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Fourier Expansion, Integral Representation and Explicit Formula at Rational Arguments of the Tangent Polynomials of Higher-Order

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Abstract. In this paper, Fourier series expansion of Tangent polynomials are derived and the integral representation and explicit formula at rational arguments of these polynomials are established.
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1. Introduction

For $r \in \mathbb{N}$, the higher-order tangent polynomials, $T_n^r(x)$ $(n \geq 0)$, are defined by the following generating function (see [1])

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_n^r(x)\frac{t^n}{n!}, \quad |2t| < \pi. \tag{1}$$

When $r = 1$, the above equation gives the generating function for the classical tangent polynomials (see [2]).

The study of tangent polynomials has become an interesting area for many mathematicians for they possess significant properties that can be found in the field of mathematics and physics (see [3], [4]). Analogues, explicit identities and symmetric properties for tangent polynomials are derived in (see [5], [6], [7]).

In this paper, the researchers derive the Fourier expansion and integral representation of the tangent polynomials of order r, $r \in \mathbb{Z}^+$ and present an explicit formula of these polynomials at rational arguments using the method of Luo [8].

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2. Fourier Expansions for Tangent polynomials of Higher-order

In this section, we give Fourier expansion for tangent polynomials of higher order.

Theorem 2.1. *For* $0 \leq x \leq 1$,

$$T_n^r(x) = 2 \cdot n! \left(\frac{2}{\pi}\right)^{r+n} \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} (-1)^j \binom{r+n-j-1}{r-j-1} \frac{\pi^j}{j!} B_j^r\left(\frac{x}{2}\right) \times \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}, \quad (2)$$

where $B_j^r\left(\frac{x}{2}\right)$ *denotes the Bernoulli polynomials of order* r *defined by*

$$\left(\frac{w}{e^w - 1}\right)^r e^{xw} = \sum_{j=0}^{\infty} B_j^r(x) \frac{w^n}{n!}.$$

Proof. For $r \geq 2$,

$$Res\left(f(t), t = t_k\right) = \frac{1}{(r-1)!} \lim_{t \to t_k} \frac{d^{r-1}}{dt^{r-1}} (t - t_k)^r \left(\frac{2}{e^{2t}+1}\right)^r \frac{e^{xt}}{t^{n+1}}.$$

Consider the function

$$(t - t_k)^r \left(\frac{2}{e^{2t}+1}\right)^r \frac{e^{xt}}{t^{n+1}} = 2^r \frac{(t-t_k)^r}{(e^{2t}+1)^r} \frac{e^{xt}}{t^{n+1}}.$$

Writing $\left(e^{2t}+1\right)^r$ as

$$\begin{aligned} \left(e^{2t}+1\right)^r &= (-1)^r \left(e^{2t}(-1) - 1\right)^r \\ &= (-1)^r \left(e^{2t} \cdot e^{-2t_k} - 1\right)^r \\ &= (-1)^r \left(e^{2(t-t_k)} - 1\right)^r, \end{aligned}$$

and since $e^{-2t_k} = e^{-2(2k+1)\frac{\pi}{2}i} = e^{-(2k+1)\pi i} = -1$, we have

$$\begin{aligned} (t-t_k)^r \left(\frac{2}{e^{2t}+1}\right)^r \frac{e^{xt}}{t^{n+1}} &= \frac{2^r (t-t_k)^r}{(-1)^r \left(e^{2(t-t_k)} - 1\right)^r} \cdot \frac{e^{xt}}{t^{n+1}} \\ &= (-1)^r \frac{(2(t-t_k))^r}{(e^{2(t-t_k)} - 1)^r} \cdot \frac{e^{xt}}{t^{n+1}} \\ &= (-1)^r \left(\sum_{n=0}^{\infty} B_n^r \, \frac{(2(t-t_k))^n}{n!}\right) e^{xt} \, t^{-(n+1)}, \end{aligned}$$

where B_n^r denotes the Bernoulli numbers of order r defined by the generating function

