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e^*-Essential submodule

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Abstract. The purpose of this paper is to introduce a new concept in a module M over a ring R, this concept is called e^*-essential submodule, which is a generalization of an essential submodule. We will introduce some examples and properties about this concept such that, what is the inverse image of e^*-essential submodule, the intersection of e^*-essential submodules and direct sum of e^*-essential submodules. We will show the relationship between e^*-essential submodule and Noetherian R-module. Also we will define e^*-closed submodule with some properties

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1. Introduction

Let R be a ring with identity, M be a right R-module and $E(M)$ be the injective hull of M. A submodule N of an R-module M is called a small submodule of M ($N \ll M$) if for any submodule A of M such that $M = N + A$, then $A = M$ [5]. Leonard defines a module M to be small if it is a small submodule of some R-module and he shows that M is small if and only if M is small in its injective hull [1]. Recall that a submodule A of R-module B is called essential in B if every nonzero submodule of B has nonzero intersection with A [5], [3] and [4].

Oscan in [2], introduced the concept of cosingular submodule as the following: $Z^*(M) = \{m \in M | mR \ll E(M)\}$. An R-module M is called cosingular $Z^*(M) = M$.

As in [6], we will used the Oscan presented to generalize the essential submodule, to introduce the concepte e^*-essential and investigate some properties.

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2. e^*-Essential submodule

Definition 1.

- *Let M be R-module, a submodule A of M is said to be e*-essential if $A \cap B \neq 0$ for each nonzero cosingular submodule B of M. denoted by $A \leq_{e^*} B$.*
- *A right ideal B of a ring R is e*-essential in R if and only if B is e*-essential submodule of R_R.*
- *An R-homomorphism $f : A \to B$ is said e*-essential if and only if, $Im(f)$ is e*-essential submodule in B.*
- *We may deduce the following from the definition:*
  1. *$A \leq_{e^*} M$ if $A \cap K = 0$, then $K = 0$ where K is cosingular submodule in M.*
  2. *If $M \neq 0$ and $L \leq_{e^*} M$ then $L \neq 0$.*

Examples and Remarks 1.

1. *Every essential submodule is e*-essential, but the converse need not to be true in general. For example, in $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module, the only cosingular submodle of $\mathbb{Z}_6$ is $\{\overline{0}\}$. Hence every submodule K of $\mathbb{Z}_6$ is e*-essential, since $K \cap \{\overline{0}\} = 0$. Therefore, $\{\overline{0}, \overline{2}, \overline{4}\}$ is e*-essential which is not essential submodule in $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module since there is a nonzero submodule $\{\overline{0}, \overline{3}\}$ but $\{\overline{0}, \overline{2}, \overline{4}\} \cap \{\overline{0}, \overline{3}\} = 0$.*
2. *For any R-module M, we have $M \leq_{e^*} M$.*
3. *Every nonzero submodule of $\mathbb{Z}$ as $\mathbb{Z}$-module is cosingular [2]. Hence, $n\mathbb{Z} \cap m\mathbb{Z} = nm\mathbb{Z} \neq 0$ for each $n \neq 0$ and $m \neq 0$. So that every submodule of $\mathbb{Z}$ is e*-essential.*
4. *In $\mathbb{Z}_6$ as $\mathbb{Z}$-module every submodule is cosingular [2], but $\{\overline{0}, \overline{2}, \overline{4}\}$ is not e*-essential since $\{\overline{0}, \overline{2}, \overline{4}\} \cap \{\overline{0}, \overline{3}\} = 0$ where $\{\overline{0}, \overline{3}\}$ a nonzero cosingular submodule.*
5. *The image of e*-essential need not be e*-essential for example. Let $f : \mathbb{Z} \to \mathbb{Z}_2$ be a $\mathbb{Z}$-homomorphism defined by $f(x) = \begin{cases} \overline{0} & if \quad x even \\ \overline{1} & if \quad x odd \end{cases}$*
   So $f(2\mathbb{Z}) = \{\overline{0}\}$. Hence, $2\mathbb{Z}$ is e-essential in $\mathbb{Z}$ but $\{\overline{0}\}$ is not e*-essential in $\mathbb{Z}_2$, since $\{\overline{0}\} \cap \mathbb{Z}_2 = 0$ where $\mathbb{Z}_2$ is nonzero cosingular.*
6. *The quotient submodule of e*-essential submodule need not to be e*-essential, for example: $2\mathbb{Z}_{\mathbb{Z}}$ is e*-essential submodule of $\mathbb{Z}_{\mathbb{Z}}$, but $\frac{2\mathbb{Z}}{2\mathbb{Z}} = 0$ not e*-essential submodule of $\frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$.*

In the following lemma, gives a property of cosingular submodule

Lemma 1. *If K is cosingular submodule of B and $B \leq A \leq M$, then K is cosingular in A.*

Proof. Since K is cosingular submodule of B by ***Lemma 2.2*** in [2] $Z^*(K) = K \cap Z^*(B)$ and again since B is a submodule of A. So that, $Z^*(K) = K \cap (B \cap Z^*(A))$, from hypothesis $Z^*(K) = K$. Hence, $K \leq Z^*(A)$. Therefore, $Z^*(K) = K \cap Z^*(A) = K$, i.e. K is cosingular in A.

