Revisiting Domination, Hop Domination, and Global Hop Domination in Graphs

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Abstract. A set $S \subseteq V(G)$ is a hop dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. It is a global hop dominating set of G if it is a hop dominating set of both G and the complement $\overline{G}$ of G. The minimum cardinality of a hop dominating (global hop dominating) set of G, denoted by $\gamma_h(G)$ (resp. $\gamma_{gh}(G)$), is called the hop domination (resp. global hop domination) number of G. In this paper, we give some realization results involving domination, hop domination, and global hop domination parameters. Also, we give a rectification of a result found in a recent paper of the authors and use this to prove some results in this paper.

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1. Introduction

Domination has been a topic of interest to many researchers in the field of Graph Theory. By imposing certain additional conditions or formulating similar conditions from the standard concept, a variant can then be obtained. Indeed, the domination concept yielded several variations which have been investigated by researchers. Some of these variants can be found in [1], [2], [3], [4], [6], [7], [8], [9], [10] and [14].

Shortly after Natarajan and Ayyaswamy [13] introduced and studied the concept of hop domination in a graph, some variants of the concept emerged. The concept and some of its variants are studied in [5], [11], [12], [15], and [16]. In this paper, we show that the standard domination and hop domination parameters are generally non-comparable. It

is shown that the absolute difference of these parameters can be made arbitrarily large. Further, we rectify a result found in [16] and use the corrected result to prove some results in this paper.

Let $G = (V(G), E(G))$ be a simple undirected graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. A vertex v is called a *leaf* in G if $deg_G(v) = 1$. The *open hop neighbourhood* of a vertex v of G is the set $N_G(v, 2) = \{w \in V(G) : d_G(v, w) = 2\}$ and its *closed hop neighbourhood* is the set $N_G[v, 2] = N_G(v, 2) \cup \{v\}$. The *open hop neighbourhood* of a subset S of $V(G)$ is the set $N_G(S, 2) = \cup_{v \in S} N_G(v, 2)$ and its *closed hop neighbourhood* is the set $N_G[S, 2] = N_G(S, 2) \cup S$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. A vertex v of G is a *dominating vertex* if $\{v\}$ is a dominating set of G. The smallest cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of G with with cardinality $\gamma(G)$ is called a *γ-set* of G.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for each $x \in V(G) \setminus S$, there exists $z \in S$ such that $d_G(x, z) = 2$. The smallest cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A hop dominating set of G with cardinality $\gamma_h(G)$ is called a *γ_h-set* of G. A set $S \subseteq V(G)$ is a *global hop dominating set* of G if it is a hop dominating set of G and $\overline{G}$. The smallest cardinality of a global hop dominating set of G, denoted by $\gamma_{gh}(G)$, is called the *global hop domination number* of G. A global hop dominating set of G with cardinality $\gamma_{gh}(G)$ is called a *γ_{gh}-set* of G.

2. Results

We note that although hop domination is, in some sense, a variation of the standard domination concept, the associated parameters are, in general, not comparable. Our first simple result says that the absolute difference of the domination number and hop domination number can be made arbitrarily large.

Proposition 1. *Each of the following statements holds.*

(i) *For each integer $n \geq 1$, there exists a connected graph G such that $\gamma_h(G) - \gamma(G) = n$.*

(ii) *For each integer $n \geq 1$, there exists a connected graph G such that $\gamma(G) - \gamma_h(G) = n$.*

Proof. (i) Let $G = K_{n+1}$. Then $\gamma(G) = 1$ and $\gamma_h(G) = n+1$. Hence, $\gamma_h(G) - \gamma(G) = n$.

(ii) Consider the star $K_{1,n+2}$ with vertices $v_0, v_1, v_2, \ldots, v_{n+1}, v_{n+2}$, where v_0 is the central vertex. Let G (see Figure 1) be the graph obtained from $K_{1,n+2}$ by adding $n + 2$ pendant edges $v_1 w_1, v_2 w_2, \ldots, v_{n+1} w_{n+1}, v_{n+2} w_{n+2}$. Let $S_1 = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\}$
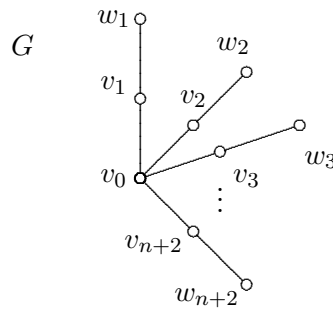
Figure 1

and $S_2 = \{v_0, v_1\}$. Clearly, S_1 and S_2 are γ-set and γ_h-set of G, respectively. Thus, $\gamma(G) - \gamma_h(G) = (n+2) - 2 = n$. $\qquad\square$

The next result is, in fact, a realization problem.

Theorem 1. *Let a and b be positive integers. Then each of the following statements holds.*

(i) If $2 \le a \le b$, then there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma(G) = b$.

(ii) If $3 \le a \le b$, then there exists a graph H such that $\gamma(H) = a$ and $\gamma_h(H) = b$.

Proof. *(i)* Suppose first that $a = b$. Consider the graph G in Figure 2. Clearly, $S_1 = \{x_1, x_2, \ldots, x_a\}$ is a γ_h-set and $S_2 = \{y_1, y_2, \ldots, y_a\}$ is a γ-set of G. Hence, $\gamma_h(G) = \gamma(G) = a$.
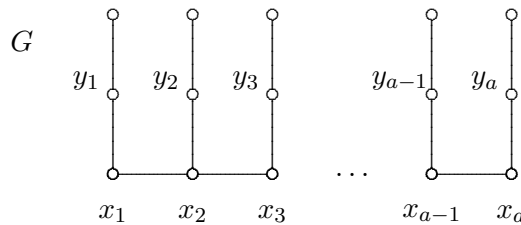

Figure 2

Next, suppose $a < b$ and let $m = b - a$. Consider the graph G in Figure 3. One can easily see that set $S = \{x_1, x_2, \ldots, x_a\}$ is a γ_h-set of G. Hence, $\gamma_h(G) = |S| = a$. The set $S' = \{y_1, y_2, \ldots, y_a, z_1, z_2, \ldots, z_m\}$ is a γ-set of G and so $\gamma(G) = |S'| = a + m = b$.
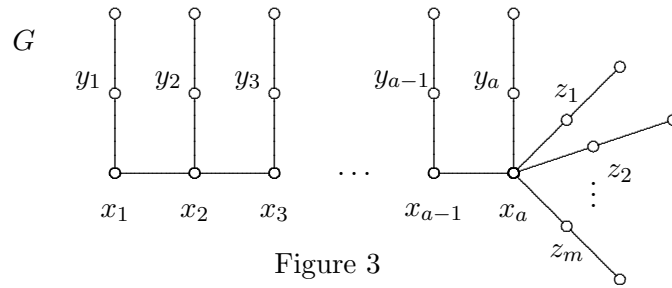
Figure 3