$$\left(\frac{w}{e^w-1}\right)^r=\sum_{n=0}^{\infty}B_n^r\,\frac{w^n}{n!}.$$

To get the derivative, applying the Leibniz Rule yields

$$\begin{aligned}
&\frac{d^{r-1}}{dt^{r-1}}\left\{(t-t_k)^r\left(\frac{2}{e^{2t}+1}\right)^r\frac{e^{xt}}{t^{n+1}}\right\}\\
&\quad=\frac{d^{r-1}}{dt^{r-1}}\left\{(-1)^r\left(\sum_{n=0}^{\infty}B_n^r\,\frac{(2(t-t_k))^n}{n!}\right)e^{xt}\;t^{-(n+1)}\right\}\\
&\quad=(-1)^r\frac{d^{r-1}}{dt^{r-1}}\left\{\left(e^{xt}\sum_{n=0}^{\infty}B_n^r\,\frac{(2(t-t_k))^n}{n!}\right)t^{-(n+1)}\right\}\\
&\quad=(-1)^r\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n+1)}\cdot\frac{d^j}{dt^j}\left(e^{xt}\sum_{n=0}^{\infty}B_n^r\,\frac{(2(t-t_k))^n}{n!}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{d^j}{dt^j}\left(e^{xt}\sum_{n=0}^{\infty}B_n^r\,\frac{2^n(t-t_k)^n)}{n!}\right)\\
&\quad=\sum_{l=0}^{j}\binom{j}{l}x^{j-l}e^{xt}\sum_{n=l}^{\infty}B_n^r\frac{2^n}{n!}\;(n)_l\;(t-t_k)^{n-l}\\
&\quad=e^{xt}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=l}^{\infty}2^nB_n^r\,\frac{(t-t_k)^{n-l}}{(n-l)!},
\end{aligned}$$

$$\begin{aligned}
&\frac{d^{r-1}}{dt^{r-1}}\left((t-t_k)^r\left(\frac{2}{e^{2t}+1}\right)^r\frac{e^{xt}}{t^{n+1}}\right)\\
&\quad=(-1)^r\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n+1)}\\
&\qquad\times e^{xt}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=l}^{\infty}2^nB_n^r\,\frac{(t-t_k)^{n-l}}{(n-l)!}.
\end{aligned}$$

Thus,

$$\begin{aligned}
Res\,(f(t),t=t_k)&=\frac{1}{(r-1)!}\lim_{t\to t_k}(-1)^r\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{d^{r-1-j}}{dt^{r-1-j}}t^{-(n+1)}\\
&\qquad\times\lim_{t\to t_k}e^{xt}\sum_{l=0}^{j}\binom{j}{l}x^{j-l}\sum_{n=l}^{\infty}2^nB_n^r\,\frac{(t-t_k)^{n-l}}{(n-l)!}.
\end{aligned}$$

Note that $B_n^r \frac{(t-t_k)^{n-l}}{(n-l)!} \to 0$ as $t \to t_k$ except when $n = l$. This gives

$$\begin{aligned}
Res\,(f(t), t = t_k) &= \frac{1}{(r-1)!}(-1)^r \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j}(n+r-1-j)_{r-1-j}\ t_k^{-(n+r-j)} \\
&\quad \times e^{xt_k} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} 2^l B_l^r \\
&= \frac{(-1)^r}{(r-1)!} \sum_{j=0}^{r-1} \frac{(r-1)!}{j!(r-1-j)!}(-1)^{r-1-j}(n+r-1-j)_{r-1-j}\ t_k^{-(n+r-j)} \\
&\quad \times e^{xt_k} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} 2^l B_l^r \\
&= \sum_{j=0}^{r-1} (-1)^{j-1} \binom{n+r-1-j}{r-1-j} \frac{t_k^{j-n-r}}{j!} e^{xt_k} 2^j \sum_{l=0}^{j} \binom{j}{l} \frac{x^{j-l}}{2^{j-l}} B_l^r.
\end{aligned}$$

Recall that $B_j^r\left(\frac{x}{2}\right) = \sum_{l=0}^{j} \binom{j}{l} B_l^r \left(\frac{x}{2}\right)^{j-l}$. Thus,

$$\begin{aligned}
Res\,(f(t), t = t_k) &= \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{n+r-1-j}{r-1-j} \frac{t_k^{j-n-r}}{j!} e^{xt_k} B_j^r\left(\frac{x}{2}\right) \\
&= \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{n+r-1-j}{r-1-j} \frac{B_j^r\left(\frac{x}{2}\right)}{j!} \frac{e^{xt_k}}{t_k^{n+r-j}} \\
&= \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{r+n-j-1}{r-j-1} \frac{B_j^r\left(\frac{x}{2}\right)}{j!} \frac{e^{xt_k}}{t_k^{r+n-j}}.
\end{aligned}$$