Now, we will prove some properties which e^*-essential submodule satisfied:

Proposition 1. *Let $A \leq B \leq M$, then $A \leq_{e^*} M$ if and only if $A \leq_{e^*} B \leq_{e^*} M$*

Proof. $\Rightarrow$) Let $K \neq 0$ be a cosingular submodule of B, hence $K \leq M$ since $A \leq_{e^*} M$. Therefore $A \cap K \neq 0$. Hence, $A \leq_{e^*} B$. Now, for $B \leq_{e^*} M$, let $0 \neq L$ be a consigular submodule of M. Hence, $A \cap L \neq 0$ and since $A \leq B$ so that, $B \cap L \neq 0$.
$\Leftarrow$) Let N be a nonzero cosingular submodule of M. Since $B \leq_{e^*} M$. Hence, $B \cap N \neq 0$, so that $B \cap N$ is a nonzero cosingular submodule of B (since $B \cap N \leq N$ and by ***Lemma 2.2*** in [2] $Z^*(B \cap N) = (B \cap N) \cap Z^*(N) = (B \cap N) \cap N = B \cap N$. Since $A \leq_{e^*} B$ then $A \cap B \cap N \neq 0$ and $A \cap N \neq 0$. Therefore, $A \leq_{e^*} M$.

Corollary 1. *If $A_1 \leq A_2 \leq A_3 \leq M$ and $A_1 \leq_{e^*} M$, then $A_2 \leq_{e^*} A_3$.*

Proof. Let L be a nonzero cosingular in A_3. By ***Lemma 1***, we have that L is cosingular in M and since $A_1 \leq_{e^*} M$. Thus, $A_1 \cap L \neq 0$ and since $A_1 \leq A_2$. Therefore, $A_2 \cap L \neq 0$, i.e. $A_2 \leq_{e^*} A_3$.

Proposition 2. *Let $f : M \to M'$ be R-homomorphism, if $A \leq_{e^*} M'$, then $f^{-1}(A) \leq_{e^*} M$.*

Proof. Let $A \leq_{e^*} M'$. Hence, $f^{-1}(A) \leq M$, suppose that $f^{-1}(A)$ is not e^*-essential submodule of M, i.e. there exists a nonzero cosingular submodule B of M such that $f^{-1}(A) \cap B = 0$. Since $ker(f|_B) = f^{-1}(A) \cap B = 0$. Thus, $B \cong f(B)$. Also, we have that $A \cap f(B) = 0$ since if not, i.e. there exists $0 \neq x = f(b) \in A \cap f(b)$. Hence, $0 \neq b \in f^{-1}(A) \cap B$ which is contradiction. Since B is cosingular by ***lemma 2.6*** in [2], $f(B)$ is cosingular also $A \leq_{e^*} M'$. Hence, $f(B) = 0$ which is contradiction. Therefore, $f^{-1}(A) \leq_{e^*} M$.

Proposition 3. *If $A \leq_{e^*} B \leq M$ and $A' \leq_{e^*} B' \leq M$, then $A \cap A' \leq_{e^*} B \cap B'$.*

Proof. Let K be a nonzero cosingular submodule of $B \cap B'$. By ***Lemma 1*** K be a nonzero cosingular submodule of B and B'. Since $A \leq_{e^*} B$. So that, $A \cap K \neq 0$ and since $A \cap K$ is a nonzero submodule of cosingular K. Hence, $A \cap K$ is cosingular. But, $A' \leq_{e^*} B'$. Hence, $A' \cap (A \cap K) \neq 0$. Therefore, $A \cap A' \leq_{e^*} B \cap B'$.

Corollary 2. *Let $B_j \leq_{e^*} M$ for each $j = 1, ..., n$, then $\cap_{i=1}^{n} \leq_{e^*} M$*

Proof. The prove by induction on n.

