**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 14, No. 4, 2021, 1517-1529
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# The Bivariate Extended Poisson Distribution of type 1

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**Abstract.** In this paper, we will construct the bivariate extended Poisson distribution which generalizes the univariate extended Poisson distribution. This law will be obtained by the method of the product of its marginal laws by a factor. This method was demonstrated in [7]. Thus we call the bivariate extended Poisson distribution of type 1 the bivariate extended Poisson distribution obtained by the method of the product of its marginal distributions by a factor. We will show that this distribution belongs to the family of bivariate Poisson distributions and and will highlight the conditions relating to the independence of the marginal variables. A simulation study was realised.

**2020 Mathematics Subject Classifications**: 62E15, 62H10, 60E05

**Key Words and Phrases**: Extended Poisson distribution, bivariate Poisson distribution, estimation and statistical testing

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## 1. Introduction

Several authors have studied bivariate Poisson laws, in particular, Berkhout and Plug[2] and Lakshminarayna et al.[7]. Then, [3] has highlighted the weighted bivariate Poisson law having as a basic law, the bivariate Poisson law according to Berkhout and Plug[2]; a law that allows to generate all the bivariate Poisson laws. The bivariate Poisson distribution according to Berkhout and Plug[2] is rightly considered the standard distribution in $\mathbb{N}^2$ as is the Poisson distribution in $\mathbb{N}$. In this paper, we will construct the bivariate extended Poisson distribution which generalizes the univariate extended Poisson distribution. This law is obtained by the method of the product of its marginal laws by a factor. This method was demonstrated in [7], thus we call the bivariate extended Poisson distribution of type

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i4.4151

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1 the bivariate extended Poisson distribution obtained by the method of the product of its marginal distributions by a factor. We have shown that this distribution belongs to the family of bivariate Poisson distributions and have highlighted the conditions relating to the independence of the marginal variables. A simulation study was realised.

## 2. A review of laws

### 2.1. The Univariate Extended Poisson distribution

**Definition 1.** *The laws following probability mass function*

$$P(Y=y)=\begin{cases} \dfrac{1-e^{-\theta}}{\beta}, & y=0,\\ \left(\dfrac{\theta^y}{y!}e^{-\theta}\right)\beta^{-1}\left(\dfrac{\beta}{\theta}y-1\right), & y=1,2,\ldots \end{cases} \quad \forall\ \theta>0,\ \textit{and}\ \beta\geq\theta, \tag{1}$$

*is renamed as the univariate extended Poisson distribution (see [5]).*

It can be written in the form [8]

$$P(Y=y)=\frac{\theta^y}{y!}e^{-\theta}\left(\frac{\beta}{\theta}y-1\right)\beta^{-1}\left\{\frac{1-e^{-\theta}}{\left(\frac{\beta}{\theta}y-1\right)e^{-\theta}}\right\}^{\delta_0(y)}, \quad y\in\mathbb{N},\ \forall\,\theta>0 \text{ and } \beta\geq\theta, \tag{2}$$

where $\delta_0$ is the Dirac function in 0. $\theta$ is the canonic parameter. This distribution has the following characteristics

$$E_\theta(Y)=1+\frac{\beta-1}{\beta}\theta, \tag{3}$$

$$var(Y)=\frac{\beta-1}{\beta^2}\theta^2+\frac{\beta+1}{\beta}\theta. \tag{4}$$

**Under- or over-dispersed distribution**

The following facts are immediate.

**Proposition 1.** *The Fisher dispersion index of the variable $Y$ which follows the extended Poisson distribution of parameters $(\theta,\beta)$ noted $I(Y)$ is such that*

- $I(Y)>1$ *if* $\dfrac{\beta}{1+\sqrt{\beta}}<\theta\leq\beta$, *i.e. the extended Poisson distribution is overdispersed ;*

- $I(Y) < 1$ *if* $0 < \theta < \dfrac{\beta}{1+\sqrt{\beta}}$, *i.e. the extended Poisson distribution is underdispersed ;*
- $I(Y) = 1$ *if* $\theta = \dfrac{\beta}{1+\sqrt{\beta}}$, *i.e. the extended Poisson distribution is equiderdispersed.*

