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A Note on an Octuple Integral in terms of the Lerch Function

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Abstract. The known exact expression for an octuple integral relating to research in the fields of mathematics and physics is summarized. A new closed form expression for this integral is given in terms of the Lerch function.

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Key Words and Phrases: Octuple integral, Riemann zeta function, Cauchy integral, Lerch function

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1. Significance Statement

Octuple integrals are used and evaluated in may areas of mathematics and physics. Some areas of interest where these integrals are used are in multipupil in phase microscopy, where pairs of Fourier transforms are evaluated [8], the kinetic theory of simple and composite monatomic gases : viscosity, thermal conduction, and diffusion [1], statistical characteristics of the laser-radiation-intensity fluctuations in rainfall [6], the velocity distribution function, and on the stresses in a non-uniform rarefied monatomic gas [5], and some applications of Marcel Riesz's Integrals of Fractional Order [2].

In current literature octuple integrals expressed in terms of a closed form solution do not appear to be tabulated. In this work the authors derive and evaluate a octuple integral in terms of the Lerch function and derive special cases of this integral transform in terms of special constants. It is our hope that researchers will find such evaluations useful for potential research requiring these formulae.

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2. Introduction

The octuple integral derived in this manuscript is given by

$$\int_{\mathbb{R}_+^8} (rs)^{\frac{m-1}{2}}(r+s)^{-m/2}(tz)^{-\frac{m}{2}-1}(t+z)^{\frac{m+1}{2}}(uv)^{-\frac{m}{2}-\frac{1}{2}} \\ (u+v)^{m/2}(xy)^{m/2}(x+y)^{\frac{1}{2}(-m-1)}e^{-p(r+u+x+z)-q(s+t+v+y)} \\ \log^k\left(\frac{a\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)dxdydzdrdsdtdudv \quad (1)$$

where the parameters $k, a \in \mathbb{C}, Re(p,q) > 0$ are general complex numbers with $-1/2 \geq Re(m) \geq -1$. The derivation of the definite integral follows the method used by us in [10] which involves Cauchy's integral formula. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \quad (2)$$

where C is in general an open contour in the complex plane where the bilinear concomitant [10] has the same value at the end points of the contour. The method in [10] involves using a form of equation (2) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying equation (2) by a function and performing some substitutions and taking the infinite sum so that the contour integrals are the same.

3. Definite integral of the contour integral

We use the method in [10]. The variable of integration in the contour integral is $z = w + m$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the first or second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using equation (2) we replace y by

$$\log\left(\frac{a\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right) \quad (3)$$

then multiply by both sides by

$$(rs)^{\frac{m-1}{2}}(r+s)^{-m/2}(tz)^{-\frac{m}{2}-1}(t+z)^{\frac{m+1}{2}}(uv)^{-\frac{m}{2}-\frac{1}{2}}(u \\ +v)^{m/2}(xy)^{m/2}(x+y)^{\frac{1}{2}(-m-1)}e^{-p(r+u+x+z)-q(s+t+v+y)} \quad (4)$$

and take the definite octuple integral over $x, y, z, r, s, t, u, v \in [0, \infty)$ to get;

$$\int_{\mathbb{R}_+^8} (rs)^{\frac{m-1}{2}}(r+s)^{-m/2}(tz)^{-\frac{m}{2}-1}(t+z)^{\frac{m+1}{2}}(uv)^{-\frac{m}{2}-\frac{1}{2}}$$
$$(u+v)^{m/2}(xy)^{m/2}(x+y)^{\frac{1}{2}(-m-1)}e^{-p(r+u+x+z)-q(s+t+v+y)}$$
$$\log^k\left(\frac{a\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)\frac{dxdydzdrdsdtdudv}{\Gamma(k+1)}$$

$$= \frac{1}{2\pi i}\int_C\int_{\mathbb{R}_+^8} a^w w^{-k-1}(rs)^{\frac{1}{2}(m+w-1)}(r+s)^{\frac{1}{2}(-m-w)}$$
$$(tz)^{\frac{1}{2}(-m-w)-1}(t+z)^{\frac{1}{2}(m+w+1)}(uv)^{\frac{1}{2}(-m-w)-\frac{1}{2}}(u+v)^{\frac{m+w}{2}}$$
$$(xy)^{\frac{m+w}{2}}(x+y)^{\frac{1}{2}(-m-w-1)}e^{-p(r+u+x+z)-q(s+t+v+y)}dwdxdydzdrdsdtdudv$$

