EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 1, 2022, 36-46
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Pythagorean Fuzzy Small Submodules

Areej Almuhaimeed

Department of Mathematics, College of Science, Taibah University, Medina, Saudi Arabia

---

Abstract. In this paper, we introduce the notion of a pythagorean fuzzy small submodule. We prove various characterisations for pythagorean fuzzy small submodules. We provide a relation between a pythagorean fuzzy small submodule and a basic small submodule. In addition, some important properties regarding pythagorean fuzzy small submodules are investigated.

2020 Mathematics Subject Classifications: 03E72, 03B52, 94D05, 08A72

Key Words and Phrases: Pythagorean fuzzy set, pythagorean fuzzy small submodule, homomorphism,

---

1. Introduction

In 1965, Zadeh [16] introduced the concept of fuzzy set which was a generalisation of the classical set. This encourages many researchers to investigate set theory in fuzzy setting. Pythagorean fuzzy set is one of the most important fuzzy sets. Its importance lies behind the fact that this set can be applied in order to characterized uncertain data accurately.

This kind of fuzzy sets has been widely investigated. Peng [11] introduced several operators on a pythagorean fuzzy set and discussed its properties. Yager [15] introduced the concept of a pythagorean fuzzy subset as a generalization of an intuitionistic fuzzy subset. In [4], lattices which have been suggested for pythagorean fuzzy sets were characterized and then the results extended to the unit disc of the complex plane.

Moreover, it can be applied on many areas, for instance, decision making, information measures and aggregation operators. Yager used pythagorean memmbership in decision making [14]. In [10], some algorithms in decision making problems were presented. Grag [5] presented some generalised aggregation operators in order to illustrate a group decision making problem. In [6], he presented an improved score function for solving multi-criteria decision-making in the environment of pythagorean fuzzy set. In [8], hesitant pythagorean fuzzy set was investigated and applied to some methods for multiple criteria decision making. A new approach in computing the weight of decision makers is presented in [9] using properties of pythagorean fuzzy sets. Distance and similarity measures of pythagorean

DOI: https://doi.org/10.29020/nybg.ejpam.v15i1.4170

Email address: aamuhaimeed@taibahu.edu.sa (A. Alhumaimeed)

fuzzy set was presented and applied to decision making, see [17]. For more application of this concept in decision making, see [13] and [12].

The study of pythagorean fuzzy sets is a step in order to study q-rung orthopair fuzzy sets as a generalization of pythagorean fuzzy sets see [7], [1] and [2].

In this paper, we introduce the notion of pythagorean submodule. In addition, we present the concept pythagorean small submodule and investigate some results regarding this concept. Moreover, we find a relationship between small submodule and pythagorean fuzzy small submodule. We also study homomorphism between pythagorean fuzzy modules.

2. Preliminaries

Definition 1. *A pythagorean fuzzy set (PFS) P of universe of discourse X is of the form $P = \{(a, \eta_P(a), \hat{\eta}_P(a)) : a \in X\}$, where $\eta_P(a)$ and $\hat{\eta}_P(a)$ are the membership and non-membership values of a respectively in which*

$$0 \leq \eta_P(a) \leq 1, \quad 0 \leq \hat{\eta}_P(a) \leq 1$$

and

$$0 \leq \eta_P(a)^2 + \hat{\eta}_P(a)^2 \leq 1,$$

for every $a \in X$.

We prsent some basic notions regarding pythagorean fuzzy sets.

Definition 2. *Let P, S be pythagorean fuzzy sets in a fixed set X. Then*

- *P is a subset of S if for all $a \in X$, we have*

$$\eta_P^2(a) \leq \eta_S^2(a) \quad \text{and} \quad \hat{\eta}_P^2(a) \geq \hat{\eta}_S^2(a)$$

.