(*ii*) The case $a = b$ is similar to the first case of (*i*). Suppose $a < b$ and let $m = b - a$. Consider graph $H = G \cup K_{m+1}$, where $V(K_{m+1}) = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$ and G is the graph in Figure 4. Then $S = \{y_1, y_2, \ldots, y_{a-1}, v_1\}$ is a γ-set and $S' = \{x_1, x_2, \ldots, x_{a-1}\} \cup$
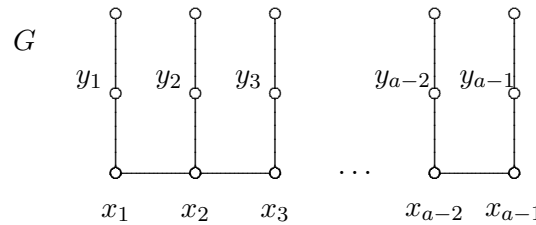

Figure 4

$V(K_{m+1})$ is a γ_h-set of H. Therefore, $\gamma(H) = a$ and $\gamma_h(H) = a - 1 + m + 1 = a + m = b$. ☐

It must be clear that every global hop dominating set is a hop dominating a set and a set is a global hop dominating set of a graph G if and only if it is a global hop dominating set of $\overline{G}$. The following remark is immediate from these facts.

Remark 1. *For any graph G, $\gamma_h(G) \leq \gamma_{gh}(G)$ and $\gamma_{gh}(G) = \gamma_{gh}(\overline{G})$.*

It was pointed out in [16] that $1 \leq \gamma_{gh}(G) \leq |V(G)|$ for any graph G and that $\gamma_{gh}(G) = 1$ if and only if $G = K_1$. The next result is a rectification of Theorem 3.3 in [16].

Theorem 2. (Theorem 3.3 in [16]) *Let G be a graph of order $n \geq 1$. Then $\gamma_{gh}(G) = n$ if and only if every component of G or $\overline{G}$ is complete. Moreover, if G is connected, then for each $v \in V(G)$, we have*

   (*i*) $V(G) \setminus N_G(v)$ *is an independent set, and*

   (*ii*) $N_G(v) = N_G(a)$ *for each $a \in V(G) \setminus N_G(v)$.*

Proof. Suppose $\gamma_{gh}(G) = n$. Assume first that G is disconnected and suppose that G has a component C which is not complete. Then there exist distinct vertices $x, y \in V(C)$ such that $d_G(x, y) = d_C(x, y) = 2$. Let $S = V(G) \setminus \{x\}$. Then S is a hop dominating set of G. Let $z \in C$ such that $[x, z, y]$ is an x-y geodesic in G. Let C' be a component of

G with $C' \neq C$ and pick any $w \in C'$. Then $[x, w, z]$ is an x-z geodesic in $\overline{G}$. It follows that $d_{\overline{G}}(x, z) = 2$. Thus, S is a hop dominating set of $\overline{G}$, showing that S is a global hop dominating set of $\overline{G}$. Therefore, $\gamma_{gh}(G) \leq |S| = n - 1$, a contradiction. Accordingly, every component of G is complete.

Next, suppose that G is connected. Suppose further that $\overline{G}$ is connected. Then, clearly, $G \neq K_n$. Let $u, v \in V(G)$ be such that $d_G(u, v) = 2$ and let $[u, p, v]$ be a u-v geodesic in G. Then $S^* = V(G) \setminus \{u\}$ is a hop dominating set of G. Since $up \notin E(\overline{G})$, it follows that $d_{\overline{G}}(u, p) \geq 2$. It follows that there exists $q \in S$ such that $d_{\overline{G}}(u, q) = 2$. This shows that S^* is hop dominating set of $\overline{G}$. Thus, S^* is a global hop dominating set of G and $\gamma_{gh}(G) \leq |S^*| = n - 1$, a contradiction. Therefore, $\overline{G}$ is disconnected. Since $\gamma_{gh}(\overline{G}) = \gamma_{gh}(G) = n$, this would imply that every component of $\overline{G}$ is complete (as in the first case applied to $\overline{G}$).

For the converse, suppose first that every component of G is complete. Then, clearly, $S = V(G)$ is the only hop dominating set of G. It follows that S is the only global hop dominating set of G. If every component of $\overline{G}$ is complete, then S is the only global hop dominating set of G. Therefore, $\gamma_{gh}(G) = n$.

Now suppose G is connected and let $v \in V(G) = V(\overline{G})$. Suppose there exist distinct vertices $a, b \in V(G) \setminus N_G(v)$ such that $ab \in E(G)$. Then $[a, v, b]$ is an a-b geodesic in $\overline{G}$, implying that $S_a = V(G) \setminus \{a\}$ is a hop dominating set of $\overline{G}$. Now, since $a \in V(G) \setminus N_G(v)$, it follows that $d_G(a, v) \geq 2$. This implies that there exists $w \in S_a$ such that $d_G(a, w) = 2$, showing that S_a is also a hop dominating set of G. Hence, $\gamma_{gh}(G) \leq |S_a| = n - 1$, a contradiction. Therefore, $V(G) \setminus N_G(v)$ is an independent set, showing that (i) holds. Next, let $a \in V(G) \setminus N_G(v)$. Let C_v be the component of $\overline{G}$ with $v \in C_v$. Since $a \in N_{\overline{G}}(v)$ and C_v is complete, $N_G(a) = N_G(v)$. This shows that (ii) holds. $\qquad \square$

The next result is a consequence of Theorem 2.

Corollary 1. $\gamma_{gh}(K_n) = \gamma_{gh}(K_{1,n-1}) = n$ *for all integer* $n \geq 2$.

Theorem 3. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma_{gh}(G) = b$.*

Proof. Consider the following cases:

Case 1. $a = b$

Let $G = K_a$. Then $\overline{G} = \overline{K}_a$. By Theorem 2, $\gamma_h(G) = \gamma_{gh}(G) = a$.

