



Atomic Solution of Fractional Abstract Cauchy Problem of High Order in Banach Spaces

Fatema Bekraoui¹, Mohammed Al Horani², Roshdi Khalil^{3,*}

¹ *Department of Mathematics, School of Science, The University of Jordan, Amman, Jordan*

Abstract. The Abstract Cauchy problem, which is a vector valued differential equation is an important equation in many branches of science. Authors usually study and discuss first and second order abstract Cauchy problem. In this paper, we try to find atomic solutions of the fractional abstract Cauchy problem with order 3α , where $\alpha \in (0, 1)$. It turned out that there are so many cases to consider in order to determine atomic solution.

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1. Introduction

Let X be a Banach space and $I = [0, 1]$. Let $C(I)$ be the Banach space of all real valued continuous functions on I under the sup-norm, and $C(I, X)$ be the Banach space of all continuous functions defined on I with values in X . A classical and important differential equation that appears in physics and other branches of applied sciences is the so called Abstract Cauchy problem of the first order.

One form of such equation is

$$\begin{aligned} Bu' &= Au(t) + f(t)z \\ u(0) &= x_0. \end{aligned}$$

Here u is a continuously differentiable function in $C(I, X)$ and A, B are densely defined linear operators on the codomain of u .

If $f = 0$ or $z = 0$, then the equation is homogeneous otherwise it is called non-homogeneous. Now in the non-homogeneous problem we have two cases. The first type, if u is unknown and f is given and this is called the direct problem. The second type, u

*Corresponding author.

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Email addresses: fat9180303@ju.edu.jo (F. Bekraoui), horani@ju.edu.jo (M. Al Horani), roshdi@ju.edu.jo (R. Khalil)

and f are unknowns and it is called the inverse problem.

If B is not invertible, then the equation is called degenerate otherwise it is called non-degenerate.

It was Hille, in 1952 who introduced the Abstract Cauchy Problem. All the work since then was trying to solve the abstract Cauchy problem of the first order or second order. Most of the analysis

of the Abstract Cauchy problem was using theory of semigroups of operators. We refer to [1], [2]-[4], for more on the Abstract Cauchy problem.

In this paper, we will use tensor product technique to find what is called atomic solution for the fractional Abstract Cauchy Problem, of the third order. Indeed, we will study the fractional abstract Cauchy problem of the form:

$$\begin{cases} u^{(3\alpha)}(t) + Au^{(2\alpha)}(t) + Bu^{(\alpha)}(t) + Cu(t) = f(t) \\ u(0) = x_0, u^{(\alpha)}(0) = x_1, u^{(2\alpha)}(0) = x_2. \end{cases} \tag{1}$$

where, A, B, C are closed operators with domain in range of u , and f is a given vector valued function with range in X . Here $u^{(\alpha)}$ denotes the α -conformable derivative of u .

Here is the definition of the conformable derivative given in [5]:

For $g : [0; \infty) \rightarrow \mathbb{R}$ and $0 < \alpha \leq 1$, the conformable fractional derivative of g of order α is defined by

$$D^\alpha(g)(t) = \lim_{\varepsilon \rightarrow 0} \frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}$$

We often write $u^{(\alpha)}$ for $D^\alpha u$

For all $t > 0$, if g is α differentiable on $(0; b)$ where $b > 0$ and $\lim_{t \rightarrow 0^+} g^{(\alpha)}(t)$ exists, then one can define $g^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} g^{(\alpha)}(t)$.

The α fractional integral of a function f starting from $a \geq 0$ is:

$$I_\alpha^a(f(t)) = I^\alpha(t^{1-\alpha} f(t)) = \int_a^t \frac{f(s)}{s^{1-\alpha}} ds$$

For more on Conformable fractional derivative we refer to [6]-[19].

Now, we need some basic facts from theory of tensor product of Banach spaces. Let X and Y be Banach spaces, X^* denote the dual of X . For $x \in X$ and $y \in Y$ define the map $x \otimes y : X^* \rightarrow Y$ as: $x \otimes y(x^*) = \langle x, x^* \rangle y$, for all $x^* \in X^*$.