- $\eta^2_{P\cap S}(a) = \min\{\eta^2_P(a), \eta^2_S(a) : a \in X\}$ *and*
  $\hat{\eta}^2_{P\cap S}(a) = \max\{\hat{\eta}^2_P(a), \hat{\eta}^2_S(a)\}$.
- $\eta^2_{P\cup S}(a) = \max\{\eta^2_P(a), \eta^2_S(a)\}$ *and*
  $\hat{\eta}^2_{P\cup S}(a) = \min\{\hat{\eta}^2_P(a), \hat{\eta}^2_S(a)\}$.
- $\eta^2_{P+S}(a) = \eta^2_P(a) + \eta^2_S(a) - \eta^2_P(a)\eta^2_S(a)$ *and* $\hat{\eta}^2_{P+S}(a) = \hat{\eta}^2_P(a)\hat{\eta}^2_S(a)$.

Now, we are able to introduce the definition of a pythagorean fuzzy submodule.

Definition 3. *Let M be an R-module and P a pythagorean fuzzy subset of M. Then P is called a pythagorean fuzzy submodule of M, denoted by $P \leq_{PF} M$, if the following conditions are satisfied:*

(1) $\eta^2_P(0) = 1$ *and* $\hat{\eta}^2_P(1) = 0$.

(2) $\eta_P^2(a+b) \geq \min\{\eta_P^2(a), \eta_P^2(b)\}$ *for all* $a, b \in M$ *and* $\hat{\eta}_P^2(a+b) \leq \max\{\hat{\eta}_P^2(a), \hat{\eta}_P^2(b)\}$ *for all* $a, b \in M$.

(3) $\eta_P^2(ra) \geq \eta_P^2(a)$ *and* $\hat{\eta}_P^2(ra) \leq \hat{\eta}_P^2(a)$ *for all* $a \in M$ *and* $r \in R$

Recall that for a module M, we define the pythagorean fuzzy set $\chi_M^{PF} = (\chi_M, \chi_M^c)$ in which

$$\chi_M(a) = \begin{cases} 1 & \text{if } a \in M \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_M^c(a) = \begin{cases} 0 & \text{if } a \in M \\ 1 & \text{otherwise} \end{cases}$$

Definition 4. *Let* M *be a module and* P *be a pythagorean fuzzy subset of* M*. Then*

(1) $P^\star = \eta_P^\star \cap \hat{\eta}_P^\star$, *where*

$$\eta_P^\star = \{a \in M : \eta_P(a) > 0\}$$
$$\hat{\eta}_P^\star = \{a \in M : \hat{\eta}(a) < 1\}$$

(2) $P_\star = \eta_{\star P} \cap \hat{\eta}_{\star P}$, *where*

$$\eta_{\star P} = \{a \in M : \eta_P(a) = 1\}$$
$$\hat{\eta}_{\star P} = \{a \in M : \hat{\eta}(a) = 0\}$$

3. Pythagorean Fuzzy Small Submodule

Recall that a submodule N of a module M is called a small submodule of M, denoted by $N \ll M$, if $N+S \neq M$ for every proper submodule S of M. Clearly, the zero submodule is a small submodule of any module M. Moreover, a small submodule of a module M should be a proper submodule. Now, we present some well-known properties regarding the concept of small submodules.

Theorem 1. *[3] Suppose that* M *is a module and* S, T, N *are submodules of* M *such that* $S \leq T$*. Then*

(1) $S + N \ll M$ *if and only if* $S \ll M$ *and* $N \ll M$.

(2) $T \ll M$ *if and only if* $S \ll M$ *and* $\frac{T}{S} \ll \frac{M}{S}$.

(3) If $S \ll T$*, then* $S \ll M$.

Now, we are ready to introduce the main concept in this paper.

Consider a module M. Then a PFS, $P = (\eta_P, \hat{\eta}_P)$, is called a pythagorean fuzzy small submodule of M, denoted by $P \ll_{PF} M$, if $P + S \neq \chi_M^{PF}$ for any PSF $S \neq \chi_M^{PF}$. That is whenever $P + S = \chi_M^{PF}$, then $S = \chi_M^{PF}$.

