



Extended Apelblat integrals for fractional calculus

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Abstract. A quadruple integral involving the logarithmic, exponential, polynomial and Gamma functions is derived in terms of the Hurwitz-Lerch zeta function. Special cases of this integral are evaluated in terms of special functions and fundamental constants. Almost all Hurwitz-Lerch zeta functions have an asymmetrical zero-distribution. The majority of the results in this work are new.

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1. Significance Statement

Apelblat [1] evaluated a triple integral involving the Volterra function in terms of the Gamma function which is used in Fractional Calculus. In this work the authors expand on Apelblat's work by deriving a quadruple integral involving the Volterra function and express it in terms of the Hurwitz-Lerch zeta function. Our hope is researchers will find the results in this work useful where applicable.

2. Introduction

In this paper we derive the quadruple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{y^{m-1} z^{-m} e^{-bz-px} x^{\alpha+u+y} \log^k \left(\frac{ay}{z} \right)}{\Gamma(u+y+\alpha+1)} dx dy du dz \quad (1)$$

where the parameters k, a, p, b, α and m are general complex numbers and the Volterra function is given in equation (1.2.1) in [1]. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations

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follow the method used by us in [5]. This method involves using a form of the generalized Cauchy’s integral formula given by

$$\frac{y^k}{\Gamma(k + 1)} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, u and z , then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of x, y, u and z and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [5]. The variable of integration in the contour integral is $t = w + m$. The cut and contour are in the first quadrant of the complex t -plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy’s integral formula we form the quadruple integral by replacing y by $\log\left(\frac{ay}{z}\right)$ and multiplying by $\frac{y^{m-1}z^{-m}e^{-bz-px}x^{\alpha+u+y}}{\Gamma(u+y+\alpha+1)}$ then taking the definite integral with respect to $x \in [0, \infty), y \in [0, \infty), u \in [0, \infty)$ and $z \in [0, \infty)$ to obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{y^{m-1}z^{-m}e^{-bz-px}x^{\alpha+u+y} \log^k\left(\frac{ay}{z}\right)}{\Gamma(k+1)\Gamma(u+y+\alpha+1)} dx dy du dz \\ &= \frac{1}{2\pi i} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_C \frac{a^w w^{-k-1} y^{m+w-1} z^{-m-w} e^{-bz-px} x^{\alpha+u+y}}{\Gamma(u+y+\alpha+1)} dw dx dy du dz \\ &= \frac{1}{2\pi i} \int_C \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{a^w w^{-k-1} y^{m+w-1} z^{-m-w} e^{-bz-px} x^{\alpha+u+y}}{\Gamma(u+y+\alpha+1)} dx dy du dz dw \\ &= \frac{1}{2\pi i} \int_C \pi a^w w^{-k-1} p^{-\alpha-1} b^{m+w-1} \csc(\pi(m+w)) \log^{-m-w-1}(p) dw \end{aligned} \tag{3}$$

from equation (9.67.9) in [1] and equation (3.326.2) in [3] where $Re(\alpha) > -1, Re(p) > 1, Re(w+m) > 0$ and using the reflection Formula (8.334.3) in [3] for the Gamma function. We are able to switch the order of integration over t, x, y, u and z using Fubini’s theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty)$.

4. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [2] has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Infinite sum of the Contour Integral

Using Equation (2) and replacing y by $\log(a) + \log(b) - \log(\log(p)) + i\pi(2y + 1)$ then multiplying both sides by $-2i\pi b^{m-1} p^{-\alpha-1} \log^{-m-1}(p)$ taking the infinite sum over $y \in [0, \infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned} & -\frac{1}{\Gamma(k+1)} (2i\pi)^{k+1} e^{i\pi m} b^{m-1} p^{-\alpha-1} \log^{-m-1}(p) \\ & \Phi\left(e^{2im\pi}, -k, \frac{-i \log(a) - i \log(b) + i \log(\log(p)) + \pi}{2\pi}\right) \\ & = -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C 2i\pi a^w w^{-k-1} p^{-\alpha-1} b^{m+w-1} e^{i\pi(2y+1)(m+w)} \log^{-m-w-1}(p) dw \tag{6} \\ & = -\frac{1}{2\pi i} \int_C \sum_{y=0}^{\infty} 2i\pi a^w w^{-k-1} p^{-\alpha-1} b^{m+w-1} e^{i\pi(2y+1)(m+w)} \log^{-m-w-1}(p) dw \\ & = \frac{1}{2\pi i} \int_C \pi a^w w^{-k-1} p^{-\alpha-1} b^{m+w-1} \csc(\pi(m+w)) \log^{-m-w-1}(p) dw \end{aligned}$$

from equation (1.232.2) in [3] where $Im(w + m) > 0$ in order for the sum to converge.

5. Definite Integral in terms of the Hurwitz-Lerch zeta function

Theorem 1. For all $k, a \in \mathbb{C}, Re(p) > 1, Re(b) > 0, Re(\alpha) > -1, 1/2 < Re(m) > 1$,

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{y^{m-1} z^{-m} e^{-bz-px} x^{\alpha+u+y} \log^k\left(\frac{ay}{z}\right)}{\Gamma(u+y+\alpha+1)} dx dy dudz \\ & = (2i\pi)^{k+1} e^{i\pi m} (-b^{m-1}) p^{-\alpha-1} \log^{-m-1}(p) \tag{7} \\ & \Phi\left(e^{2im\pi}, -k, \frac{-i \log(a) - i \log(b) + i \log(\log(p)) + \pi}{2\pi}\right) \end{aligned}$$

Proof. Observe the right-hand side of equation (3) is equal to the right-hand side of equation (6) so we may equate the left-hand sides to yield the stated result.