*Proof.* Indeed,

$$\begin{aligned} var(Y) - E(Y) &= \frac{\beta-1}{\beta^2}\theta^2 + \frac{\beta+1}{\beta}\theta - 1 - \frac{\beta-1}{\beta}\theta, \\ &= \frac{\theta^2\beta - (\beta-\theta)^2}{\beta^2}, \\ &= \frac{(\theta\sqrt{\beta}+\beta-\theta)[\theta(1+\sqrt{\beta})-\beta]}{\beta^2}. \end{aligned}$$

Since $\beta \geq \theta > 0$ then the sign of $var(Y) - E(Y)$ depends only on $\theta(1+\sqrt{\beta}) - \beta$. Then $I(Y)$ is greater, smaller or equal to 1 depending on whether $var(Y) - E(Y)$ is positive, negative or null respectively. We are assured of the answer

Let us recall the result of [5].

**Proposition 2.** *The moments generating function of the extended Poisson distribution is equal to*

$$M_Y(t) = \frac{1 - \left(1 - \beta e^t\right) e^{\theta\left(e^t - 1\right)}}{\beta}, \quad t \in [-1, 1]. \tag{5}$$

Now, we have the following result.

**Proposition 3.**

$$E_\theta\left[e^{-Y}\right] = \frac{1 - \left(1 - \beta e^{-1}\right) e^{\theta\left(e^{-1} - 1\right)}}{\beta}, \tag{6}$$

$$E_\theta\left[\left(e^{-Y}\right)^2\right] = \frac{1 - \left(1 - \beta e^{-2}\right) e^{\theta\left(e^{-2} - 1\right)}}{\beta}, \tag{7}$$

$$E_\theta\left[Y e^{-Y}\right] = \frac{e^{-1} e^{\theta\left(e^{-1} - 1\right)}\left[\beta - \theta\left(1 - \beta e^{-1}\right)\right]}{\beta}. \tag{8}$$

*Proof.* Indeed, expression (6) is obvious because it is equal to $M_Y(-1)$. Ditto for expression (7) which is equal to $M_Y(-2)$. And for expression (8), we have $E_\theta(Ye^{tY}) = \dfrac{d}{dt}M_Y(t)$. Since $\dfrac{d}{dt}M_Y(t) = \dfrac{e^t e^{\theta\left(e^t - 1\right)}\left[\beta - \theta\left(1 - \beta e^t\right)\right]}{\beta}$, by setting t=-1, we are assured of the answer.

## 2.2. The Bivariate Poisson distribution according to Berkhout and Plug[2]

**Definition 2.** *Let $Y_j$ $(j = 1, 2)$ be a random variable that follows the Poisson distribution of parameter $\theta_j$ $(j = 1, 2)$. The vector $(Y_1, Y_2)$ follows the bivariate Poisson distribution according to Berkhout and Plug [2] if its mass function $f_{BP}$ is equal to*

$$f_{BP}(y_1, y_2; \theta_1, \theta_2) = \left(\frac{\theta_1^{y_1}}{y_1!}e^{-\theta_1}\right)\left(\frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}\right), \quad (y_1, y_2) \in \mathbb{N}^2,\ (\theta_1, \theta_2) \in \mathbb{R}_+^{*2}, \tag{9}$$

*under conditions*

$$\ln \theta_1 = x'\rho_1, \tag{10}$$

$$\ln \theta_2 = x'\rho_2 + \eta y_1, \tag{11}$$

*where $\rho_1$, $\rho_2$ and $\eta$ are parameters, $x = (x_1, \ldots, x_p)$ is a vector of deterministic variables or factors.*

The generalized model (10) has the response variable $Y_1$ and the model (11) the variable $Y_2$. The expression (10) induces that $P(Y_1 = y_1; \theta_1) = \frac{\theta_1^{y_1}}{y_1!}e^{-\theta_1}$ is a marginal law while the model (11) induces that

$$\begin{aligned} P(Y_2 = y_2; \theta_2) &= P(Y_2 = y_2/Y_1 = y_1), \\ &= \frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}, \\ &= \frac{e^{y_2(x'\rho_2 + \eta y_1)}}{y_2!}e^{-(x'\rho_2 + \eta y_1)}, \end{aligned}$$

is a conditional law.

When $\eta = 0$ then the conditional probability $P(Y_2 = y_2/Y_1 = y_1)$ is not depend of observation $y_1$ and the variables $Y_1$ and $Y_2$ are independent.