Taking $t_k = \frac{1}{2}(2k+1)\pi i$, we get

$$Res\,(f(t), t = t_k) = \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{r+n-j-1}{r-j-1} \frac{B_j^r\left(\frac{x}{2}\right)}{j!} \frac{e^{\frac{1}{2}(2k+1)\pi i x}}{\left(\frac{1}{2}(2k+1)\pi i\right)^{r+n-j}}$$

$$= \frac{1}{\left(\frac{1}{2}\pi i\right)^{r+n}} \sum_{j=0}^{r-1} (-1)^{j-1} 2^j \binom{r+n-j-1}{r-j-1} \frac{\left(\frac{1}{2}\pi i\right)^j}{j!} B_j^r\left(\frac{x}{2}\right) \frac{e^{\frac{1}{2}(2k+1)\pi i x}}{(2k+1)^{r+n-j}}.$$

This gives

$$T_n^r(x) = n! \left(\frac{2}{\pi i}\right)^{r+n} \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{r-1} (-1)^j \binom{r+n-j-1}{r-j-1} \frac{(\pi i)^j}{j!} B_j^r\left(\frac{x}{2}\right) \right) \frac{e^{\frac{1}{2}(2k+1)\frac{\pi i}{2}x}}{(2k+1)^{r+n-j}}. \quad (3)$$

Now, from (3), we look at

$$i^{-(r+n-j)}\sum_{k\in\mathbb{Z}}\frac{e^{(2k+1)\frac{\pi i}{2}x}}{(2k+1)^{r+n-j}}. \tag{4}$$

Noting that $i^{-(r+n-j)} = e^{-(r+n-j)\pi i/2}$ and $(-1)^{r+n-j} = e^{(r+n-j)\pi i}$, we see that

$$\begin{aligned}
i^{-(r+n-j)}\sum_{k\in\mathbb{Z}}\frac{e^{(2k+1)\frac{\pi i}{2}x}}{(2k+1)^{r+n-j}} &= i^{-(r+n-j)}\left\{\sum_{k=0}^{\infty}\frac{e^{(2k+1)\frac{\pi i}{2}x}}{(2k+1)^{r+n-j}} + (-1)^{r+n-j}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)\frac{\pi i}{2}x}}{(2k+1)^{r+n-j}}\right\}\\
&= \sum_{k=0}^{\infty}\frac{e^{[(2k+1)x/2-(r+n-j)/2]\pi i} + e^{-[(2k+1)x/2-(r+n-j)/2]\pi i}}{(2k+1)^{r+n-j}}\\
&= \sum_{k=0}^{\infty}\frac{2\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}\\
&= 2\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}.
\end{aligned} \tag{5}$$

Replacing (4) with (5) in (3), we get the desired formula (2).

3. Integral representation for Tangent polynomials of Higher-order

In this section, we establish an integral representation for tangent polynomials of higher order.

Theorem 3.1. *For $n \in \mathbb{N}$, $r \geq 2$, and $0 \leq \Re(x) \leq 1$,*

$$\begin{aligned}
T_n^r(x) = 2^{r+n}\sum_{j=0}^{r-1}\frac{(-1)^j}{j!}\cdot\frac{B_j^r\left(\frac{x}{2}\right)}{(r-j-1)!}&\left\{\int_0^{\infty}\frac{e^{\pi t}\cos\left[\pi x/2-(r+n-j)\pi/2\right]}{\cosh(2\pi t)-\cos(\pi x)}\ t^{r+n-j-1}\ dt\right.\\
&\left.-\int_0^{\infty}\frac{e^{-\pi t}\cos\left[\pi x/2+(r+n-j)\pi/2\right]}{\cosh(2\pi t)-\cos(\pi x)}\ t^{r+n-j-1}\ dt\right\}.
\end{aligned} \tag{6}$$

Proof. From (2), we get

$$\begin{aligned}
T_n^r(x) &= 2\cdot n!\left(\frac{2}{\pi}\right)^{r+n}\sum_{k=0}^{\infty}\sum_{j=0}^{r-1}(-1)^j\ \frac{(r+n-j-1)!}{(r-j-1)!\ n!}\cdot\frac{\pi^j}{j!}\ B_j^r\left(\frac{x}{2}\right)\\
&\qquad\times\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}\\
&= 2^{r+n+1}\sum_{j=0}^{r-1}\frac{(-1)^j\ B_j^r\left(\frac{x}{2}\right)}{(r-j-1)!\ j!}\cdot\frac{(r+n-j-1)!}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}.
\end{aligned} \tag{7}$$

We look at

$$\frac{(r+n-j-1)!}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}$$
$$=\frac{1}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]\frac{(r+n-j-1)!}{(2k+1)^{r+n-j}}. \quad (8)$$