Clearly, $x \otimes y$ is a bounded linear operator and $\|x \otimes y\| = \|x\| \|y\|$, [18]. Such an operator $x \otimes y$ is called an atom. The set $X \otimes Y = span\{x \otimes y : x \in X \text{ and } y \in Y\}$ is a subspace of $L(X^*, Y)$. The following lemma, see [20]-[22], is needed in our paper.

Theorem 1. *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that*

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$$

Then either x_1, x_2 or y_1, y_2 are linearly dependent.

One can easily prove that:

Lemma 1. *If*

$$x_1 \otimes y_1 = x_2 \otimes y_2,$$

then x_1, x_2 are dependent and y_1, y_2 are dependent too.

2. Main Result

Now we are interested in finding atomic solution of problem (1). That is a solution of the form $v \otimes x$, with $v(t) \in R$, and $x \in X$. Also, we assume $f = g \otimes z$. Substitute in (1) to get:

$$v^{(3\alpha)}(t) \otimes x + v^{(2\alpha)}(t) \otimes Ax + v^{(\alpha)}(t) \otimes Bx + v(t) \otimes Cx = g(t) \otimes z. \tag{2}$$

We assume the conditions $v(0) = 1, v^{(\alpha)}(0) = 1, v^{(2\alpha)}(0) = 1$.

There are many cases to consider.

Once again, the general form of third-order fractional Abstract Cauchy problem is

$$\begin{cases} u^{(3\alpha)}(t) + Au^{(2\alpha)}(t) + Bu^{(\alpha)}(t) + Cu(t) = h(t) \\ u(0) = x_0, u^{(\alpha)}(0) = x_1, u^{(2\alpha)}(0) = x_2. \end{cases}$$

Where u is 3α differentiable function from I to X and A, B , and C are closed linear operator on X . Notice, we use $u^{(3\alpha)}$ to denote $D^\alpha D^\alpha D^\alpha u$

We are interested in finding an atomic solution of equation (1), where the right hand side of the equation is an atom.

Now let the atomic solution we are looking for be $u(t) = v(t)x = v \otimes x$. In this case we let $h(t) = f(t)z = f \otimes z$, f is a scalar valued function.

Here v, x are unknowns and f, z are given

Let us rewrite (2) in the form

$$v^{(3\alpha)}(t) \otimes x + v^{(2\alpha)}(t) \otimes Ax + v^{(\alpha)}(t) \otimes Bx + v(t) \otimes Cx = f(t) \otimes z, \tag{*}$$

$$v(0) = v^{(\alpha)}(0) = v^{(2\alpha)}(0) = 1. \tag{**}$$

There are many cases to consider.

A. The first Case

$$\begin{cases} (a) v^{(3\alpha)} = v^{(2\alpha)} = v^{(\alpha)} = v. \\ \text{or} \\ (b) x = Ax = Bx = Cx \end{cases}$$

Let us consider the case where

$$v^{(3\alpha)} = v^{(2\alpha)} = v^{(\alpha)} = v. \tag{3}$$

Now we are looking for v that satisfies (1). From (3) we have the following situations:

1. $v^{(3\alpha)} = v^{(2\alpha)}$,
2. $v^{(2\alpha)} = v^{(\alpha)}$,
3. $v^{(\alpha)} = v$,
4. $v^{(3\alpha)} = v^{(\alpha)}$,
5. $v^{(3\alpha)} = v$,
6. $v^{(2\alpha)} = v$,

Situation 1.

$$v^{(3\alpha)} = v^{(2\alpha)}$$

By the results in [12], the associated characteristic equation is $r^2(r - 1) = 0$. Hence $r = 0, 0, 1$

Thus

$$v(t) = c_1 + c_2 \frac{1}{\alpha} t^\alpha + c_3 \exp \frac{1}{\alpha} t^\alpha$$