$$= \frac{1}{2\pi i}\int_{\mathbb{R}_+^8}\int_C a^w w^{-k-1}(rs)^{\frac{1}{2}(m+w-1)}(r+s)^{\frac{1}{2}(-m-w)}$$
$$(tz)^{\frac{1}{2}(-m-w)-1}(t+z)^{\frac{1}{2}(m+w+1)}(uv)^{\frac{1}{2}(-m-w)-\frac{1}{2}}(u+v)^{\frac{m+w}{2}}$$
$$(xy)^{\frac{m+w}{2}}(x+y)^{\frac{1}{2}(-m-w-1)}e^{-p(r+u+x+z)-q(s+t+v+y)}dxdydzdrdsdtdudvdw$$

$$= -\frac{1}{2\pi i}\int_C \frac{2\pi^4 a^w w^{-k-1}\csc(\pi(m+w))}{p^2q^2}dw \quad (5)$$

from equation (3.1.3.9) in [9] where $-1 < Re(w+m) < 1$ and using the reflection formula for the gamma function. The logarithmic function is given for example in section (4.2) in [3]. We are able to switch the order of integration over $w+m$ and x,y,z,r,s,t,u,v using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

4. The Lerch function

We use section (25.14) in [3] where $\Phi(z,s,v)$ is the Lerch function which is a generalization of the Hurwitz zeta $\zeta(s,v)$ and Polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (6)$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt, \quad (7)$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Infinite sum of the contour integral

In this section we will again use Cauchy's integral formula (2) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed using equation (2) and replace y by $\log(a) + i\pi(2y+1)$ and multiply both sides by $\frac{4i\pi^4}{p^2q^2}$ then take the infinite sum over $y \in [0, \infty)$ simplifying in terms of the Lerch function to get

$$\begin{aligned}\frac{i^{k+1}2^{k+2}\pi^{k+4}e^{i\pi m}\Phi\left(e^{2im\pi}, -k, \frac{\pi - i\log(a)}{2\pi}\right)}{p^2q^2\Gamma(k+1)} &= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C \frac{4i\pi^4 a^w w^{-k-1} e^{i\pi(2y+1)(m+w)}}{p^2q^2}dw \\ &= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty}\frac{4i\pi^4 a^w w^{-k-1} e^{i\pi(2y+1)(m+w)}}{p^2q^2}dw \\ &= -\frac{1}{2\pi i}\int_C \frac{2\pi^4 a^w w^{-k-1}\csc(\pi(m+w))}{p^2q^2}dw \end{aligned} \tag{8}$$

from equation(1.232.3) in [4] where $Im(w+m) > 0$ in order for the sum to converge.

Theorem 1. *For all $k, a \in \mathbb{C}, Re(p,q) > 0, -1/2 < Re(m) < -1$,*

$$\begin{aligned}&\int_{\mathbb{R}_+^8}(rs)^{\frac{m-1}{2}}(r+s)^{-m/2}(tz)^{-\frac{m}{2}-1}(t+z)^{\frac{m+1}{2}}(uv)^{-\frac{m}{2}-\frac{1}{2}} \\ &\quad (u+v)^{m/2}(xy)^{m/2}(x+y)^{\frac{1}{2}(-m-1)}e^{-p(r+u+x+z)-q(s+t+v+y)} \\ &\quad \log^k\left(\frac{a\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)dxdydzdrdsdtdudv \\ &= \frac{i^{k+1}2^{k+2}\pi^{k+4}e^{i\pi m}\Phi\left(e^{2im\pi}, -k, \frac{\pi - i\log(a)}{2\pi}\right)}{p^2q^2}\end{aligned} \tag{9}$$

Proof. Observe the right-hand side of equation (5) is equal to the right-hand side of equation (8) so we may equate the left-hand sides and simplify the gamma function to yield the stated result.

Example 1. *The degenerate case.*

$$\begin{aligned}&\int_{\mathbb{R}_+^8}(rs)^{\frac{m-1}{2}}(r+s)^{-m/2}(tz)^{-\frac{m}{2}-1}(t+z)^{\frac{m+1}{2}}(uv)^{-\frac{m}{2}-\frac{1}{2}}(u+v)^{m/2}(xy)^{m/2}(x+y)^{\frac{1}{2}(-m-1)} \\ &\quad e^{-p(r+u+x+z)-q(s+t+v+y)}dxdydzdrdsdtdudv\end{aligned}$$

$$= -\frac{2\pi^4 \csc(\pi m)}{p^2 q^2} \quad (10)$$

Proof. Use equation (9) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [7].