Theorem 2. *Let M be a module and P be a submodule of M. Then $P \ll M$ if $\chi_P^{PF} \ll_{PF} M$.*

Proof. Suppose that $\chi_P^{PF} \ll_{PF} M$ and $P + S = M$ for some proper submodule S of M. Then for any $m \in M$ there exist $a \in P$ and $b \in S$ such that $a + b = m$. We obtain

$$\begin{aligned}\eta^2_{\chi_P^{PF}+\chi_S^{PF}}(m) &= \chi_P^2(m) + \chi_S^2(m) - \chi_P^2(m)\chi_S^2(m)\\ &\geq \min\{\chi_P^2(a) + \chi_S^2(a) - \chi_P^2(a)\chi_S^2(a), \chi_P^2(b) + \chi_S^2(b) - \chi_P^2(b)\chi_S^2(b)\}\\ &= 1\end{aligned}$$

This means that $\eta^2_{\chi_P^{PF}+\chi_S^{PF}} = \chi_M^2$. Similarly,

$$\begin{aligned}\hat{\eta}^2_{\chi_P^{PF}+\chi_S^{PF}}(m) &= \chi_P^{c^2}(m)\chi_S^{c^2}(m)\\ &\leq \max\{\chi_P^{c^2}(a)\chi_S^{c^2}(a), \chi_P^{c^2}(b)\chi_S^{c^2}(b)\}\\ &= 0\end{aligned}$$

This means that $\hat{\eta}^2_{\chi_P^{PF}+\chi_S^{PF}} = \chi_M^{c^2}$. Thus $\chi_P^{PF} + \chi_S^{PF} = \chi_M^{PF}$, but this contradicts the facts that $\chi_P^{PF} \ll_{PF} M$ and $\chi_S^{PF} \neq \chi_M^{PF}$ as S is a proper submodule of M. Therefore, P is a small submodule of M.

Theorem 3. *Let M be a module and P be a pythagorean fuzzy submodule of M. If $P \ll_{PF} M$, then $P_\star \ll M$.*

Proof. Assume that $P \ll_{PF} M$. In order to see that $P_\star \ll M$, suppose that $P_\star + S = M$ for a submodule S of M. We aim to prove that $P + \chi_S^{PF} = \chi_M^{PF}$. Let $m \in M$. Then $m = a + b$, for some $a \in P_\star$ and $b \in S$. Then

$$\begin{aligned}\eta_{P+\chi_S^{PF}}(m) =&\eta_P^2(m) + \chi_S^2(m) - \eta_P^2(m)\chi_S^2(m)\\ \geq&\min\{\eta_P^2(a) + \chi_S^2(a) - \eta_P^2(a)\chi_S^2(a), \eta_P^2(b) + \chi_S^2(b) - \eta_P^2(b)\chi_S^2(b)\}\\ =&1\end{aligned}$$

Moreover,

$$\begin{aligned}\hat{\eta}_{P+\chi_S^{PF}}(m) =&\hat{\eta}_P^2(m)\chi_S^{c^2}(m)\\ \leq&\max\{\hat{\eta}_P^2(a)\chi_S^{c^2}(a), \hat{\eta}_P^2(b)\chi_S^{c^2}(b)\}\\ =&0\end{aligned}$$

Thus $P + \chi_S^{PF} = \chi_M^{PF}$. By hypothesis, $\chi_S^{PF} = \chi_M^{PF}$. Therefore, $S = M$.

Example 1. *Consider the $\mathbb{Z}$-module $\mathbb{Z}_{10}$ and the submodule $S = \langle \bar{5} \rangle$. Let P be a pythagorean fuzzy submodule of $\mathbb{Z}_{10}$ defined as follows*

$$\eta_P(m) = \begin{cases} 1 & \text{if } m \in S \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

and

$$\hat{\eta}_P(m) = \begin{cases} 0 & \text{if } m \in S \\ \frac{1}{6} & \text{otherwise} \end{cases}$$

It is clear that $P_\star$ is not a small submodule of $\mathbb{Z}_{10}$ as $P_\star + \langle \bar{2} \rangle = \mathbb{Z}_{10}$. Thus P is not a pythagorean fuzzy small submodule of $\mathbb{Z}_{10}$.

Corollary 1. *Let P, S be two pythagorean fuzzy submodules of a module M in which $P \subseteq S$. Then $P \ll_{PF} S$ if and only if $P_\star \ll S^\star$.*

Proof. Clear.