Case 2. $a < b$

Let $k = b - a$. Let $V(K_{a-1}) = \{x_1, x_2, \ldots, x_{a-1}\}$ and consider the graph G in Figure 5 obtained from $\langle v \rangle + K_{a-1}$ by adding the edges $x_i y_j$ for $i \in \{1, 2, \ldots, a - 1\}$ and $j \in \{1, 2, \ldots, k\}$ ($\langle v \rangle$ is the graph induced by $\{v\}$).
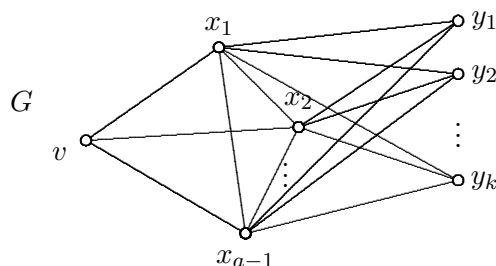
Figure 5

Let $S = \{v\} \cup V(K_{a-1}) = \{v, x_1, x_2, \ldots, x_{a-1}\}$. Since every vertex x_i $(1 \le i \le a-1)$ is a dominating vertex of G, it follows that each x_i is in every γ_h-set of G. Since $d_G(v, y_j) = 2$ for all $j \in \{1, 2, \ldots, k\}$, it follows that S is a γ_h-set of G. Hence, $\gamma_h(G) = a$. Now, the complement $\overline{G}$ of G is the graph isomorphic to $(K_1 + K_k) \cup \overline{K}_{a-1}$. By Theorem 2, $\gamma_{gh}(G) = |V(G)| = (k+1) + (a-1) = b$. $\square$

Corollary 2. *For each positive integer n, there exists a connected graph G such that $\gamma_{gh}(G) - \gamma_h(G) = n$. In other words, the difference $\gamma_{gh} - \gamma_h$ can be made arbitrarily large.*

Proof. Let n be a positive integer. By Theorem 3, there exists a connected graph G such that $\gamma_h(G) = n + 1$ and $\gamma_{gh}(G) = 2n + 1$. Hence, $\gamma_{gh}(G) - \gamma_h(G) = n$. $\square$

For a graph G, the complementary prism, denoted by $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. In simple terms, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$.

The next result gives bounds for the domination number of the complementary prism of a graph.

Theorem 4. [10] *For any graph G, $\max\{\gamma(G), \gamma(\overline{G})\} \le \gamma(G\overline{G}) \le \gamma(G) + \gamma(\overline{G})$.*

Theorem 5. *Let G be a connected graph of order n. Then each of following holds.*

(i) *If G is a non-trivial graph such that $\gamma(G) = 1$, then $\gamma(G\overline{G}) = 1 + \gamma(\overline{G} \setminus \overline{v})$, where v is a dominating vertex of G. In particular, $\gamma(K_n\overline{K}_n) = n$.*

(ii) *If $n \ge 1$, then $\gamma_h(G\overline{G}) = 2$. In particular, $\{v, \overline{v}\}$ is γ_h-set of $G\overline{G}$ for each $v \in V(G)$.*

(iii) *If $n \ge 2$, then $\gamma_{gh}(G\overline{G}) \le \min\{n, 2\gamma_{gh}(G)\}$.*

Proof. (i) Let v be a dominating vertex of G and let D be a dominating set of $\overline{G} \setminus \overline{v}$. Since $(V(G) \setminus \{v\}) \cup \{\overline{v}\} \subseteq N_{G\overline{G}}(v)$ and $V(\overline{G}) \setminus \{\overline{v}\} \subseteq N_{G\overline{G}}(D)$, $S = D \cup \{v\}$ is a dominating set of $G\overline{G}$. This implies that $\gamma(G\overline{G}) \le 1 + |D| = 1 + \gamma(\overline{G} \setminus \overline{v})$.

Suppose now that S_0 is a γ-set of $G\overline{G}$. Since v is a dominating vertex of G, $\overline{v}$ is an isolated vertex of $\overline{G}$ (and so a leaf in $G\overline{G}$). Hence, $v \in S_0$ or $\overline{v} \in S_0$. Suppose $v \notin S_0$. Then $\overline{v} \in S_0$. Suppose $S_1 = S_0 \cap V(G) = \varnothing$. Since $N_{G\overline{G}}(w) \cap V(\overline{G}) = \{\overline{w}\}$ for each $w \in V(G)$, it follows that $S_0 = V(\overline{G})$. Hence, $\gamma(G\overline{G}) = |S_0| = n \geq 1 + \gamma(\overline{G} \setminus \overline{v})$. Suppose $S_1 \neq \varnothing$ and let $S_2 = S_0 \cap (V(\overline{G}) \setminus \{\overline{v}\})$. If S_2 is a dominating of $\overline{G} \setminus \overline{v}$, then $\gamma(G\overline{G}) = |S_0| \geq 1 + \gamma(\overline{G} \setminus \overline{v})$. Suppose $R = V(\overline{G} \setminus \overline{v}) \setminus N_{\overline{G}\setminus\overline{v}}[S_2] \neq \varnothing$ and set $R_G = \{w \in V(G) \setminus \{v\} : \overline{w} \in R\}$. Then necessarily, $R_G \subseteq S_1$. Thus, $\gamma(G\overline{G}) = |S_0| = 1 + |S_1| + |S_2| \geq 1 + |R_G| + |S_2| \geq 1 + \gamma(\overline{G} \setminus \overline{v})$.

Next, suppose that $v \in S_0$. Since S_0 is a γ-set of $G\overline{G}$ and $\overline{v}$ is a leaf of $G\overline{G}$, $\overline{v} \notin S$. Let $D_1 = (V(G) \setminus \{v\}) \cap S_0$. If $D_1 = \varnothing$, then $D_2 = (V(\overline{G}) \cap S_0$ must be a dominating set of $\overline{G} \setminus \overline{v}$. It follows that $\gamma(G\overline{G}) = |S_0| \geq 1 + \gamma(\overline{G} \setminus \overline{v})$. Suppose that $D_1 \neq \varnothing$. If $D_2 = \varnothing$, then $D_1 = V(G) \setminus \{v\}$. It follows that $S_0 = V(G)$ and $\gamma(G\overline{G}) = |S_0| = n \geq 1 + \gamma(\overline{G} \setminus \overline{v})$. Suppose $D_2 \neq \varnothing$ and let $D_2^* = V(\overline{G} \setminus \overline{v}) \setminus N_{\overline{G}\setminus\overline{v}}[D_2]$. Since S_0 is a γ-set and $v \in S_0$, it follows that $D_2^* = \{\overline{x} : x \in D_1\}$ and $|D_2^*| = |D_1|$. Clearly, $D' = D_2 \cup D_2^*$ is a dominating set of $\overline{G} \setminus \overline{v}$ and so $\gamma(G\overline{G}) = |S_0| = 1 + |D_1| + |D_2| = 1 + |D'| \geq 1 + \gamma(\overline{G} \setminus \overline{v})$. This proves the desired equality. Thus, in particular, if $G = K_n$, then $\gamma(K_n\overline{K_n}) = n$.