The bivariate Poisson distribution according to Berkhout and Plug[2] has the characteristics (see [1]).

$$E_{\theta_1}(Y_1) = var(Y_1) = \theta_1, \tag{12}$$

$$E_{\theta_2}(Y_2) = e^{x'\rho_2 + a_2 + \theta_1(e^{\eta} - 1)}, \tag{13}$$

$$var(Y_2) = E_{\theta_2}(Y_2) + [E_{\theta_2}(Y_2)]^2 \left(e^{\theta_1(e^{\eta}-1)} - 1\right), \tag{14}$$

$$cov(Y_1, Y_2) = \theta_1 E_{\theta_2}(Y_2)\left(e^{\eta} - 1\right). \tag{15}$$

Expression (14) shows that the variable $Y_2$ is overdispersed. And the covariance is negative, null or positive depending on whether the parameter $\eta$ is negative, null or positive.

## 2.3. The Bivariate Poisson distribution according to Lakshminarayna et al.[7]

In [7], the authors defined the bivariate Poisson law as the product of its marginal laws by a multiplier factor.

**Definition 3.** *Let $Y_1$ and $Y_2$ be two Poisson random variables with respective parameters $\theta_1$ and $\theta_2$. The bivariate distribution of the couple $(Y_1, Y_2)$, denoted $f_{LPS}$, has the mass function*

$$f_{LPS}(y_1, y_2; \theta_1, \theta_2, \alpha) = \left(\frac{\theta_1^{y_1}}{y_1!}e^{-\theta_1}\right)\left(\frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}\right)\left[1 + \alpha\left(e^{-y_1} - e^{-d\theta_1}\right)\left(e^{-y_2} - e^{d\theta_2}\right)\right], \tag{16}$$

*with* $e^{-d\theta_i} = E_{\theta_i}\left(e^{-Y_i}\right), \quad y_i \in \mathbb{N}, \quad \theta_i \in \mathbb{R}_+^* \ (i = 1, 2), \quad \alpha \in \mathbb{R}_+^*$ *and* $d = 1 - e^{-1}$.

This distribution has the following characteristics

$$E_{\theta_1}(Y_1) = var(Y_1) = \theta_1, \tag{17}$$

$$cov(Y_1, Y_2) = \theta_1\theta_2 d^2 e^{-d(\theta_1 + \theta_2)}. \tag{18}$$

In [4], the authors showed that the bivariate Poisson distribution according to Lakshminarayna et al.[7] is a distribution of the bivariate Poisson family and that it converges to the bivariate Poisson distribution according to Berkhout and Plug[2].

# 3. The Bivariate Extended Poisson distribution of type 1

Based on the work [7], we define the bivariate extended Poisson of type 1 distribution as follows.

**Definition 4.** *Let us consider $Y_1$ and $Y_2$ two univariate extended Poisson variables with respective parameters $(\theta_1, \beta_1)$ and $(\theta_2, \beta_2)$. The bivariate Poisson distribution of the pair $(Y_1, Y_2)$, denoted $f_{BEP,1}$, has the mass function (see [7])*

$$f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) = \left(\prod_{j=1}^{2}\left[\left(\frac{\theta_j^{y_j}}{y_j!}e^{-\theta_j}\right)\left(\frac{\beta_j}{\theta_j}y_j - 1\right)\beta_j^{-1}\left\{\frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_j - 1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right]\right) \times$$

$$g(y_1, y_2; \theta_1, \theta_2, \alpha), \tag{19}$$

*where* $g(y_1, y_2; \theta_1, \theta_2, \alpha) = [1 + \alpha (e^{-y_1} - c_1)(e^{-y_2} - c_2)]$, *with* $c_j = E_{\theta_j}\left(e^{-Y_j}\right)$, $y_j \in \mathbb{N}$, $\theta_j \in \mathbb{R}_+^*$, $\beta_j \geq \theta_j$ $(j = 1, 2)$ *and* $\alpha \in \mathbb{R}$.

The initials "BEP,1" stand for Bivariate Extended Poisson of type 1. We have the following result.

**Proposition 4.**

*(i) The marginal laws of* $Y_1$ *and* $Y_2$ *are extended Poisson laws of respective parameters* $(\theta_1, \beta_1)$ *and* $(\theta_2, \beta_2)$.