Theorem 4. *Let M be a module, S be a submodule of M and P is a pythagorean fuzzy submodule of M in which $P \subseteq \chi_S^{PF}$. If $P|_S$ is a pythagorean fuzzy small submodule of S, then P is pythagorean fuzzy small submodule of M.*

Proof. Assume that T is a pythagorean fuzzy submodule of M such that $P + T = \chi_M^{PF}$. In order to see that $P|_S + (T|_S \cap \chi_S^{PF})$, let $a \in S$. Then we obtain

$$\begin{aligned}
\eta^2_{P|_S+(T|_S\cap\chi_S^{PF})}(a) =& \eta^2_{P|_S}(a) + \eta^2_{T|_S\cap\chi_S^{PF}}(a) - \eta^2_{P|_S}(a)\eta^2_{T|_S\cap\chi_S^{PF}}(a) \\
=& \eta^2_{P|_S}(a) + \min\{\eta^2_{T|_S}(a), \chi^2_S(a)\} - \eta^2_{P|_S}(a)\min\{\eta^2_{T|_S}(a), \chi^2_S(a)\} \\
=& \min\{\eta^2_P(a), \chi^2_S(a)\} + \min\{\eta^2_T(a), \chi^2_S(a)\} \\
& - \min\{\eta^2_P(a), \chi^2_S(a)\}\min\{\eta^2_T(a), \chi^2_S(a)\} \\
=& \eta^2_P(a) + \eta^2_T(a) - \eta^2_P(a)\eta^2_T(a) \\
=& \eta^2_{P+T}(a) \\
=& \chi^2_M(a) \\
=& 1 \\
=& \chi^2_S(a)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\eta}^2_{P|_S+(T|_S\cap\chi_S^{PF})}(a) =& \hat{\eta}^2_{P|_S}(a)\hat{\eta}^2_{T|_S\cap\chi_S^{PF}}(a) \\
=& \hat{\eta}^2_{P|_S}(a)\max\{\hat{\eta}^2_{T|_S}(a), \chi^{c^2}_{S^{PF}}(a)\} \\
=& \max\{\hat{\eta}^2_P(a), \chi^{c^2}_{S^{PF}}(a)\}\max\{\hat{\eta}^2_T(a), \chi^{c^2}_{S^{PF}}(a)\}
\end{aligned}$$

$$
\begin{aligned}
&=\hat{\eta}_P^2(a)\hat{\eta}_T^2(a)\\
&=\hat{\eta}_{P+T}^2(a)\\
&=\chi_M^{c^2}(a)\\
&=0\\
&=\chi_S^{c^2}(a)
\end{aligned}
$$

This implies that $P|_S+(T|_S\cap\chi_S^{PF})=\chi_S^{PF}$. By hypothesis, we conclude that $T|_S\cap\chi_S^{PF}=\chi_S^{PF}$. Thus $\chi_S^{PF}\subseteq T|_S$. Then $\chi_M^{PF}=P+T\subseteq T\subseteq\chi_M^{PF}$. Therefore, $T=\chi_M^{PF}$ and P is pythagorean fuzzy small submodule of M.

As a consequence of the above theorem, we have:

Corollary 2. *Let M be a module and, P and S are pythagorean fuzzy submodules of M in which $P\subseteq S$. If P is pythagorean fuzzy small submodule of S, then P is pythagorean fuzzy small submodule of M.*

Proof. Clear.

Remark 1. *The converse of theorem 4 need not be true in general. That is if M is a module, S be a submodule of M and P is a pythagorean fuzzy small submodule of M in which $P\subseteq\chi_S^{PF}$, then it is not true in general that $P|_S$ is a pythagorean fuzzy small submodule of S. For example take $P|_S=S$.*