(ii) If $n = 1$, then $G\overline{G} = K_2$. Hence, $\gamma_h(G\overline{G}) = 2$. Suppose $n \geq 2$. Let $v \in V(G)$ and let $S = \{v, \overline{v}\}$. Let $w \in V(G) \setminus \{v\}$. If $wv \in E(G)$, then $[w, v, \overline{v}]$ is a w-$\overline{v}$ geodesic in $G\overline{G}$. If $wv \notin E(G)$, then $\overline{w}\,\overline{v} \in E(\overline{G})$. It follows that $[w, \overline{w}, \overline{v}]$ is a w-$\overline{v}$ geodesic in $G\overline{G}$. Next, let $\overline{z} \in V(\overline{G}) \setminus \{\overline{v}\}$. If $\overline{z}\,\overline{v} \in E(\overline{G})$, then $[\overline{z}, \overline{v}, v]$ is a $\overline{z}$-v geodesic in $G\overline{G}$. If $\overline{z}\,\overline{v} \notin E(\overline{G})$, then $zv \in E(G)$ and $[\overline{z}, z, v]$ is a $\overline{z}$-v geodesic in $G\overline{G}$. Thus, S is a hop dominating set of G. Since $G\overline{G}$ is non-trivial, it follows that $\gamma_h(G\overline{G}) = 2$.

(iii) Suppose $n \geq 2$. Let $S = V(G)$ and let $\overline{v} \in V(G\overline{G}) \setminus S = V(\overline{G})$. Since G is connected and non-trivial, we may choose any $w \in V(G) \cap N_G(v)$. Consequently, $[\overline{v}, v, w]$ is a $\overline{v}$-w geodesic in $G\overline{G}$. Hence, $d_{G\overline{G}}(\overline{v}, w) = 2$. Since $\overline{v}$ was arbitrarily chosen, it follows that $S = V(G)$ is a hop dominating set of $G\overline{G}$. Next, let $\overline{y} \in V(\overline{G\overline{G}}) \setminus S$. Then $y\overline{y} \notin E(\overline{G\overline{G}})$. Pick any $x \in V(G) \setminus \{y\}$. If $xy \in E(G\overline{G})$, then $\overline{x}\,\overline{y} \in E(\overline{G\overline{G}})$. Since $y\overline{x} \in E(\overline{G\overline{G}})$, it follows that $d_{\overline{G\overline{G}}}(\overline{y}, y) = 2$. If $xy \notin E(G\overline{G})$, then $xy \in E(\overline{G\overline{G}})$. Since $x\overline{y} \in E(\overline{G\overline{G}})$, $d_{\overline{G\overline{G}}}(\overline{y}, y) = 2$. This shows that S is also a hop dominating set of $\overline{G\overline{G}}$. Therefore, S is a global hop dominating set of $G\overline{G}$ and $\gamma_{gh}(G\overline{G}) \leq |S| = n$.

Now let S_G be a global hop dominating set of G and let $S_{\overline{G}} = \{\overline{v} : v \in S_G\}$. Then clearly, $S' = S_G \cup S_{\overline{G}}$ is a hop dominating set of both $G\overline{G}$ and $\overline{G\overline{G}}$, that is, S' is a global hop dominating set of $G\overline{G}$. Thus, in particular, if S_G is a γ_{gh}-set of G, then $S' = S_G \cup S_{\overline{G}}$ is a global hop dominating set of $G\overline{G}$. This implies that $\gamma_{gh}(G\overline{G}) \leq |S'| = 2\gamma_{gh}(G)$. Combining this with the first inequality, we find that the assertion in (iii) holds. $\qquad\square$

The bound given in Theorem 5(iii) is sharp. To see this, consider the graph $G = P_3$ and the graph H obtained from C_4 by adding a pendant edge. It can be verified that $\gamma_{gh}(G\overline{G}) = |V(G)| = 3$ and $\gamma_{gh}(H\overline{H}) = 2\gamma_{gh}(H) = 4 < 5 = |V(H)|$.

Theorem 6. *Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete multipartite graph such that $m_1 \leq m_2 \leq$*

$\ldots \leq m_k$ *and* $k \geq 2$. *Then*

$$\gamma(G\overline{G}) = \begin{cases} k & if \ m_1 = 1 \\ k+1 & if \ m_1 = 2 \\ k+2 & if \ m_1 \geq 3. \end{cases}$$

Proof. Let $U_1, U_2, \ldots, U_k$ be the partite sets of G with $|U_i| = m_i$ for each $i \in \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$, let $\overline{U}_i = \{\overline{v} : v \in U_i\}$. Then the induced graphs of the sets $\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_k$ are exactly the (complete) components of $\overline{G}$ in $G\overline{G}$. Suppose first that $m_1 = 1$, say $U_1 = \{v\}$. Then v is a dominating vertex of G and so by Theorem 5(i), $\gamma(G\overline{G}) = 1 + \gamma(\overline{G} \setminus \overline{v})$. Since $\overline{G} \setminus \overline{v}$ is the disjoint union of complete graphs $\langle \overline{U}_2 \rangle, \langle \overline{U}_3 \rangle, \ldots, \langle \overline{U}_k \rangle$, it follows that $\gamma(\overline{G} \setminus \overline{v}) = k - 1$. Thus, $\gamma(G\overline{G}) = k$.