*(ii)*

$$cov(Y_1, Y_2) = \alpha cov\left(Y_1, e^{-Y_1}\right) cov\left(Y_2, e^{-Y_2}\right). \tag{20}$$

Let be $P(Y_j = y_j) = \left(\frac{\beta_j}{\theta_j} y_j - 1\right) \beta^{-1} \left[\frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta} y_j - 1\right) e^{-\theta_j}}\right]^{\delta_0(y_j)}$ $\forall\ \theta_j > 0,\ \beta_j \geq \theta_j$, $j = 1, 2$, the marginal law of variable $Y_j$ $j = 1, 2$. It follows the result.

**Corollary 1.**

$$f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) = P(Y_1 = y_1) P(Y_2 = y_2) \left[1 + \alpha \left(e^{-y_1} - c_1\right)\left(e^{-y_2} - c_2\right)\right], \tag{21}$$

$c_j = E_{\theta_j}\left(e^{-Y_j}\right)$, $y_j \in \mathbb{N}$, $\theta_j \in \mathbb{R}_+^*$, $\beta_j \geq \theta_j$ $(j = 1, 2)$ *and* $\alpha \in \mathbb{R}$.

This result confirms that the definition 4 is rigorously correct.

**Corollary 2.** *When* $\alpha = 0$, *the variables* $Y_1$ *and* $Y_2$ *are independent.*

*Proof.* [Proof of the proposition 5]

(i)

$$\begin{aligned} P(Y_1 = y_1) =& \sum_{y_2} f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha), \\ &= \left(\frac{\theta_1^{y_1}}{y_1!} e^{-\theta_1}\right) \left(\frac{\beta_1}{\theta} y_1 - 1\right) \beta_1^{-1} \left\{\frac{1 - e^{-\theta_1}}{\left(\frac{\beta_1}{\theta_1} y_1 - 1\right) e^{-\theta_1}}\right\}^{\delta_0(y_1)} \times \\ &\sum_{y_2} \left(\frac{\theta_2^{y_2}}{y_2!} e^{-\theta_2}\right) \left(\frac{\beta_2}{\theta} y_2 - 1\right) \beta_2^{-1} \left\{\frac{1 - e^{-\theta_2}}{\left(\frac{\beta_2}{\theta_2} y_2 - 1\right) e^{-\theta_2}}\right\}^{\delta_0(y_2)} + \end{aligned}$$

$$\alpha\left(e^{-y_1}-c_1\right)\sum_{y_2}\left(\frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}\right)\left(\frac{\beta_2}{\theta}y_2-1\right)\beta_2^{-1}\left\{\frac{1-e^{-\theta_2}}{\left(\frac{\beta_2}{\theta_2}y_2-1\right)e^{-\theta_2}}\right\}^{\delta_0(y_2)}\times$$
$$\left(e^{-y_2}-c_2\right),$$
$$=\left(\frac{\theta_1^{y_1}}{y_1!}e^{-\theta_1}\right)\left(\frac{\beta_1}{\theta}y_1-1\right)\beta_1^{-1}\left\{\frac{1-e^{-\theta_1}}{\left(\frac{\beta_1}{\theta_1}y_1-1\right)e^{-\theta_1}}\right\}^{\delta_0(y_1)}+$$
$$\left(e^{-y_1}-c_1\right)E_{\theta_2}\left(e^{-Y_2}-c_2\right).$$

Since $E_{\theta_2}\left(e^{-Y_2}-c_2\right)=0$, we are sure of the answer.

By symmetry, we have

$$P(Y_2=y_2)=\left(\frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}\right)\left(\frac{\beta_2}{\theta}y_2-1\right)\beta_2^{-1}\left\{\frac{1-e^{-\theta_2}}{\left(\frac{\beta_2}{\theta_2}y_2-1\right)e^{-\theta_2}}\right\}^{\delta_0(y_2)}.$$

(ii) $cov(Y_1,Y_2)=E_{\theta_1,\theta_2}(Y_1Y_2)-E_{\theta_1}(Y_1)E_{\theta_2}(Y_2)$.

