y) = f(x \circledast_G y)$. Thus, $f(a) = f(a')$, where $a' \in x \circledast_G y$. So, $z = \mu(f(a'))$, where $a' \in x \circledast_G y$. Hence, $z \in A$ which means that $B \subseteq A$. Therefore, $A = B$. □

Theorem 10. *Let $f : G \to H$ be a hyper epimorphism of hyper* UP*-algebras G and H and μ be a fuzzy subset of H. If μ^f is a fuzzy hyper* UP*-subalgebra of G, then μ is a fuzzy hyper* UP*-subalgebra of H.*

Proof. Let f be a hyper epimorphism of hyper UP-algebras G and H and suppose μ^f is a fuzzy hyper UP-subalgebra of G. Let $x, y \in H$. Since f is onto, there exist $x', y' \in G$ such that $f(x') = x$ and $f(y') = y$. Hence, $\mu(x) = \mu^f(x')$ and $\mu(y) = \mu^f(y')$. Let $z \in x \circledast_H y$. Then f is onto implies that there exists $z' \in G$ such that $f(z') = z$, that is, $\mu^f(z') = \mu(z)$. Note that $f(z') = z \in x \circledast_H y = f(x') \circledast_H f(y')$. Hence, by Lemma 5, $z' \in x' \circledast_G y'$. By Definition 8 and the assumption that μ^f is a fuzzy hyper UP-subalgebra of G, we have

$$\begin{aligned}\mu(z) = \mu^f(z') &\geq \inf\nolimits_{a \in x' \circledast_G y'}\{\mu^f(a)\} \\ &\geq \min\{\mu^f(x'), \mu^f(y')\} \\ &= \min\{\mu(f(x')), \mu(f(y'))\} \\ &= \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Thus, $\inf_{z \in x \circledast_H y}\{\mu(z)\} \geq \min\{\mu(x), \mu(y)\}$. Hence, μ is a fuzzy hyper UP-subalgebra of H. □

Theorem 11. *Let $f : G \to H$ be a hyper homomorphism of hyper* UP*-algebras G and H. If μ is a fuzzy hyper* UP*-filter of H, then μ^f is a fuzzy hyper* UP*-filter of G.*

Proof. Let $x \in G$. Then $f(x) \in H$. Since $\mu : H \longrightarrow [0, 1]$ is a fuzzy hyper UP-filter of $H, \mu(0_H) = \mu(f(0_G)) \geq \mu(f(x))$, that is, $\mu^f(0_G) \geq \mu^f(x)$ for all $x \in G$. Next, let $x, y \in G$. Then $f(x), f(y) \in H$. Again, by our assumption on μ, and by Lemma 5,

$$\begin{aligned}\mu(f(y)) = \mu^f(y) &\geq \min\{\mu(f(x)), \inf_{f(z) \in f(x) \circledast_H f(y)}\{\mu(f(z))\}\} \\ &= \min\{\mu^f(x), \inf_{z \in x \circledast_G y}\{\mu^f(z)\}\}.\end{aligned}$$

Hence, μ^f is a fuzzy hyper UP-filter of G. □

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