Next, suppose that $m_1 = 2$, say $U_1 = \{u, v\}$. By Theorem 4, $\max\{\gamma(G), \gamma(\overline{G})\} = k \leq \gamma(G\overline{G}) \leq k + 2 = \gamma(G) + \gamma(\overline{G})$. For each $i \in \{2 \ldots, k\}$, choose any $v_i \in U_i$ and let $S = \{u, \overline{v}\} \cup \{\overline{v}_i : i \in \{2, 3, \ldots, k\}\}$. Then S is a dominating set of $G\overline{G}$. Hence, $\gamma(G\overline{G}) \leq |S| = 2 + k - 1 = k + 1$. Let S' be a γ-set of $G\overline{G}$. If $S' \cap V(\overline{G}) = \varnothing$ or $S' \cap V(G) = \varnothing$, then $|S'| = |V(G)| = \sum_{i=1}^{k} m_i \geq 2k > k + 1$ which is not possible. Thus, $S' \cap V(\overline{G}) \neq \varnothing$ and $S' \cap V(G) \neq \varnothing$. Clearly, $S' \cap (U_i \cup \overline{U}_i) \neq \varnothing$ for all $i \in \{1, 2, \ldots, k\}$. Suppose $S' \cap U_1 \neq \varnothing$. If $|S' \cap U_1| = 1$, say $v \in S' \cap U_1$, then $V(G) \setminus \{u, v\} \subseteq N_{G\overline{G}}(v)$. Since S' is a γ-set of $G\overline{G}$ and $u \notin S'$, $|S' \cap \overline{U}_1| = 1$. We may assume that $\overline{u} \in S'$. Let $j \in \{2, 3, \ldots, k\}$ and suppose that $S \cap \overline{U}_j = \varnothing$. Then necessarily, $U_j \subseteq S'$. Pick any $\overline{w} \in \overline{U}_j$ and let $S_w = (S' \setminus U_j) \cup \{\overline{w}\}$. Then S_w is a dominating set of $G\overline{G}$. Since $|U_j| \geq 2$, $\gamma(G\overline{G}) = |S'| > |S_w|$, a contradiction. Therefore, $S' \cap \overline{U}_j \neq \varnothing$ for each $j \in \{2, 3, \ldots, k\}$. Moreover, because S' is a γ-set of $G\overline{G}$, $|S' \cap \overline{U}_j| = 1$ for all $j \in \{2, 3, \ldots, k\}$. Thus, $\gamma(G\overline{G}) = |S'| \geq k + 1$. If $|S' \cap U_1| = 2$, then $S' \cap \overline{U}_1 = \varnothing$. Since $S' \cap (U_i \cup \overline{U}_i) \neq \varnothing$ for all $i \in \{2, \ldots, k\}$, it follows that $\gamma(G\overline{G}) = |S'| \geq k + 1$. Suppose now that $S' \cap U_1 = \varnothing$. If $|S' \cap \overline{U}_1| = 1$, then there exists $j \neq 1$ such that $S' \cap U_j \neq \varnothing$. If $|S' \cap U_j| = 1$, then $|S' \cap \overline{U}_j| \neq 0$. Since $S' \cap (U_i \cup \overline{U}_i) \neq \varnothing$ for all $i \in \{2, \ldots, j-1, j+2, \ldots, k\}$, it follows that $\gamma(G\overline{G}) = |S'| \geq k + 1$. If $|S' \cap U_j| \geq 2$, then clearly, $\gamma(G\overline{G}) = |S'| \geq k + 1$. Therefore, $\gamma(G\overline{G}) = k + 1$.

Finally, let $m_1 \geq 3$. Let S be a γ-set of $G\overline{G}$. Since $\gamma(G\overline{G}) \leq k + 2$ (by Theorem 4) and $|V(G)| = \sum_{i=1}^{k} m_i > k + 2$, $S \cap V(\overline{G}) \neq \varnothing$ and $S \cap V(G) \neq \varnothing$. Again, $S \cap (U_i \cup \overline{U}_i) \neq \varnothing$ for each $i \in \{1, 2, \ldots, k\}$. Suppose $S \cap U_1 = \varnothing$. Then $S \cap \overline{U}_1 \neq \varnothing$. If $|S \cap \overline{U}_1| \geq 3$, then $\gamma(G\overline{G}) = |S| \geq k + 2$. So suppose that $|S \cap \overline{U}_1| \leq 2$. Then there exists $j \neq 1$ such that $S \cap U_j \neq \varnothing$. If $S \cap U_j = U_j$, then $\gamma(G\overline{G}) = |S| \geq k + 2$. If $S \cap U_j \neq U_j$, then $S \cap \overline{U}_j \neq \varnothing$. Hence, if $|S \cap \overline{U}_1| = 2$, then $\gamma(G\overline{G}) = |S| \geq k + 2$. Suppose now that $|S \cap \overline{U}_1| = 1$. Suppose further that $|S \cap (U_j \cup \overline{U}_j)| = 2$. Then there exists $r \neq 1, j$ such that $|S \cap (U_r \cup \overline{U}_r)| \geq 2$. Thus, $\gamma(G\overline{G}) = |S| \geq k + 2$.

Next, suppose that $S \cap U_1 \neq \varnothing$. Suppose $S \cap U_j = \varnothing$ for all $j \geq 2$. Since S is a γ-set of $G\overline{G}$ and $U_j \subseteq N_{G\overline{G}}(S \cap U_1)$ for all $j \geq 2$, $|S \cap \overline{U}_j| = 1$ for all $j \geq 2$ and $|S| = k - 1 + |U_1|$. If $m_1 = |U_1| = 3$, then $|S| = k - 1 + 3 = k + 2$. However, if $m_1 \geq 4$, then $|S| = k - 1 + |U_1| \geq k + 3$, contrary to the fact that $k + 2$ is an upper bound of $|S|$. Hence, for $m_1 \geq 4$, there exists $j \geq 2$ such that $|S \cap (U_j \cup \overline{U}_j)| \geq 2$. Since $|S \cap (U_1 \cup \overline{U}_1)| \geq 2$, S

is a γ-set, $\gamma(G\overline{G}) = |S| \geq k+2$. Note that we obtain the same implications if $m_1 = 3$ and $S \cap U_j \neq \varnothing$ for some $j \geq 2$.

Accordingly, $\gamma(G\overline{G}) = k+2$. $\square$

Given a graph G with $\gamma(G) = 1$, we denote by $Dom(G)$ the set $\{v \in V(G) : \{v\}$ is a dominating set of $G\}$.

Lemma 1. *Let G be a graph with $\gamma(G) = 1$ and let S be a global hop dominating set of $G\overline{G}$. If $v \in Dom(G)$, then $v \in S$ or $\overline{v} \in S$.*

Proof. Let $v \in Dom(G)$ and suppose that $v, \overline{v} \notin S$. Since S is a global hop dominating set of $G\overline{G}$, it is a hop dominating set of $\overline{G\overline{G}}$. As $\overline{v} \in V(\overline{G\overline{G}}) \setminus S$, there exists $z \in S$ such that $d_{\overline{G\overline{G}}}(\overline{v}, z) = 2$. However, $d_{\overline{G\overline{G}}}(\overline{v}, x) = 1$ for all $x \in V(\overline{G\overline{G}}) \setminus \{v, \overline{v}\}$. Since $v \notin S$, it follows that such a vertex z does not exist, contrary to the assumption that S is a hop dominating set of of $\overline{G\overline{G}}$. Therefore, $v \in S$ or $\overline{v} \in S$. $\square$

Corollary 3. *For each positive integer $n \geq 2$, $\gamma_{gh}(K_n\overline{K}_n) = n$.*