We have

$$E_{\theta_1,\theta_2}(Y_1Y_2)=\sum_{y_1}\sum_{y_2}y_1y_2f_{BEP,1}(y_1,y_2;\theta_1,\theta_2,\alpha),$$
$$=\sum_{y_1}\sum_{y_2}y_1y_2\prod_{j=1}^{2}\left[\left(\frac{\theta_j^{y_j}}{y_j!}e^{-\theta_j}\right)\left(\frac{\beta_j}{\theta_j}y_j-1\right)\beta_j^{-1}\left\{\frac{1-e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_1-1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right]+$$
$$\sum_{y_1}\sum_{y_2}y_1y_2\prod_{j=1}^{2}\left[\left(\frac{\theta_j^{y_j}}{y_j!}e^{-\theta_j}\right)\left(\frac{\beta_j}{\theta_j}y_j-1\right)\beta_j^{-1}\left\{\frac{1-e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_j-1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right]\times$$
$$\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right),$$
$$=E_{\theta_1}(Y_1)E_{\theta_2}(Y_2)+\alpha E_{\theta_1}\left[Y_1\left(e^{-Y_1}-c_1\right)\right]E_{\theta_2}\left[Y_2\left(e^{-Y_2}-c_2\right)\right],$$
$$=E_{\theta_1}(Y_1)E_{\theta_2}(Y_2)+\alpha\left[E_{\theta_1}\left(Y_1e^{-Y_1}\right)-E_{\theta_1}(Y_1)E_{\theta_1}\left(e^{-Y_1}\right)\right]\times$$
$$\left[E_{\theta_2}\left(Y_2e^{-Y_2}\right)-E_{\theta_2}(Y_2)E_{\theta_2}\left(e^{-Y_2}\right)\right].$$

And

$$cov(Y_1, Y_2) = \alpha \left[E_{\theta_1}\left(Y_1 e^{-Y_1}\right) - E_{\theta_1}(Y_1)E_{\theta_1}\left(e^{-Y_1}\right)\right] \times \left[E_{\theta_2}\left(Y_2 e^{-Y_2}\right) - E_{\theta_2}(Y_2)E_{\theta_2}\left(e^{-Y_2}\right)\right].$$

Ultimately, we have $cov(Y_1, Y_2) = \alpha cov\left(Y_1, e^{-Y_1}\right) cov\left(Y_2, e^{-Y_2}\right)$. We are sure of the answer.

**Proposition 5.** *Under conditions* (10) *and* (11) *we have*

$$f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) = \prod_{j=1}^{2}\left[\left(\frac{\beta_j}{\theta_j}y_j - 1\right)\beta_j^{-1}\left\{\frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_j - 1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right] \times g(y_1, y_2; \theta_1, \theta_2, \alpha) \times f_{BP}(y_1, y_2, \theta_1, \theta_2). \quad (22)$$

Expression (22) confirms that the bivariate Poisson extended distribution is a member of the family of bivariate Poisson distributions (see [1]).

*Proof.* Indeed, we have

$$f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) = \left(\frac{\theta_1^{y_1}}{y_1!}e^{-\theta_1}\right)\left(\frac{\theta_2^{y_2}}{y_2!}e^{-\theta_2}\right) \times \left(\prod_{j=1}^{2}\left[\left(\frac{\beta_j}{\theta_j}y_j - 1\right)\beta_j^{-1}\left\{\frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_j - 1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right]\right) \times g(y_1, y_2; \theta_1, \theta_2, \alpha),$$

And under conditions (10) and (11) we are assured of the result.

## 3.1. Estimation of parameters $\theta_1,\ \theta_2,\ \beta_1,\ \beta_2,\ \alpha$

The parameters $\theta_1,\ \theta_2,\ \beta_1,\ \beta_2$ and $\alpha$ will be estimated by the maximum likelihood method. Let us consider an n-sample $(y_{1,1}, y_{2,1}),\ (y_{1,2}, y_{2,2}),\ \ldots, (y_{1,n}, y_{2,n})$ of the couple of random variables $(Y_1, Y_2)$ of density $f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha)$. The log-likelihood function $L((y_1, y_2), \theta_1, \theta_2, \beta_1, \beta_2, \alpha)$ is given by

$$L((y_1, y_2), \theta_1, \theta_2, \beta_1, \beta_2, \alpha) = \sum_{i=1}^{n} \ln f_{BEP,1}(y_{1,j}, y_{2,j}; \theta_1, \theta_2, \beta_1, \beta_2, \alpha).$$

The following system of normal equations

$$\begin{aligned} \frac{\partial}{\partial \theta_j} L((y_1, y_2), \theta_1, \theta_2, \beta_1, \beta_2, \alpha) &= 0, \\ \frac{\partial}{\partial \beta_j} L((y_1, y_2), \theta_1, \theta_2, \beta_1, \beta_2, \beta, \alpha) &= 0, \\ \frac{\partial}{\partial \alpha} L((y_1, y_2), \theta_1, \theta_2, \beta_1, \beta_2, \alpha) &= 0, \end{aligned}$$

is used to calculate the estimators $\widehat{\theta}_j,\ \widehat{\beta}_j,\ (j = 1, 2)$ and $\widehat{\alpha}$ using the package maxLik for the statistical environment R (see [6]).