Applying the integral formula

$$\int_0^\infty t^n e^{-at}\,dt=\frac{n!}{a^{n+1}},$$

for $n\geq 0$ and $\Re(a)>0$, then (8) becomes

$$\begin{aligned}
&\frac{(r+n-j-1)!}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}\\
&=\frac{1}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]\int_0^\infty t^{r+n-j-1}e^{-(2k+1)t}\,dt\\
&=\frac{1}{\pi^{r+n-j}}\int_0^\infty t^{r+n-j-1}\sum_{k=0}^{\infty}e^{-(2k+1)t}\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]dt\\
&=\frac{1}{\pi^{r+n-j}}\int_0^\infty t^{r+n-j-1}\sum_{k=0}^{\infty}e^{-(2k+1)t}\Big\{\cos\left[(2k+1)\pi x/2\right]\cos\left[(r+n-j)\pi/2\right]\\
&\qquad+\sin\left[(2k+1)\pi x/2\right]\sin\left[(r+n-j)\pi/2\right]\Big\}\,dt\\
&=\frac{1}{\pi^{r+n-j}}\int_0^\infty\Big\{\cos\left[(r+n-j)\pi/2\right]\sum_{k=0}^{\infty}e^{-(2k+1)t}\cos\left[(2k+1)\pi x/2\right]\\
&\qquad+\sin\left[(r+n-j)\pi/2\right]\sum_{k=0}^{\infty}e^{-(2k+1)t}\sin\left[(2k+1)\pi x/2\right]\Big\}t^{r+n-j-1}\,dt.
\end{aligned} \quad (9)$$

By making use of

$$\sum_{k=0}^{\infty}e^{-(2k+1)t}\sin\left[(2k+1)x\right]=\frac{\sin x\cosh t}{\cosh(2t)-\cos(2x)},$$

and

$$\sum_{k=0}^{\infty}e^{-(2k+1)t}\cos\left[(2k+1)x\right]=\frac{\cos x\sinh t}{\cosh(2t)-\cos(2x)},$$

which may be deduced from

$$\sum_{k=0}^{\infty}e^{(xi-t)(2k+1)}=\frac{\cos x\sinh t+i\sin x\cosh t}{\cosh(2t)-\cos(2x)},$$

for $t > 0$, (9) then becomes

$$\frac{(r+n-j-1)!}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}$$
$$=\frac{1}{\pi^{r+n-j}}\int_0^{\infty}\left\{\cos\left[(r+n-j)\pi/2\right]\frac{\cos\frac{\pi x}{2}\ \sinh t}{\cosh(2t)-\cos(\pi x)}\right.$$
$$\left.+\sin\left[(r+n-j)\pi/2\right]\frac{\sin\frac{\pi x}{2}\ \cosh t}{\cosh(2t)-\cos(\pi x)}\right\}t^{r+n-j-1}\,dt. \quad (10)$$

Applying the transformation $t = \pi t$, (10) becomes

$$\frac{(r+n-j-1)!}{\pi^{r+n-j}}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\pi x/2-(r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}$$
$$=\frac{1}{\pi^{r+n-j}}\int_0^{\infty}\left\{\cos\left[(r+n-j)\pi/2\right]\frac{\cos\frac{\pi x}{2}\ \sinh \pi t}{\cosh(2\pi t)-\cos(\pi x)}\right.$$
$$\left.+\sin\left[(r+n-j)\pi/2\right]\frac{\sin\frac{\pi x}{2}\ \cosh \pi t}{\cosh(2\pi t)-\cos(\pi x)}\right\}\pi^{r+n-j}t^{r+n-j-1}\,dt$$
$$=\int_0^{\infty}\left\{\frac{\cos\left[(r+n-j)\pi/2\right]\ \cos\frac{\pi x}{2}\ \left(e^{\pi s}-e^{-\pi s}\right)}{2\left[\cosh(2\pi t)-\cos(\pi x)\right]}\right.$$
$$\left.+\frac{\sin\left[(r+n-j)\pi/2\right]\ \sin\frac{\pi x}{2}\ \left(e^{\pi s}+e^{-\pi s}\right)}{2\left[\cosh(2\pi t)-\cos(\pi x)\right]}\right\}t^{r+n-j-1}\,dt$$
$$=\frac{1}{2}\int_0^{\infty}\left\{\frac{e^{\pi s}\big(\cos\left[(r+n-j)\pi/2\right]\ \cos\frac{\pi x}{2}+\sin\left[(r+n-j)\pi/2\right]\ \sin\frac{\pi x}{2}\big)}{\cosh(2\pi t)-\cos(\pi x)}\right.$$
$$\left.-\frac{e^{-\pi s}\big(\cos\left[(r+n-j)\pi/2\right]\ \cos\frac{\pi x}{2}-\sin\left[(r+n-j)\pi/2\right]\ \sin\frac{\pi x}{2}\big)}{\cosh(2\pi t)-\cos(\pi x)}\right\}t^{r+n-j-1}\,dt$$
$$=\frac{1}{2}\int_0^{\infty}\frac{e^{\pi s}\cos\left[\pi x/2-(r+n-j)\pi/2\right]-e^{-\pi s}\cos\left[\pi x/2+(r+n-j)\pi/2\right]}{\cosh(2\pi t)-\cos(\pi x)}\ t^{r+n-j-1}\,dt. \quad (11)$$

Applying (11) to (7), we get the desired formula (6).