Proof. By Theorem 5(*iii*), $\gamma_{gh}(K_n\overline{K}_n) \leq n$. Let S be a γ_{gh}-set of $K_n\overline{K}_n$. Since $Dom(K_n) = V(K_n)$, $\gamma_{gh}(K_n\overline{K}_n) = |S| \geq n$ by Lemma 1. Therefore, $\gamma_{gh}(K_n\overline{K}_n) = n$. $\square$

Theorem 7. *Let $G = K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph such that $1 \leq m_1 \leq m_2 \leq \ldots \leq m_k$, where $k \geq 2$ and $m_j \geq 2$ for some j with $1 \leq j \leq k$. Then*

$$\gamma_{gh}(G\overline{G}) = \begin{cases} k & \text{if } m_1 = m_2 = 1 \text{ and } k \geq 4 \\ k+1 & \text{otherwise.} \end{cases}$$

Proof. Let $U_1, U_2, \ldots, U_k$ be the partite sets of G with $|U_i| = m_i$ for each $i \in \{1, 2, \ldots, k\}$. Again, for each $i \in \{1, 2, \ldots, k\}$, let $\overline{U}_i = \{\overline{v} : v \in U_i\}$. Choose any $v_i \in U_i$ for each $i \in \{1, 2, \ldots, k\}$. Suppose $m_1 = m_2 = 1$ and $k \geq 4$. Then $U_1 = \{v_1\}$ and $U_2 = \{v_2\}$. Let $S = \{v_1, v_2, \overline{v_3}, \overline{v_4}, \ldots, \overline{v_k}\}$. Then $d_{G\overline{G}}(\overline{v_1}, v_2) = 2$ and $d_{G\overline{G}}(\overline{v_2}, v_1) = 2$. For each $j \in \{3, 4, \ldots, k\}$, we have $d_{G\overline{G}}(\overline{w}, v_1) = 2$ for each $\overline{w} \in \overline{U}_j \setminus S$ and $d_{G\overline{G}}(x, \overline{v_r}) = 2$ for each $x \in U_j$, where $r \geq 3$ and $r \neq j$. Thus, S is a hop dominating set of $G\overline{G}$. On the other hand, $d_{\overline{G\overline{G}}}(\overline{v_1}, v_1) = 2$ and $d_{\overline{G\overline{G}}}(\overline{v_2}, v_2) = 2$. For each $j \in \{3, 4, \ldots, k\}$, we have $d_{\overline{G\overline{G}}}(\overline{w}, \overline{v_j}) = 2$ for each each $\overline{w} \in \overline{U}_j \setminus S$ and $d_{\overline{G\overline{G}}}(x, v_1) = 2$ for each each $x \in U_j$. Thus, S is a hop dominating set of $\overline{G\overline{G}}$. Therefore, S is a global hop dominating set of $G\overline{G}$ and $\gamma_{gh}(G\overline{G}) \leq |S| = k$.

Next, let S_0 be a γ_{gh}-set of $G\overline{G}$. Suppose there exists $j \in \{1, 2, \ldots, k\}$ such that $S_0 \cap (U_j \cup \overline{U}_j) = \varnothing$. Let $\overline{w} \in \overline{U}_j$. It follows from the adjacency in $\overline{G\overline{G}}$ that $d_{\overline{G\overline{G}}}(\overline{w}, p) = 1$ for all $p \in V(\overline{G\overline{G}}) \setminus (\overline{U}_j \cup \{w\})$. Hence, by assumption, there exists no $q \in S_0$ with $d_{\overline{G\overline{G}}}(\overline{w}, q) = 2$, contrary to the fact that S_0 is a hop dominating set of $\overline{G\overline{G}}$. Therefore, $S_0 \cap (U_j \cup \overline{U}_j) \neq \varnothing$ for each $j \in \{1, 2, \ldots, k\}$. Consequently, $\gamma_{gh}(G\overline{G}) = |S| \geq k$. Accordingly, $\gamma_{gh}(G\overline{G}) = k$.

Suppose now that the conditions $m_1 = m_2 = 1$ and $k \geq 4$ do not hold. Let $S' = \{v_1, \overline{v}_1, \overline{v}_2, \overline{v}_3, \ldots, \overline{v}_k\}$. By Theorem 5(ii), $\{v_1, \overline{v}_1\}$ is a hop dominating set of $G\overline{G}$. Hence, S is a hop dominating set of $G\overline{G}$.

Let $v' \in U_j \setminus \{v_1\}$, where $j \in \{1, 2, \ldots, k\}$. Then $[v', \overline{v}_1, \overline{v}_2]$ is a v'-$\overline{v}_2$ geodesic in $\overline{\overline{G\overline{G}}}$. Hence, $d_{\overline{G\overline{G}}}(v', \overline{v}_2) = 2$. Let $j \in \{1, 2, \ldots, k\}$ and pick any $i \in \{1, 2, \ldots, k\} \setminus \{j\}$. Then for $\overline{y} \in \overline{U}_j \setminus \{\overline{v}_j\}$, we find that $[\overline{y}, \overline{v}_i, \overline{v}_j]$ is a $\overline{y}$-$\overline{v}_j$ geodesic in $\overline{G\overline{G}}$. Hence, $d_{\overline{G\overline{G}}}(\overline{y}, \overline{v}_j) = 2$. This shows that S' is a hop dominating set of $\overline{G\overline{G}}$. Therefore, S' is a global hop dominating set of $G\overline{G}$ and $\gamma_{gh}(G\overline{G}) \leq |S'| = k + 1$.