6. Special Cases

In this section we will evaluate equation (7) for various parameter values in terms of the Riemann zeta function $\zeta(s)$, equation (25.2.1) in [2], Apéry’s constant $\zeta(3)$, equation (25.6.9) in [2], $\log(2)$ and π .

Example 1. *The degenerate case.*

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{y^{m-1} z^{-m} e^{-bz-px} x^{\alpha+u+y}}{\Gamma(u+y+\alpha+1)} dx dy dudz = \pi b^{m-1} \csc(\pi m) p^{-\alpha-1} \log^{-m-1}(p) \tag{8}$$

Proof. Use equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [4].

Example 2.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-ex-z} x^{u+y+1} \log^k\left(\frac{-y}{z}\right)}{\sqrt{y}\sqrt{z}\Gamma(u+y+2)} dx dy dudz = -\frac{i^{k+2} (1-2^{k+1}) (2\pi)^{k+1} \zeta(-k)}{e^2} \tag{9}$$

Proof. Use equation (7) and set $a = -1, p = e, b = 1, m = 1/2, \alpha = 1$ and simplify using entry (2) in Table below (64:7) and entry (4) in table below (64:12:7) in [4].

Example 3.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-ex-z} x^{u+y+1}}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2\left(\frac{y}{z}\right) + \pi^2)} dx dy dudz = \frac{\log(2)}{e^2 \pi} \tag{10}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-ex-z} x^{u+y+1} \log\left(\frac{y}{z}\right)}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2\left(\frac{y}{z}\right) + \pi^2)} dx dy dudz = 0 \tag{11}$$

Proof. Use equation (9) and apply l’Hopital’s rule to the right-hand side as $k \rightarrow -1$ simplify using equation (25.6.11) in [2] and rationalize the denominator and equate real and imaginary parts.

Example 4.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-ex-z} x^{u+y+1} (\pi^2 - 3 \log^2(\frac{y}{z}))}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2(\frac{y}{z}) + \pi^2)^3} dx dy du dz = \frac{3\zeta(3)}{16e^2\pi^3} \quad (12)$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-ex-z} x^{u+y+1} \log(\frac{y}{z}) (\log^2(\frac{y}{z}) - 3\pi^2)}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2(\frac{y}{z}) + \pi^2)^3} dx dy du dz = 0 \quad (13)$$

Proof. Use equation (9) and set $k = -3$ using entry (2) in Table below (64:7) and rationalize the denominator and simplify into real and imaginary parts.

Example 5.

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-px} p^{-z} x^{u+y+1} \log^k(-\frac{y}{z})}{\sqrt{y}\sqrt{z}\Gamma(u+y+2)} dx dy du dz \\ = -\frac{i^k (2^{k+1} - 1) (2\pi)^{k+1} \zeta(-k)}{p^2 \log^2(p)} \end{aligned} \quad (14)$$

Proof. Use equation (7) and set $a = -1, b = \log(p), m = 1/2, \alpha = 1$ and simplify using entry (2) in Table below (64:7) and entry (4) in Table below (64:12:7) in [4].

Example 6.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-2x} 2^{-z} x^{u+y+1}}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2(\frac{y}{z}) + \pi^2)} dx dy du dz = \frac{1}{4\pi \log(2)} \quad (15)$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-2x} 2^{-z} x^{u+y+1} \log(\frac{y}{z})}{\sqrt{y}\sqrt{z}\Gamma(u+y+2) (\log^2(\frac{y}{z}) + \pi^2)} dx dy du dz = 0 \quad (16)$$

Proof. Use equation (14) and apply l'Hopital's rule to the right-hand side as $k \rightarrow -1$ and set $p = 2$ and rationalize the denominator and simplify into real and imaginary parts.

Example 7.

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{z^{-m-n} e^{-bx-z \log(b)} (z^m y^n - y^m z^n) x^{\alpha+u+y}}{y \log(\frac{y}{z}) \Gamma(u+y+\alpha+1)} dx dy du dz \\ = b^{-\alpha-1} \log^{-m-n-2}(b) (2 \tanh^{-1}(e^{i\pi m}) \log^{m+n}(b) - 2 \tanh^{-1}(e^{i\pi n}) \log^{m+n}(b)) \end{aligned} \quad (17)$$

Proof. Use equation(7) and form a second equation by replacing $m \rightarrow n$ and taking their difference. Next set $k = -1, a = 1, b = \log(b), p = b$ and simplify using entry (3) in Table below (64:12:7) in [4].

Example 8.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-2x} 2^{-z} (\sqrt[12]{z} - \sqrt[12]{y}) x^{u+y+2}}{\sqrt[3]{y} z^{3/4} \Gamma(u+y+3) \log\left(\frac{y}{z}\right)} dx dy du dz = \frac{\log(9 - 6\sqrt{2})}{16 \log^2(2)} \quad (18)$$

Proof. Use equation (17) and set $b = 2, \alpha = 2, n = 2/3, m = 3/4$ and simplify.

Example 9.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-2x} 2^{-z} (\sqrt[4]{y} - \sqrt[4]{z}) x^{u+y+1}}{\sqrt{y} z^{3/4} \Gamma(u+y+2) \log\left(\frac{y}{z}\right)} dx dy du dz = \frac{\coth^{-1}(\sqrt{2})}{4 \log^2(2)} \quad (19)$$

Proof. Use equation (17) and set $b = 2, \alpha = 1, n = 1/2, m = 3/4$ and simplify.

7. Discussion

In this paper, we have presented a novel method for deriving a quadruple integral involving the Volterra function along with some interesting definite integrals using our contour integration method. The results presented were numerically verified for both real and imaginary complex values of the parameters in the integrals using Mathematica by Wolfram.

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