4. Explicit formula for Tangent polynomials of Higher-order at rational arguments

In this section, we obtain an explicit formula for tangent polynomials of higher order at rational arguments by applying the Fourier expansion (2). Here let $\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$ denote the set of nonpositive integers.

Theorem 4.1. *For $n, q \in \mathbb{N}$ and $p \in \mathbb{Z}$,*

$$T_n^r\left(\frac{2p}{q}\right)=\frac{2\cdot n!}{(q\pi)^{r+n}}\sum_{j=0}^{r-1}(-1)^j\binom{r+n-j-1}{r-j-1}\frac{(2q\pi)^j}{j!}B_j^r\left(\frac{p}{q}\right)$$

$$\times \sum_{l=1}^{q} \zeta\left(r+n-j, \frac{2l-1}{2q}\right) \cos\left[\frac{(2l-1)p\pi}{q} - \frac{(r+n-j)\pi}{2}\right], \tag{12}$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \tag{13}$$

for $\Re(s) > 1$ *and* $a \notin \mathbb{Z}_0^-$ *is Hurwitz zeta function.*

Proof. We look at

$$\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}. \tag{14}$$

Replacing k with $k-1$:

$$\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}} = \sum_{k=1}^{\infty} \frac{\cos\left[(r+n-j)\pi/2 - (2k-1)\pi x/2\right]}{(2k-1)^{r+n-j}}. \tag{15}$$

Applying the elementary series identity

$$\sum_{k=1}^{\infty} f(k) = \sum_{l=1}^{q}\sum_{k=0}^{\infty} f(qk+l), \quad q \in \mathbb{N},$$

(15) becomes

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}} \\
&\quad = \sum_{l=1}^{q}\sum_{k=0}^{\infty} \frac{\cos\left[(r+n-j)\pi/2 - (2qk+2l-1)\pi x/2\right]}{(2qk+2l-1)^{r+n-j}} \\
&\quad = \sum_{l=1}^{q}\sum_{k=0}^{\infty} \frac{\cos\left[(r+n-j)\pi/2 - (2l-1)\pi x/2 - qk\pi x\right]}{\left[2q\left(k+\frac{2l-1}{2q}\right)\right]^{r+n-j}} \\
&\quad = \sum_{l=1}^{q}\sum_{k=0}^{\infty} \frac{\cos\left[(r+n-j)\pi/2 - (2l-1)\pi x/2 - qk\pi x\right]}{(2q)^{r+n-j}} \cdot \frac{1}{\left(k+\frac{2l-1}{2q}\right)^{r+n-j}}.
\end{aligned} \tag{16}$$

Setting $x = 2p/q$, (16) becomes

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}} \\
&\quad = \sum_{l=1}^{q}\sum_{k=0}^{\infty} \frac{\cos\left[(r+n-j)\pi/2 - (2l-1)p\pi/q - 2\pi(pk)\right]}{(2q)^{r+n-j}} \cdot \frac{1}{\left(k+\frac{2l-1}{2q}\right)^{r+n-j}}
\end{aligned}$$

$$= \frac{1}{(2q)^{r+n-j}} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \cos\left[(r+n-j)\pi/2 - (2l-1)p\pi/q\right] \cdot \frac{1}{\left(k + \frac{2l-1}{2q}\right)^{r+n-j}}$$

$$= \frac{1}{(2q)^{r+n-j}} \sum_{l=1}^{q} \cos\left[(r+n-j)\pi/2 - (2l-1)p\pi/q\right] \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{2l-1}{2q}\right)^{r+n-j}}. \quad (17)$$

By (13), (17) becomes

$$\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\pi x/2 - (r+n-j)\pi/2\right]}{(2k+1)^{r+n-j}}$$

$$= \frac{1}{(2q)^{r+n-j}} \sum_{l=1}^{q} \cos\left[\frac{(2l-1)p\pi}{q} - \frac{(r+n-j)\pi}{2}\right] \zeta\left(r+n-j, \frac{2l-1}{2q}\right). \quad (18)$$

Replacing (14) with (18) in (2), we obtain the desired formula (12).

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