Example 2.

$$\int_{\mathbb{R}_+^8} \frac{\sqrt[4]{r+s}\sqrt[4]{t+z}e^{-r-2s-2t-u-2v-x-2y-z}}{(rs)^{3/4}(tz)^{3/4}\sqrt[4]{uv}\sqrt[4]{u+v}\sqrt[4]{xy}\sqrt[4]{x+y}\left(\log^2\left(\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)+\pi^2\right)} dxdydzdrdsdtdudv$$

$$= \frac{1}{2}\pi^2 \log(2) \quad (11)$$

and

$$\int_{\mathbb{R}_+^8} \frac{\sqrt[4]{rs}\sqrt[4]{r+s}\sqrt[4]{tz}\sqrt[4]{t+z}(uv)^{3/4}(xy)^{3/4}e^{-r-2s-2t-u-2v-x-2y-z}}{rstuvxyz\sqrt[4]{u+v}\sqrt[4]{x+y}\left(\log^2\left(\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)+\pi^2\right)}$$
$$\log\left(\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right) dxdydzdrdsdtdudv \quad (12)$$
$$= 0$$

Proof. Use equation (9) and set $a = -1, p = 1, q = 2, m = -1/2$ and simplify in terms of the Riemann zeta function using entry (2) in Table below (64:7) and entry (4) in Table below (64:12:7) in [7]. Next apply l'Hopital's rule to the right-hand side as $k \to -1$ rationalize the denominator equate real and imaginary parts and simplify.

Example 3.

$$\int_{\mathbb{R}_+^8} \frac{(r+s)^{3/8}\sqrt[8]{t+z}e^{-r-2s-2t-u-2v-x-2y-z}}{(rs)^{7/8}(tz)^{5/8}\sqrt[8]{uv}(u+v)^{3/8}(xy)^{3/8}\sqrt[8]{x+y}\left(\log^2\left(-\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)+\pi^2\right)} \quad (13)$$
$$dxdydzdrdsdtdudv$$

$$= \frac{\pi^2(\pi + \log(4))}{8\sqrt{2}}$$

and

$$\int_{\mathbb{R}_+^8} \frac{(r+s)^{3/8}\sqrt[8]{t+z}e^{-r-2s-2t-u-2v-x-2y-z}\log\left(\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)}{(rs)^{7/8}(tz)^{5/8}\sqrt[8]{uv}(u+v)^{3/8}(xy)^{3/8}\sqrt[8]{x+y}\left(\log^2\left(\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)+\pi^2\right)} \quad (14)$$
$$dxdydzdrdsdtdudv$$

$$= \frac{\pi^3(\log(4) - \pi)}{8\sqrt{2}}$$

Proof. Use equation (9) and set $k=-1, a=-1, p=1, q=2, m=-1/2$ and simplify in terms of the polylogarithm function $Li_n(z)$ function using entry (2) in in Table below (64:12:7) in [7] and equation (25.12.10) in [3] and rationalize the denominator equate real and imaginary parts and simplify.

Example 4.

$$\int_{\mathbb{R}_+^8} \frac{(r+s)^{3/8}\sqrt[8]{t+z}e^{-r-s-t-u-v-x-y-z}}{(rs)^{7/8}(tz)^{5/8}\sqrt[8]{uv}(u+v)^{3/8}(xy)^{3/8}\sqrt[8]{x+y}\sqrt{\log\left(-\frac{\sqrt{rs}\sqrt{t+z}\sqrt{u+v}\sqrt{xy}}{\sqrt{r+s}\sqrt{tz}\sqrt{uv}\sqrt{x+y}}\right)}} dxdydzdrdsdtdudv$$

$$= (-1-i)\sqrt{2}\pi^{7/2}\left(\zeta\left(\frac{1}{2},\frac{1}{4}\right)-i\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right) \quad (15)$$

Proof. Use equation (9) and set $k=-1/2, a=-1, p=q=1, m=-3/4$ and simplify in terms of the Hurwitz zeta function using entry (4) in Table below (64:12:7) in [7].

6. Discussion

In this paper, we have presented a novel method for deriving a new octuple integral along with some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram. Some of the challenges encountered were in the numerical evaluation of the integrals. We know from our method the definite integral is equal to the Lerch function so this is a new way of computing this octuple integral. We tried various numerical methods in the Mathematica software to achieve the best possible result relative to the Lerch function.

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