From (**), we obtain $\begin{cases} c_1 + c_2 + c_3 = 1 \\ c_2 + c_3 = 1 \\ c_3 = 1 \end{cases}$. Thus $c_1 = 0, c_2 = 0, c_3 = 1$. Hence

$$v(t) = \exp \frac{1}{\alpha} t^\alpha. \tag{4}$$

Situation 4.

$$v^{(3\alpha)} = v^{(\alpha)}$$

Again using the results in [12], the associated characteristic equation is $r(r^2 - 1) = 0$. Hence $r = 0, 1, -1$. So

$$v(t) = c_1 + c_2 \exp \left(-\frac{1}{\alpha} t^\alpha \right) + c_3 \exp \frac{1}{\alpha} t^\alpha$$

Using the conditions in (**) we get

$$\begin{cases} c_1 + c_2 + c_3 = 1, \\ -c_2 + c_3 = 1, \\ c_2 + c_3 = 1. \end{cases}$$

Thus, $c_1 = 0, c_2 = 0$, and $c_3 = 1$. So

$$v(t) = \exp \frac{1}{\alpha} t^\alpha$$

Situation 5.

$$v^{(3\alpha)} = v$$

Another use of the result [12], the associated characteristic equation is $r^3 - 1 = 0$, so $(r - 1)(r^2 + r + 1) = 0$. Hence $r_1 = 1, r_2 = \frac{-1 + i\sqrt{3}}{2}$ and $r_3 = \frac{-1 - i\sqrt{3}}{2}$. So

$$v(t) = \exp\left(-\frac{1}{2\alpha}t^\alpha\right) \left(c_1 \cos \frac{\sqrt{3}t^\alpha}{2\alpha} + c_2 \sin \frac{\sqrt{3}t^\alpha}{2\alpha} \right) + c_3 \exp \frac{1}{\alpha}t^\alpha.$$

Using (**) to get

$$\begin{cases} c_1 + c_3 = 1 \\ \frac{-1}{2}c_1 + c_2 + c_3 = 1 \\ -\frac{3}{4}c_1 - c_2 + c_3 = 1 \end{cases},$$

from which we get $c_1 = 0, c_2 = 0$, and $c_3 = 1$. Thus

$$v(t) = \exp \frac{1}{\alpha}t^\alpha.$$

If we do the other situations we get the same solution: $v(t) = \exp \frac{1}{\alpha}t^\alpha$. Thus if there is an atomic solution for this case, then v must equal to $\exp \frac{1}{\alpha}t^\alpha$. But from Lemma 1, we must have $f(t) = \exp \frac{1}{\alpha}t^\alpha$ in order to get an atomic solution.

Now, substitute $v(t) = \exp \frac{1}{\alpha}t^\alpha$ in (*), we get

$$\exp \frac{1}{\alpha}t^\alpha [x + Ax + Bx + Cx] = f(t) \otimes z. \tag{***}$$

It remains to find x . From equation (***), we have two atoms are equal. Thus the first coordinates are equal and the second coordinates are equal by Lemma 1. Thus we get

$$x + Ax + Bx + Cx = z.$$

This is

$$(I + A + B + C)x = z. \tag{5}$$

Hence, for the atomic solution to exist we must have x to satisfy (5), noting that z is given. So the image of x under $I + A + B + C$ must be z .

Let

$$x = Ax = Bx = Cx = z. \tag{6}$$

So,

$$x = z$$

and it is an eigenvector (a fixed point) for A, B , and C . Now substitute in equation (*) to get

$$\left[v^{(3\alpha)}(t) + v^{(2\alpha)}(t) + v^{(\alpha)}(t) + v(t) \right] \otimes x = f(t) \otimes z. \tag{7}$$

From equation (7), since $x = z$, we get

$$v^{(3\alpha)} + v^{(2\alpha)} + v^{(\alpha)} + v = f. \tag{8}$$

This is a linear fractional non-homogenous differential equation. Thus using result in [12], we find the homogenous solution v_h and a particular solution v_p , and so, the general solution will be $v_g = v_h + v_p$.