## Student's t test to test $\alpha$=0

To ensure the independance of variables $Y_1$ and $Y_2$, we must perform a statistical test that allow discriminate between the following hypotheses [9]

- Null hypothesis $H_0:\ \alpha = 0$, vs
- alternative hypothesis $H_1:\ \alpha \neq 0$.

Let $\widehat{\alpha} = \widehat{\alpha}_n$ the maximum likelihood estimator of $\alpha$. The variable

$$U = \sqrt{n} \frac{\widehat{\alpha}_n - \alpha}{\sqrt{I^{-1}(\alpha; \theta_1, \theta_2, \beta_1, \beta_2)}}, \tag{23}$$

follows, when $n$ is large, the normal distribution $\mathcal{N}(0, 1)$, with $I(\alpha; \theta_1, \theta_2, \beta_1, \beta_2)$ the amount of information provided by the pair $(Y_1, Y_2)$ at parameter $\alpha$.

The result is as follows.

**Proposition 6.**

*(i)*

$$I(\alpha; \theta_1, \theta_2, \beta_1, \beta_2) = E_{\theta_1,\theta_2} \left[ \frac{\left(e^{-Y_1} - c_1\right)^2 \left(e^{-Y_2} - c_2\right)^2}{\left[1 + \alpha \left(e^{-Y_1} - c_1\right) \left(e^{-Y_2} - c_2\right)\right]^2} \right]. \tag{24}$$

*(ii) And under the null hypothesis*

$$I(0; \theta_1, \theta_2, \beta_1, \beta_2) = var\left(e^{-Y_1}\right) var\left(e^{-Y_2}\right). \tag{25}$$

The estimator of $I(0; \theta 1, \theta 2, \beta 1, \beta 2)$ which we denote $\widehat{I}(0)$ will be calculated following two approaches:

**First approach: the substitution method**

Knowing the estimators $\widehat{\theta}_{1n}, \widehat{\theta}_{2n}, \widehat{\beta}_{1n}$ and $\widehat{\beta}_{2n}$ we can estimate $I(0;\theta_1,\theta_2,\beta_1,\beta_2)$ by $\widehat{I}(0) = I(0;\widehat{\theta}_{1n}, \widehat{\theta}_{2n}, \widehat{\beta}_{1n}, \widehat{\beta}_{2n})$.

**Second approach: basic statistics**

Let $\overline{y}_{jn} = \dfrac{1}{n}\sum_{i=1}^{n} e^{-y_{ji}}$ and $s_{jn}^{'2} = \dfrac{1}{n-1}\sum_{i=1}^{n}\left(e^{-y_{ji}} - \overline{y}_{jn}\right)^2$ $(j=1,2)$ the empirical mean and the empirical unbiased variance of the sample $\left(e^{-Y_{j1}}, \ldots, e^{-Y_{ji}}, \ldots, e^{-Y_{jn}}\right)$ $(j = 1,2)$ of size $n$ of the variable $e^{-Y_j}$ $(j = 1,2)$. Since the variances $var\left(e^{-Y_1}\right)$ and $var\left(e^{-Y_2}\right)$ can be estimated by the respective empirical unbiased variances $s_{1n}^{'2}$ and $s_{2n}^{'2}$, then we can estimate $I(0;\theta_1,\theta_2,\beta_1,\beta_2)$ by $\widehat{I}(0) = s_{1n}^{'2} \times s_{2n}^{'2}$.

**Test statistics**:
The test statistic

$$T = \sqrt{n}\frac{\widehat{\alpha}_n}{\sqrt{\widehat{I}^{-1}(0)}}, \tag{26}$$

follows under the null hypothesis when $n$ is large, the Student's law of degree of freedom $n$.

**Decision**: let $x = P(> |T|)$ the p-value. Given a first order risk $\alpha = 5\%$, if $x < \alpha$ then $H_0$ is rejected, otherwise it is accepted.