Let S_0 be a γ_{gh}-set of $G\overline{G}$. As shown and seen earlier $S_0 \cap (U_j \cup \overline{U}_j) \neq \varnothing$ for each $j \in \{1, 2, \ldots, k\}$. Next, suppose there exists $j \in \{1, 2, \ldots, k\}$ with $m_j \geq 2$ such that $|S_0 \cap \overline{U}_j| = 0$. Let $\overline{a} \in \overline{U}_j$. Since $d_{\overline{G\overline{G}}}(\overline{a}, p) = 1$ for $p \in V(\overline{G\overline{G}}) \setminus (\overline{U}_j \cup \{a\})$ and $|S_0 \cap \overline{U}_j| = 0$, it follows that $a \in S_0$. Thus, $U_j \subseteq S_0$. Since $m_j \geq 2$, it follows that $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. Suppose that $|S_0 \cap \overline{U}_j| \neq 0$ for each j with $m_j \geq 2$. If $|S_0 \cap \overline{U}_j| \geq 2$ for some j with $m_j \geq 2$, then $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. Suppose that $|S_0 \cap \overline{U}_j| = 1$ for each j with $m_j \geq 2$. For a j satisfying this property, pick $\overline{y} \in \overline{U}_j \setminus S_0$. Since S is a hop dominating set of $G\overline{G}$ and the induced graph of $\overline{U}_j$ is a complete graph in $G\overline{G}$, it follows that there exists $r \neq j$ such $z \in U_r \cap S_0$ and $d_{G\overline{G}}(\overline{y}, z) = 2$. If an r exists such that $m_r \geq 2$, then this would imply that $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. So suppose that there exists no such r with $m_r \geq 2$. Then $m_r = 1$ and $r = 1$ or $r = 2$, say $U_r = U_1$. If $\overline{z} \in S_0$, then $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. Suppose $\overline{z} \notin S_0$. Then there exists $p \in U_s \cap S_0$ for some $s \geq 2$ such that $d_{G\overline{G}}(\overline{z}, p) = 2$. If $m_s = 1$, then $s = 2$ and $j = k = 3$ by assumption. If $\overline{p} \in S_0$, then $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. Suppose $\overline{p} \notin S_0$ and let $S_0 \cap \overline{U}_j = \{\overline{q}\}$. Since S_0 is a hop dominating set of $G\overline{G}$, $\overline{z}, \overline{p} \notin S_0$, and $k = 3$, we must have $q \in S_0$. Hence, $\gamma_{gh}(G\overline{G}) = |S_0| \geq k + 1$. Now, if $m_s \geq 2$, then $s = k$ by assumption. This implies that $|S_0 \cap (U_j \cup \overline{U}_j)| \geq 2$, showing that $\gamma_{gh}(G\overline{G}) = |S_0| \geq k+1$. Therefore, $\gamma_{gh}(G\overline{G}) = k+1$. $\square$

The shadow graph $D_2(G)$ of a graph G is the graph obtained by taking two copies of G, say G_1 and G_2, and joining each vertex $u \in V(G_1)$ to the neighbors of the corresponding vertex $u' \in V(G_2)$.

Lemma 2. *Let G be a non-trivial connected graph and let G_1 and G_2 be copies of G in the graph $D_2(G)$. If $w \in V(G_1)$ and $w' \in V(G_2)$ is the corresponding vertex of w, then*

$$N_{D_2(G)}[w, 2] = N_{G_1}[w, 2] \cup N_{G_2}[w', 2] = N_{D_2(G)}[w', 2].$$

Proof. Clearly, $N_{G_1}[w, 2] \cup N_{G_2}[w', 2] \subseteq N_{D_2(G)}[w, 2]$. Now let $x \in N_{D_2(G)}[w, 2]$. Then $x = w$ or $d_{D_2(G)}(w, x) = 2$. Suppose first that $x \in V(G_1)$. If $x = w$, then $x \in N_{G_1}[w, 2]$. Suppose that $d_{D_2(G)}(w, x) = 2$ and let $y \in V(D_2(G))$ such that $[w, y, x]$ is w-x geodesic in $D_2(G)$. If $y \in V(G_1)$, then $[w, y, x]$ is a w-x geodesic in G_1. Suppose $y \in V(G_2)$, say $y = u'$. By definition of $D_2(G)$, it follows that $u \in V(G_1)$ and $[w, u, x]$ is a w-x geodesic in G_1. Hence, $x \in N_{G_1}[w, 2]$. Next, suppose that $x = z' \in V(G_2)$. If $y \in V(G_1)$, then $[w', y', z']$ is a w-x geodesic in G_2. Suppose $y \in V(G_2)$, say $y = u'$. Then $[w', u', z']$ is w'-x geodesic in G_2. Hence, $x \in N_{G_2}[w', 2]$. Thus, $N_{D_2(G)}[w, 2] \subseteq N_{G_1}[w, 2] \cup N_{G_2}[w', 2]$, showing that

$N_{D_2(G)}[w, 2] \subseteq N_{G_1}[w, 2] \cup N_{G_2}[w', 2]$. Similarly, $N_{D_2(G)}[w', 2] \subseteq N_{G_1}[w, 2] \cup N_{G_2}[w', 2]$. Therefore, $N_{D_2(G)}[w, 2] = N_{G_1}[w, 2] \cup N_{G_2}[w', 2] = N_{D_2(G)}[w', 2]$. $\qquad\square$

A result in [13] says that $\gamma_h(D_2(G)) = \gamma_h(G)$ for any graph G. This, however, is not true if G contains an isolated vertex. Indeed, if G is the trivial graph and H is the (disjoint) union $K_1 \cup K_1 \cup P_2$, then $D_2(G) = \overline{K}_2$ and $D_2(G) = \overline{K}_2 \cup \overline{K}_2 \cup C_4$. Hence, $\gamma_h(D_2(G)) = 2 \neq 1 = \gamma_h(G)$ and $\gamma_h(D_2(H)) = 6 \neq 4 = \gamma_h(G)$.

Theorem 8. *Let G be a non-trivial graph. Then the following hold.*

(i) *If G is connected, then $\gamma_h(D_2(G)) = \gamma_h(G)$.*

(ii) *If G is disconnected with r trivial components and k non-trivial components $G_1, G_2, G_3, \ldots, G_k$, then $\gamma_h(D_2(G)) = 2r + \sum_{i=1}^{k} \gamma_h(G_k)$.*

Proof. (i) Let G_1 and G_2 be the two copies of G in the definition of $D_2(G)$. Let S be a γ_h-set of G_1 and let $v' \in V(G_2)$. If the corresponding vertex $v \in V(G_1)$ is in S, then $d_{D_2(G)}(v, v') = 2$. So suppose $v \notin S$. Since S is a hop dominating set of G_1, there exists $w \in S$ such that $d_{G_1}(v, w) = 2$. Let $[v, z, w]$ be a v-w geodesic in G_1. Then $v'z, zw \in E(D_2(G))$. Since $v \notin N_{G_1}(w)$, $v'w \notin E(D_2(G))$. Thus, $d_{D_2(G)}(w, v') = 2$. Therefore, S is a hop dominating set of $D_2(G)$ and $\gamma_h(D_2(G) \leq |S| = \gamma_h(G)$.