Proposition 4. *Let $M = M_1 \oplus M_2$ with $K_1 \leq M_1$ and $K_2 \leq M_2$, then $K_1 \leq_{e^*} M_1$ and $K_2 \leq_{e^*} M_2$ if and only if, $K_1 \oplus k_2 \leq_{e^*} M$*

Proof. $\Rightarrow$) *There exists an R-homomorphism* $\rho_1 : M_1 \oplus M_2 \to M_1$ *and* $\rho_2 : M_1 \oplus M_2 \to M_2$ *which define by* $\rho_1(m_1, m_2) = m_1$ *and* $\rho_2(m_1, m_2) = m_2$. *By* ***Proposition 2*** $\rho_1^{-1}(K_1) = K_1 \oplus M_2 \leq_{e^*} M_1 \oplus M_2$ *and* $\rho_2^{-1}(K_2) = M_1 \oplus K_2 \leq_{e^*} M_1 \oplus M_2$. *Hence, by* ***Proposition 3*** $K_1 \oplus M_2 \cap M_1 \oplus K_2 = K_1 \oplus k_2 \leq_{e^*} M$

$\Leftarrow$) *There exists an R-homomorphism* $J_1 : M_1 \to M_1 \oplus M_2$ *and* $J_2 : M_2 \to M_1 \oplus M_2$ *which define by* $J_1(m_1) = (m_1, 0)$ *and* $J_2(m_2) = (0, m_2)$. *By* ***Proposition 2*** $J_1^{-1}(K_1 \oplus k_2) = K_1 \leq_{e^*} M_1$ *and* $J_2^{-1}(K_1 \oplus k_2) = K_2 \leq_{e^*} M_2$.

In the following proposition we will give a characterization of e^*-essential submodule.

Proposition 5. *Let M be R-module and* $N \leq M$, *then N is* e^**-essential submodule of M if and only if* $N \cap xR \neq 0$ *for each nonzero cyclic cosingular submodule of M.*

Proof. $\Rightarrow$) *Clear.*

$\Leftarrow$) *Let N be a submodule of M and K be a nonzero cosingular submodule of M. Hence, there exists* $0 \neq x \in K$ *with* $xR \leq K$, *also* $Z^*(xR) = xR$. *So by hypothesis* $N \cap xR \neq 0$. *Hence,* $N \cap K \neq 0$. *Therefore,* $N \leq_{e^*} M$.

In the following proposition shows that, the composition of e^*-essential R-monomorphism is also e^*-essential R-monomorphism.

Proposition 6. *Let* $f : A \to B$ *and* $g : B \to C$ *are* e^**-essential R-monomorphism. Then,* $g \circ f : A \to C$ *is also* e^**-essential R-monomorphism.*

Proof. Let L be cosingular submodule of C such that $Im(g \circ f) \cap L = 0$. *Since g is monomorphism* $0 = kerg = g^{-1}(0) = g^{-1}(Im(g \circ f) \cap L)$. *Hence* $g^{-1}(Im(g \circ f)) \cap g^{-1}(L) = Im(f) \cap g^{-1}(L) = 0$. *Since* g^{-1} *is R-homomorphism and L is cosingular submodule of C. Hence* $g^{-1}(L)$ *is cosingular submodule of B and since* $Im(f) \leq_{e^*} B$. *Thus,* $g^{-1}(L) = 0$ *and* $Im(g) \cap L = 0$. *Since* $g : B \to C$ *is* e^**-essential. Therefore,* $L = 0$ *i.e.* $g \circ f$ *is* e^**-essential R-monomorphism.*

In the following proposition we will give another characterization of Noetherian R-module. Also, it is show the relationship between e^*-essential submodule and Noetherian R-module.

Proposition 7. *An R-module M is Noetherian if and only if, every* e^**-essential submodule of M is finitely generated.*

Proof. $\Rightarrow$) *Clear.*

$\Leftarrow$) *Let A be an essential submodule of M. Hence, A is* e^**-essential and by the hypothsis A is a finitely generated. Hence, every essential submodule is finitely generated by [3]. Therefore, M is Noetherian.*

3. e^*-Closed submodule

Definition 2. *A submodule A of R-module C is said to be e^*-closed submodule of C, if A has no proper e^*-essential extension inside C. denoted by $A \leq_{e^*C} C$.*

Examples and Remarks 2.

1. *For any module M. 0 and M always e^*-closed.*
2. *In $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module, $\{\bar{0}, \bar{2}, \bar{4}\}$ is not e^*-closed submodule since $\{\bar{0}, \bar{2}, \bar{4}\}$ is e^*-essential in $\mathbb{Z}_6$.*

In the following proposition shows that when the quotient submodule of e^*-essential submodule is e^*-essential:

Proposition 8. *Let M be R-module, If $B \leq_{e^*C} M$ and $B \leq K \leq_{e^*} M$ then $\frac{K}{B} \leq_{e^*} \frac{M}{B}$.*

Proof. *Let $\frac{L}{B}$ be cosingular submodule of $\frac{M}{B}$ with $\frac{K}{B} \cap \frac{L}{B} = 0$. Hence, $K \cap L = B$ since $K \leq_{e^*} M$. Thus, $K \cap L \leq_{e^*} M \cap L = L$. Hence $B \leq_{e^*} L \leq M$ but $B \leq_{e^*C} M$, $B = L$. Hence, $\frac{L}{B} = 0$. Therefore, $\frac{K}{B} \leq_{e^*} \frac{M}{B}$.*

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