*Proof.* [**Proof of the Proposition 6**] We have

$$\frac{\partial}{\partial\alpha}\ln f_{BEP,1} = \frac{\left(e^{-Y_1} - c_1\right)\left(e^{-Y_2} - c_2\right)}{1 + \alpha\left(e^{-Y_1} - c_1\right)\left(e^{-Y_2} - c_2\right)},$$

and

$$\frac{\partial^2}{\partial\alpha^2}\ln f_{BEP,1} = -\frac{\left(e^{-Y_1} - c_1\right)^2\left(e^{-Y_2} - c_2\right)^2}{\left[1 + \alpha\left(e^{-Y_1} - c_1\right)\left(e^{-Y_2} - c_2\right)\right]^2}.$$

Therefore

$$I(\alpha;\theta_1,\theta_2,\beta_1,\beta_2) = E_{\theta_1,\theta_2}\left[\frac{\left(e^{-Y_1} - c_1\right)^2\left(e^{-Y_2} - c_2\right)^2}{\left[1 + \alpha\left(e^{-Y_1} - c_1\right)\left(e^{-Y_2} - c_2\right)\right]^2}\right],$$

and under the null hypothesis the variables $Y_1$ and $Y_2$ are independent. This leads to

$$I(0;\theta_1,\theta_2,\beta_1,\beta_2) = var\left(e^{-Y_1}\right) var\left(e^{-Y_2}\right).$$

And we are sure of the answer. Expressions (6) and (7) allow us to calculate the variances of the variables $e^{-Y_1}$ and $e^{-Y_2}$.

# 4. Simulation study

## 4.1. Simulation and basic statistics

In this section, we realize a simulation study. On this, we consider two random variables $Y_1$ and $Y_2$ following the extended Poisson distribution of respective parameters $(\theta_1, \beta_1)$ and $(\theta_2, \beta_2)$. The table 1 contains the simulations of the variables $Y_1$ and $Y_2$ and the table 2 the basic statistics. We have the presumption that according to the Fisher indices of table 2, the variables are overdispersed. We have simulated samples of size $n = 150$.

Table 1: Simulated data, $\theta_1 = 1,\ \beta_1 = 2$ for $Y_1$ and $\theta_2 = 3,\ \beta_2 = 5$ for $Y_2$

| Count | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_{Y_1}$ | 42 | 23 | 48 | 24 | 12 | 0 | 1 | | | | | $n = 150$ |
| $N_{Y_2}$ | 25 | 3 | 22 | 23 | 29 | 24 | 10 | 5 | 5 | 3 | 1 | |

Table 2: Basic statistics

| Variable | Mean | Variance | Fisher index |
|---|---|---|---|
| $Y_1$ | 1.6333 | 1.7371 | 1.0635 |
| $Y_2$ | 3.4933 | 5.3657 | 1.5359 |

## 4.2. Estimation of model parameters and remark

Using the package maxLik for the statistical environment R (see [6]), we have the output R in table 3.

The parameter estimates are $\widehat{\theta}_1 = 1.14132,\ \widehat{\theta}_2 = 2.70274,\ \widehat{\beta}_1 = 2.64952,\ \widehat{\beta}_2 = 4.58362$ and $\widehat{\alpha} = 1.45859$. The table 3 shows that $\alpha$ is different to 0. Indeed, the corresponding p-value is equal to 0.00096, lower than at the usual significance level 0.05, so we reject the hypothesis that $\alpha = 0$ at the risk of significance 5%. For this set of simulated data, the variables $Y_1$ and $Y_2$ are dependent.

# 5. Conclusion

We constructed the bivariate extended Poisson distribution as a generalization of the univariate extended Poisson distribution by the method of the product of its marginal laws by a factor. This method was demonstrated by Lakshminarayna et al.[7]. We have shown that this distribution belongs to the family of bivariate Poisson distributions. The Student's statistical test allows to highlight the independence between the variables $Y_1$ and $Y_2$.

Table 3: Output R

```
Maximum Likelihood estimation
Newton-Raphson maximisation, 5 iterations
Return code 2: successive function values within tolerance limit
Log-Likelihood: -494.9254
5 free parameters
Estimates:
     Estimate Std. error t value  Pr(> t)
[1,]  1.14132    0.09977  11.439  < 2e-16 ***
[2,]  2.70274    0.16122  16.764  < 2e-16 ***
[3,]  2.64952    0.43437   6.100 1.06e-09 ***
[4,]  4.58362    0.81856   5.600 2.15e-08 ***
[5,]  1.45859    0.44173   3.302  0.00096 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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