Next, suppose that S' is a γ_h-set of $D_2(G)$. Let $S_1 = S' \cap V(G_1)$ and $S_2 = S' \cap V(G_2)$. If $S_1 = S'$ or $S_2 = S'$, then S' is a hop dominating set of G_1 or G_2. Hence, $\gamma_h(D_2(G) = |S'| \geq \gamma_h(G)$. Suppose $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$. If S_1 is a hop dominating set of G_1 or S_2 is a hop dominating set of G_2, then, as seen earlier, S_1 or S_2 is a hop dominating set of $D_2(G)$, contrary to the assumption that S' is a γ_h-set of $D_2(G)$. Hence, none of these two sets is a hop dominating set. Let $D_G = \{v \in V(G_1) \setminus S_1 : v \notin N_{G_1}(S_1, 2)\} = V(G_1) \setminus N_{G_1}[S_1, 2]$ and let $S_G = \{v \in V(G_1) \setminus S_1 : v' \in S_2\}$. Clearly, if $v \in S_G$, then $v' \in S_2$. Now let $y' \in S_2$. Then $N_{D_2(G)}[y', 2] = N_{D_2(G)}[y, 2]$ by Lemma 2. Since S' is a γ_h-set of $D_2(G)$, it follows that $y \in V(G_1) \setminus S_1$, that is, $y \in S_G$. Hence, $|S_G| = |S_2|$. Since S' is a hop dominating set of $D_2(G)$, $D_G \subseteq N_{D_2(G)}[S_2, 2]$. This means that if $w \in D_G$, then there exists $z' \in S_2$ such that $w \in N_{D_2(G)}[z', 2]$. Consequently, $z \in S_G$ and by Lemma 2, we have $w \in N_{D_2(G)}[z, 2] = N_{G_1}[z, 2]$. Thus, $D_G \subseteq N_{G_1}[S_G, 2]$, showing that $S_1 \cup S_G$ is a hop dominating set of G_1. Therefore, $\gamma_h(G) = \gamma_h(G_1) \leq |S_1 \cup S_G| = \gamma_h(D_2(G))$. This establishes the desired equality.

(ii) This follows from (i) and the fact that the hop domination of a (disconnected) graph is the sum of the hop domination numbers of its components. $\qquad\square$

Theorem 9. *Let G be a non-trivial connected graph. Then $\gamma_{gh}(D_2(G)) \leq 2\gamma_{gh}(G)$.*

Proof. Let G_1 and G_2 be the two copies of G in the definition of $D_2(G)$. Let S_1 be a γ_{gh}-set of G_1 and let $S_2 = \{v' \in V(G_2) : v \in S_1\}$. Then S_2 is a γ_{gh}-set of G_2. Hence, $S = S_1 \cup S_2$ is a hop dominating set of $D_2(G)$. Since S_1 and S_2 are also γ_{gh}-sets of $\overline{G}_1$ and $\overline{G}_2$, respectively, it follows that $S = S_1 \cup S_2$ is a hop dominating set of $\overline{D_2(G)}$. Hence, $S = S_1 \cup S_2$ is a global hop dominating set of $D_2(G)$ and $\gamma_{gh}(D_2(G)) \leq |S| = 2\gamma_{gh}(G)$.

The next result is easy.

Lemma 3. *Let G be a non-trivial graph. Then each of the following statements is true.*

(i) $D_2(G)$ *is not a complete graph.*

(ii) $D_2(G)$ *is connected if and only if G is connected.*

Lemma 4. *Let G be a graph of order n. Then each of the following statements is true.*

(i) *Every component of $D_2(G)$ is a complete graph if and only if $G = \overline{K}_n$.*

(ii) *Every component of $\overline{D_2(G)}$ is a complete graph if and only if $G = \overline{K}_n$ or $G = K_{m_1,m_2,\ldots,m_k}$, where $\sum_{i=1}^{k} m_i = n$.*

Proof. Let G_1 and G_2 be the two copies of G in the definition of $D_2(G)$.
(i) Suppose every component of $D_2(G)$ is a complete graph. Suppose further that G has a non-trivial component H. Then $D_2(H)$ is a component of $D_2(G)$ which is not complete by Lemma 3, a contradiction to our assumption of $D_2(G)$. Therefore, every component of G is trivial, i.e., $G = \overline{K}_n$. The converse is clear.
(ii) Suppose that every component of $\overline{D_2(G)}$ is a complete graph. If $\overline{D_2(G)}$ is connected, then $D_2(G) = \overline{K}_{2n}$. Hence, $G = \overline{K}_n$. Next, suppose that $\overline{D_2(G)}$ is disconnected with components $C_1, C_2, \ldots, C_k$. For each $i \in \{1, 2, \ldots, k\}$, let $S_{1,i} = V(G_1) \cap V(C_i)$, $S_{2,i} = V(G_2) \cap V(C_i)$ and $m_i = |S_{1,i}|$. Note that $v \in S_{1,i}$ if and only if $v' \in S_{2,i}$ and that $C_i = \langle S_{1,i} \cup S_{2,i} \rangle$ for each $i \in \{1, 2, \ldots, k\}$. Let $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. Since C_i and C_j are complete subgraphs (components) of $\overline{D_2(G)}$, it follows that in graph $D_2(G)$, $S_{1,i}$ and $S_{1,j}$ are independent subsets of $V(G_1)$ and $xy \in E(G_1)$ for each $x \in S_{1,i}$ and $y \in S_{1,j}$. It follows that G_1 is a complete multipartite graph with partite sets $S_{1,1}, S_{1,2}, \ldots, S_{1,k}$. Hence, $G = K_{m_1,m_2,\ldots,m_k}$.
The converse is clear. $\square$

Theorem 10. *Let G be a graph of order n. Then $\gamma_{gh}(D_2(G)) = 2n$ if and only if $G = \overline{K}_n$ or $G = K_{m_1,m_2,\ldots,m_k}$, where $\sum_{i=1}^{k} m_i = n$.*

Proof. By Theorem 2, $\gamma_{gh}(D_2(G)) = 2n$ if and only if every component of $D_2(G)$ or $\overline{D_2(G)}$ is complete. Thus by Lemma 4, $\gamma_{gh}(D_2(G)) = 2n$ if and only if $G = \overline{K}_n$ or $G = K_{m_1,m_2,\ldots,m_k}$, where $\sum_{i=1}^{k} m_i = n$. $\square$

Note that Theorem 10 shows that the bound given in Theorem 9 is tight.

Conclusion: The domination and hop domination parameters are, in general, not comparable. However, a result shows that the absolute difference of the domination number and hop domination number can be made arbitrarily large. On the other hand, a result shows a relationship of the hop domination and global hop domination numbers.

For any non-trivial connected graph G, it is proved that that $2\gamma_{gh}(G)$ is a tight bound for the global hop domination number of the shadow graph $D_2(G)$ of G. The authors are still unable to show that the strict inequality in Theorem 9 is also attainable. We leave to the interested readers to verify whether or not equality in this result holds.

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