Now for v_h , the associated characteristic equation is $(r + 1)(r^2 + 1) = 0$, which has the roots $-1, \pm i$. Then

$$v_h(t) = c_1 \exp\left(-\frac{1}{\alpha}t^\alpha\right) + c_2 \cos \frac{1}{\alpha}t^\alpha + c_3 \sin \frac{1}{\alpha}t^\alpha.$$

For the particular solution, we use variation of parameters introduced in [12]. Let $v_1 = \exp\left(-\frac{1}{\alpha}t^\alpha\right)$, $v_2 = \cos \frac{1}{\alpha}t^\alpha$ and $v_3 = \sin \frac{1}{\alpha}t^\alpha$. So

$$\begin{aligned} v_p(t) &= \sum_{m=1}^3 v_m(t) \int_a^t \frac{f(s)W_m^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\ &= \exp\left(-\frac{1}{\alpha}t^\alpha\right) \int_a^t \frac{f(s)W_1^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds + \cos \frac{1}{\alpha}t^\alpha \int_a^t \frac{f(s)W_2^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\ &\quad + \sin \frac{1}{\alpha}t^\alpha \int_a^t \frac{f(s)W_3^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \end{aligned}$$

where $W^\alpha = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1^{(\alpha)} & v_2^{(\alpha)} & v_3^{(\alpha)} \\ v_1^{(2\alpha)} & v_2^{(2\alpha)} & v_3^{(2\alpha)} \end{vmatrix}$ and W_m^α is the determinant obtained from W^α by replacing m^{th} column by the column $(0, 0, 1)^T$, $m = 1, 2, 3$. Thus

$$\begin{aligned} v_p(t) &= \exp\left(-\frac{1}{\alpha}t^\alpha\right) \int_a^t \frac{f(s)}{2 \exp\left(-\frac{1}{\alpha}s^\alpha\right) s^{1-\alpha}} ds - \cos \frac{1}{\alpha}t^\alpha \int_a^t \frac{f(s)(\cos s + \sin s)}{2s^{1-\alpha}} ds \\ &\quad + \sin \frac{1}{\alpha}t^\alpha \int_a^t \frac{f(s)(\cos s - \sin s)}{2s^{1-\alpha}} ds \\ &= \exp\left(-\frac{1}{\alpha}t^\alpha\right) I_\alpha^a \left(\frac{f(t)}{2 \exp\left(-\frac{1}{\alpha}t^\alpha\right)}\right) - \cos \frac{1}{\alpha}t^\alpha I_\alpha^a \left(\frac{f(t)(\cos t + \sin t)}{2}\right) \\ &\quad + \sin \frac{1}{\alpha}t^\alpha I_\alpha^a \left(\frac{f(t)(\cos t - \sin t)}{2}\right) \end{aligned}$$

Hence

$$\begin{aligned} v(t) &= \mathbf{v}_h(\mathbf{t}) + \mathbf{v}_p(t) \\ &= c_1 \exp\left(-\frac{1}{\alpha}t^\alpha\right) + c_2 \cos \frac{1}{\alpha}t^\alpha + c_3 \sin \frac{1}{\alpha}t^\alpha + \exp\left(-\frac{1}{\alpha}t^\alpha\right) I_\alpha^a \left(\frac{f(t)}{2 \exp\left(-\frac{1}{\alpha}t^\alpha\right)}\right) \end{aligned}$$

$$-\cos \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) + \sin \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right)$$

$$\begin{aligned} v^{(\alpha)} = & -c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) - c_2 \sin \frac{1}{\alpha} t^\alpha + c_3 \cos \frac{1}{\alpha} t^\alpha \\ & + \exp \left(-\frac{1}{\alpha} t^\alpha \right) \left[-I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) + \frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right] \\ & + \sin \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) + \frac{f(t) (\cos t - \sin t)}{2} \right] \\ & + \cos \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right) - \frac{f(t) (\cos t + \sin t)}{2} \right] \end{aligned}$$