Proposition 1. *Let M be a module and P,S,T be pythagorean fuzzy submodules of M. Then:*

$$(P\cap S)+(P\cap T)\subseteq P\cap(S+T).$$

Proof. Let $m\in M$. Then

$$
\begin{aligned}
\eta^2_{(P\cap S)+(P\cap T)}(m) &=\eta^2_{P\cap S}(m)+\eta^2_{P\cap T}(m)-\eta^2_{P\cap S}(m)\eta^2_{P\cap T}(m)\\
&=\min\{\eta_P^2(m),\eta_S^2(m)\}+\min\{\eta_P^2(m),\eta_T^2(m)\}\\
&\quad-\min\{\eta_P^2(m),\eta_S^2(m)\}\min\{\eta_P^2(m),\eta_T^2(m)\}\\
&\leq\min\{\eta_P^2(m),\eta_S^2(m)+\eta_T^2(m)-\eta_S^2(m)\eta_T^2(m)\}\\
&=\min\{\eta_P^2(m),\eta^2_{S+T}(m)\\
&=\eta^2_{P\cap(S+T)}(m)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\hat{\eta}^2_{(P\cap S)+(P\cap T)}(m) &=\hat{\eta}^2_{P\cap S}(m)\hat{\eta}^2_{P\cap T}(m)\\
&=\max\{\hat{\eta}_P^2(m),\hat{\eta}_S^2(m)\}\max\{\hat{\eta}_P^2(m),\hat{\eta}_T^2(m)\}\\
&\geq\max\{\hat{\eta}_P^2(m),\hat{\eta}_S^2(m)\hat{\eta}_T^2(m)\}\\
&=\hat{\eta}^2_{P\cap(S+T)}(m)
\end{aligned}
$$

Proposition 2. *Let M be a module and, P and S are pythagorean fuzzy submodules of M in which $\chi_M^{PF} = P \bigoplus_{PF} S$. Then $M = P^\star \bigoplus S^\star = P_\star \bigoplus S_\star$.*

Proof. Let $m \in M$. Then

$$\begin{aligned} 1 =& \chi_M^2(m) \\ =& \eta_{P+S}^2(m) \\ =& \eta_P^2(m) + \eta_S^2(m) - \eta_P^2(m)\eta_S^2(m) \\ =& \eta_P^2(m)(1 - \eta_S^2(m)) + \eta_S^2(m) \end{aligned}$$

This implies that $\eta_P^2(m) = 1$ or $\eta_S^2(m) = 1$, so that $\hat{\eta}_P^2(m) = 0$ or $\hat{\eta}_S^2(m) = 0$. Hence $m \in P_\star$ or $m \in S_\star$, so that $M = P_\star + S_\star$. Hence $M = P^\star + S^\star$. We aim now to show that the intersection $P^\star \cap S^\star = 0$. Assume that $m \in P^\star \cap S^\star$. Then $\eta_P^2(m), \eta_S^2(m) > 0$. Since $\chi_M^{PF} = P \bigoplus_{PF} S$, we obtain

$$\begin{aligned} 0 <& \min\{\eta_P^2(m), \eta_S^2(m)\} \\ =& \chi_0^2(m), \end{aligned}$$

which means that $m = 0$ and hence, $P_\star \cap S_\star \subseteq P^\star \cap S^\star = 0$. Therefore, the result holds.

Now, we are able to show that the convers of corollary 2 is true if S is a pythagorean fuzzy direct summand of M as follows:

Theorem 5. *Let M be a module and, P and S are pythagorean fuzzy submodules of M in which $P \subseteq S$ and S is a pythagorean fuzzy direct summand of M. Then P is pythagorean fuzzy small submodule of S if and only if P is pythagorean fuzzy small submodule of M.*

Proof. Assume that P is a pythagorean fuzzy small submodule of M. Applying theorem 3, $P_\star$ is a small submodule of M. That S is a pythagorean fuzzy direct summand of M and $P_\star \subseteq S^\star$, implies that $P_\star$ is a small submodule of $S^\star$. Applying corollary 2, the result hold.

Theorem 6. *Let M be a module and, P and S be pythagorean fuzzy submodules of M such that $P \cap S = \chi_0^{PF}$. Then*

(1) $(P \bigoplus_{PF} S)^\star = P^\star \bigoplus S^\star$.

(2) $(P \bigoplus_{PF} S)_\star = P_\star \bigoplus S_\star$.

Proof.