$$\begin{aligned} u^{(2\alpha)} = & c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) - c_2 \cos \frac{1}{\alpha} t^\alpha - c_3 \sin \frac{1}{\alpha} t^\alpha \\ & + \exp \left(-\frac{1}{\alpha} t^\alpha \right) \left[I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) + \frac{f^{(\alpha)}(t) \exp \left(-\frac{1}{\alpha} t^\alpha \right) + f(t) \exp \left(-\frac{1}{\alpha} t^\alpha \right)}{2 \exp \left(-\frac{2}{\alpha} t^\alpha \right)} \right] \\ & - f(t) + \cos \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) + \frac{f(t) (\cos t - \sin t)}{2} - \frac{f^{(\alpha)}(t) (\cos t + \sin t)}{2} \right] \\ & - \sin \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right) - \frac{f(t) (\cos t + \sin t)}{2} - \frac{f^{(\alpha)}(t) (\cos t - \sin t)}{2} \right] \end{aligned}$$

But $v(0) = 1$, $v^{(\alpha)}(0) = 1$ and $v^{(2\alpha)}(0) = 1$. Thus

$$\begin{cases} c_1 + c_2 = 1 \\ c_3 - c_1 = 1 \\ c_1 - c_2 + I_\alpha^a (f(0)) = 1 \end{cases} \iff \begin{cases} c_1 = 1 - \frac{1}{2} I_\alpha^a (f(0)) \\ c_2 = \frac{1}{2} I_\alpha^a (f(0)) \\ c_3 = 2 - \frac{1}{2} I_\alpha^a (f(0)) \end{cases}$$

Hence

$$\begin{aligned} v(t) = & \left(1 - \frac{1}{2} I_\alpha^a (f(0)) \right) \exp \left(-\frac{1}{\alpha} t^\alpha \right) + \frac{1}{2} I_\alpha^a (f(0)) \cos \frac{1}{\alpha} t^\alpha + \left(2 - \frac{1}{2} I_\alpha^a (f(0)) \right) \sin \frac{1}{\alpha} t^\alpha + \\ & \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) - \cos \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) \\ & + \sin \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right) \end{aligned}$$

Hence, the atomic solution that satisfies the conditions is

$$\left[\left(1 - \frac{1}{2} I_\alpha^a (f(0)) \right) \exp \left(-\frac{1}{\alpha} t^\alpha \right) + \frac{1}{2} I_\alpha^a (f(0)) \cos \frac{1}{\alpha} t^\alpha + \left(2 - \frac{1}{2} I_\alpha^a (f(0)) \right) \sin \frac{1}{\alpha} t^\alpha \right] \otimes z$$

$$+ \left[\exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) - \cos \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) \right. \\ \left. + \sin \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right) \right] \otimes z.$$

All other situations are handled in the same way.

B. The second Case.

$$\left\{ \begin{array}{l} \text{(i)} \quad v^{(3\alpha)}(t) \otimes x + v^{(2\alpha)}(t) \otimes Ax = v_1(t) \otimes y_1 \\ \text{and} \\ \text{(ii)} \quad v^{(\alpha)}(t) \otimes Bx + v(t) \otimes Cx = v_2(t) \otimes y_2 \end{array} \right. ,$$

This has the following situations:

Which gives four situations.

Situation 1. $\left\{ \begin{array}{l} v^{(3\alpha)} = v^{(2\alpha)} = v_1(t) \\ \text{and} \\ v^{(\alpha)} = v = v_2(t) \end{array} \right.$

Now for $v^{(3\alpha)} = v^{(2\alpha)}$, the associated characteristic equation is $r^3 - r^2 = 0$. Hence using [12], we get

$$v(t) = c_1 + c_2 \frac{t^\alpha}{\alpha} + c_3 \exp \frac{1}{\alpha} t^\alpha$$