(1) Since $P \cap S = \chi_0^{PF}$, we need to prove that $(P + S)^\star = P^\star + S^\star$. Suppose that $m \in (P + S)^\star$. By definition, $\eta_{P+S}^2(m) > 0$. This implies that

$$\begin{aligned} 0 <& \eta_P^2(m) + \eta_S^2(m) - \eta_P^2(m)\eta_S^2(m) \\ =& \eta_P^2(m)(1 - \eta_S^2(m)) + \eta_S^2(m) \end{aligned}$$

which means that $\eta_P^2(m) \neq 0$ or $\eta_S^2(m) \neq 0$. Moreover,

$$\begin{aligned}1 &> \hat{\eta}_{P+S}^2(m)\\ &= \hat{\eta}_P^2(m)\hat{\eta}_S^2(m)\end{aligned}$$

which implies that $\hat{\eta}_P^2(m) < 1$ or $\hat{\eta}_S^2(m) < 1$. Thus $m \in P^\star$ or $m \in S^\star$, so that $m \in P^\star + S^\star$. and $(P+S)^\star \subseteq P^\star + S^\star$. Now, suppose that $m = a_1 + b_1 \in P^\star + S^\star$, where $a_1 \in P^\star$ and $b_1 \in S^\star$. By definition, $\eta_P^2(a_1), \eta_S^2(b_1) > 0$. Thus

$$\begin{aligned}0 <& \min\{\eta_P^2(a_1)(1-\eta_S^2(a_1)) + \eta_S^2(a_1), \eta_P^2(b_1)(1-\eta_S^2(b_1)) + \eta_S^2(b_1)\}\\ =& \min\{\eta_P^2(a_1) + \eta_S^2(a_1) - \eta_P^2(a_1)\eta_S^2(a_1), \eta_P^2(b_1) + \eta_S^2(b_1) - \eta_P^2(b_1)\eta_S^2(b_1)\}\\ \leq& \eta_P^2(m) + \eta_S^2(m) - \eta_P^2(m)\eta_S^2(m)\\ =& \eta_{P+S}^2(m)\end{aligned}$$

Moreover, $\hat{\eta}_P^2(a_1), \hat{\eta}_S^2(b_1) < 1$ which implies that

$$\begin{aligned}1 >& \max\{\hat{\eta}_P^2(a_1)\hat{\eta}_S^2(a_1), \hat{\eta}_P^2(b_1)\hat{\eta}_S^2(b_1)\}\\ \geq& \hat{\eta}_P^2(m)\hat{\eta}_S^2(m)\\ =& \hat{\eta}_{P+S}^2(m)\end{aligned}$$

Thus $m \in (P+S)^\star$. Then $P^\star + S^\star \subseteq (P+S)^\star$ and therefore, the equality holds.

(2) Since $P \cap S = \chi_0^{PF}$, we need to prove that $(P+S)_\star = P_\star + S_\star$. Suppose that $m \in (P+S)_\star$. By definition, $\eta_{P+S}^2(m) = 1$. This implies that

$$\begin{aligned}1 &= \eta_P^2(m) + \eta_S^2(m) - \eta_P^2(m)\eta_S^2(m)\\ &= \eta_P^2(m)(1-\eta_S^2(m)) + \eta_S^2(m)\end{aligned}$$

which means that $\eta_P^2(m) = 1$ or $\eta_S^2(m) = 1$. Moreover,

$$\begin{aligned}0 &= \hat{\eta}_{P+S}^2(m)\\ &= \hat{\eta}_P^2(m)\hat{\eta}_S^2(m)\end{aligned}$$

which implies that $\hat{\eta}_P^2(m) = 0$ or $\hat{\eta}_S^2(m) = 0$. Thus $m \in P_\star$ or $m \in S_\star$, so that $m \in P_\star + S_\star$ and $(P+S)_\star \subseteq P_\star + S_\star$. Now, suppose that $m = a_1 + b_1 \in P_\star + S_\star$, where $a_1 \in P_\star$ and $b_1 \in S_\star$. By definition, $\eta_P^2(a_1), \eta_S^2(b_1) = 1$. Thus

$$\begin{aligned}1 =& \min\{\eta_P^2(a_1) + \eta_S^2(a_1) - \eta_P^2(a_1)\eta_S^2(a_1), \eta_P^2(b_1) + \eta_S^2(b_1) - \eta_P^2(b_1)\eta_S^2(b_1)\}\\ \leq& \eta_P^2(m) + \eta_S^2(m) - \eta_P^2(m)\eta_S^2(m)\\ =& \eta_{P+S}^2(m)\end{aligned}$$

Moreover, $\hat{\eta}_P^2(a_1), \hat{\eta}_S^2(b_1) = 0$ which implies that

$$\begin{aligned}0 =& \max\{\hat{\eta}_P^2(a_1)\hat{\eta}_S^2(a_1), \hat{\eta}_P^2(b_1)\hat{\eta}_S^2(b_1)\}\\ \geq& \hat{\eta}_P^2(m)\hat{\eta}_S^2(m)\\ =& \hat{\eta}_{P+S}^2(m)\end{aligned}$$

Thus $m \in (P+S)_\star$. Then $P_\star + S_\star \subseteq (P+S)_\star$ and therefore, the equality holds.

4. Homomorphism

Let P, S be two R-modules, $L \leq_{PF} P$ and $N \leq_{PF} S$. Consider an R-homomorphism

$$\psi : P \longrightarrow S$$

For $s \in S$, we define:

$$\eta_{\psi(L)}(s) = \begin{cases} \max\{\eta_L(p) : s = \psi(p)\} & \text{if } s \in \mathrm{Im}(\psi) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\eta}_{\psi(L)}(s) = \begin{cases} \min\{\eta_L(p) : s = \psi(p)\} & \text{if } s \in \mathrm{Im}(\psi) \\ 1 & \text{otherwise} \end{cases}$$

Now, we are ready to prove the following:

Theorem 7. *Let $\psi : P \longrightarrow S$ be a monomorphism of modules. If T is a pythagorean fuzzy small submodule of P, then $\psi(T)$ is a pythagorean fuzzy small submodule of S.*

Proof. Suppose that $\psi(T) + L = \chi_S^{PF}$. We aim to prove that $L = \chi_S^{PF}$. Let $s \in S$, then

$$\begin{aligned} 1 =& \eta^2_{\psi(T)+L}(s) \\ =& \eta^2_{\psi(T)}(s) + \eta^2_L(s) - \eta^2_{\psi(T)}(s)\eta^2_L(s) \end{aligned}$$

In the case that $s \notin \mathrm{Im}(\psi)$, we obtain

$$\begin{aligned} 1 =& \eta^2_{\psi(T)}(s) + \eta^2_L(s) - \eta^2_{\psi(T)}(s)\eta^2_L(s) \\ =& \eta^2_L(s) \end{aligned}$$

and so $1 = \eta^2_L(s)$ and $\hat{\eta}^2_L(s) = 0$. If $s \in \mathrm{Im}(\psi)$, we have

$$\begin{aligned} 1 =& \eta^2_{\psi(T)}(s) + \eta^2_L(s) - \eta^2_{\psi(T)}(s)\eta^2_L(s) \\ =& \max\{\eta^2_T(p) : \psi(p) = s\} + \eta^2_L(s) - \max\{\eta^2_T(p) : \psi(p) = s\}\eta^2_L(s) \\ =& \eta^2_T(p) + \eta^2_L(s) - \eta^2_T(p)\eta^2_L(s), \qquad \text{for some } p \text{ in which } \psi(p) = s \\ =& \eta^2_T(p)(1 - \eta^2_L(s)) + \eta^2_L(s) \end{aligned}$$

If $\eta^2_T(p) = 1$, then $T = \chi_P$ and this is a contradiction with the fact that T is a pythagorean fuzzy small submodule of P. Thus $\eta^2_L(s) = 1$ and $L = \chi_S$. Moreover,

$$0 = \hat{\eta}^2_{\psi(T)+L}(s)$$

$$=\hat{\eta}^2_{\psi(T)}(s)\hat{\eta}^2_L(s)$$
$$=\hat{\eta}^2_T(p)\hat{\eta}^2_L(s) \qquad \text{for some } p \text{ in which } \psi(p) = s$$

Note that ψ is one to one and so p is unique. By hypothesis $\hat{\eta}^2_T(p) \neq 0$, so that $\hat{\eta}^2_L(s) = 0$. Hence $L = \chi_S^{PF}$.

Remark 2. *(1) If ψ is not one to one, then the above theorem need not be true. For instance, take S the zero module and ψ the zero homomorphism.*

(2) The converse of the above theorem need not be true. That is if $\psi : P \longrightarrow S$ is a monomorphism of modules, T is a pythagorean fuzzy submodule of P and $\psi(T)$ is a pythagorean fuzzy small submodule of S, then it is not true in general that T is a pythagorean fuzzy small submodule of P. For example, Let P be a pythagorean fuzzy small submodule of S and consider the inclusion $P \hookrightarrow S$. Then $\psi(P) = P$ is a pythagorean fuzzy small submodule of S but P is not a pythagorean fuzzy small submodule of P.

5. Conclusion and Future Directions

In this paper, we introduce the notion of pythagorean submodule. In addition, we present the concept pythagorean small submodule and investigate some results regarding this concept. Moreover, we find a relationship between small submodule and pythagorean fuzzy small submodule. We also study homomorphism between pythagorean fuzzy modules.

This work can be extended and generalised in the environment of q-rung orthopair fuzzy sets. It can be applied in order to solve multi-criteria decision making problems.

References

[1] Muhammad Irfan Ali. Another view on q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 33(11):2139–2153, 2018.

[2] Zeeshan Ali and Tahir Mahmood. Maclaurin symmetric mean operators and their applications in the environment of complex q-rung orthopair fuzzy sets. *Computational and Applied Mathematics*, 39:1–27, 2020.

[3] Frank W Anderson and Kent R Fuller. *Rings and categories of modules*, volume 13. Springer Science & Business Media, 2012.

[4] Scott Dick, Ronald R Yager, and Omolbanin Yazdanbakhsh. On pythagorean and complex fuzzy set operations. *IEEE Transactions on Fuzzy Systems*, 24(5):1009–1021, 2015.

[5] Harish Garg. Generalised pythagorean fuzzy geometric interactive aggregation operators using einstein operations and their application to decision making. *Journal of Experimental & Theoretical Artificial Intelligence*, 30(6):763–794, 2018.

[6] Harish Garg. A linear programming method based on an improved score function for interval-valued pythagorean fuzzy numbers and its application to decision-making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 26(01):67–80, 2018.

[7] Muhammad Jabir Khan, Poom Kumam, and Meshal Shutaywi. Knowledge measure for the q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 36(2):628–655, 2021.

[8] Decui Liang and Zeshui Xu. The new extension of topsis method for multiple criteria decision making with hesitant pythagorean fuzzy sets. *Applied Soft Computing*, 60:167–179, 2017.

[9] Vahid Mohagheghi, S Meysam Mousavi, and Behnam Vahdani. Enhancing decision-making flexibility by introducing a new last aggregation evaluating approach based on multi-criteria group decision making and pythagorean fuzzy sets. *Applied Soft Computing*, 61:527–535, 2017.

[10] Xindong Peng and Ganeshsree Selvachandran. Pythagorean fuzzy set: state of the art and future directions. *Artificial Intelligence Review*, 52(3):1873–1927, 2019.

[11] Xindong Peng and Yong Yang. Some results for pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 30(11):1133–1160, 2015.

[12] Luis Pérez-Domínguez, Luis Alberto Rodríguez-Picón, Alejandro Alvarado-Iniesta, David Luviano Cruz, and Zeshui Xu. Moora under pythagorean fuzzy set for multiple criteria decision making. *Complexity*, 2018, 2018.

[13] K Rahman, S Abdullah, M Shakeel, M Sajjad Ali Khan, and Murad Ullah. Interval-valued pythagorean fuzzy geometric aggregation operators and their application to group decision making problem. *Cogent Mathematics*, 4(1):1338638, 2017.

[14] Ronald R Yager. Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, 22(4):958–965, 2013.

[15] Ronald R Yager. Properties and applications of pythagorean fuzzy sets. In *Imprecision and Uncertainty in Information Representation and Processing*, pages 119–136. Springer, 2016.

[16] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[17] Wenyi Zeng, Deqing Li, and Qian Yin. Distance and similarity measures of pythagorean fuzzy sets and their applications to multiple criteria group decision making. *International Journal of Intelligent Systems*, 33(11):2236–2254, 2018.