Using conditions in (**) we get

$$v(t) = \exp \frac{1}{\alpha} t^\alpha$$

For $v^{(\alpha)} = v$, the associated characteristic equation is $r - 1 = 0$. Hence using [6], we get

$$v(t) = c \exp \frac{1}{\alpha} t^\alpha$$

Using conditions in (**) we get

$$v(t) = \exp \frac{1}{\alpha} t^\alpha$$

Similarly for $v^{(3\alpha)} = v$

$$\text{So } \begin{cases} x + Ax = y_1 \\ \text{and} \\ Bx + Cx = y_2 \end{cases}$$

$$\text{Hence } \begin{cases} (I + A)x = y_1 \\ \text{and} \\ (B + C)x = y_2 \end{cases}$$

Now we go to our equation

$$\exp \frac{1}{\alpha} t^\alpha \otimes [x + Ax + Bx + Cx] = f(t) \otimes z$$

So, f must be equal to $\exp \frac{1}{\alpha} t^\alpha$ for the atomic solution to exist and the image of x under $[I + A + B + C] x = z$.

$$\textbf{Situation 2.} \begin{cases} v^{(3\alpha)} = v^{(2\alpha)} = v_1(t) \\ \text{and} \\ Bx = Cx = y_2 \end{cases}$$

Now for $v^{(3\alpha)} = v^{(2\alpha)} = v_1(t)$, then

$$v(t) = \exp \frac{1}{\alpha} t^\alpha$$

Substitute in the main equation we get

$$\exp \frac{1}{\alpha} t^\alpha \otimes [x + Ax] + 2 \exp \frac{1}{\alpha} t^\alpha \otimes Bx = f(t) \otimes z$$

$$\exp \frac{1}{\alpha} t^\alpha \otimes [x + Ax + 2Bx] = f(t) \otimes z$$

Then for the atomic solution to exist, $f = \exp \frac{1}{\alpha} t^\alpha$, and $[I + A + 2B] x = z$.

$$\textbf{Situation 3.} \begin{cases} x = Ax = y_1 \\ \text{and} \\ v^{(\alpha)} = v = v_2 \end{cases}$$

Now, $v^{(\alpha)} = v$ gives

$$v(t) = \exp \frac{1}{\alpha} t^\alpha$$

So

$$\exp \frac{1}{\alpha} t^\alpha \otimes [x + Ax + Bx + Cx] = f(t) \otimes z$$

$$\exp \frac{1}{\alpha} t^\alpha \otimes [2Ax + Bx + Cx] = f(t) \otimes z$$

Hence f must equal $\exp \frac{1}{\alpha} t^\alpha$ and $[2A + B + C] x = z$.

Situation 4.
$$\begin{cases} x = Ax = y_1 \\ \text{and} \\ Bx = Cx = y_2 \end{cases}$$

So $[v^{(3\alpha)} + v^{(2\alpha)}] \otimes x + [v^{(\alpha)} + v] \otimes Bx = f \otimes z$.

Hence we have two subcases:

$$1. v^{(3\alpha)} + v^{(2\alpha)} = v^{(\alpha)} + v \text{ or } 2. Bx = x.$$

If $Bx = x$, then

$$[v^{(3\alpha)} + v^{(2\alpha)} + v^{(\alpha)} + v] \otimes x = f \otimes z.$$

Hence $x = z$.

Now we solve

$$v^{(3\alpha)} + v^{(2\alpha)} + v^{(\alpha)} + v = f.$$

This is a linear fractional differential equation of order 3α . Using [12], we get the general solution $v_g = v_h + v_p$, the sum of the homogenous solution and the particular solution.

For homogenous solution v_h , the associated characteristic equation is $r^3 + r^2 + r + 1 = 0$, which has the roots $i, -i$ and -1 .

Then

$$v_h(t) = c_1 \exp\left(-\frac{1}{\alpha} t^\alpha\right) + c_2 \cos \frac{1}{\alpha} t^\alpha + c_3 \sin \frac{1}{\alpha} t^\alpha$$

For the particular solution, we use variation of parameters introduced in [12]. Let $u_1 = \exp\left(-\frac{1}{\alpha} t^\alpha\right)$, $u_2 = \cos \frac{1}{\alpha} t^\alpha$ and $u_3 = \sin \frac{1}{\alpha} t^\alpha$

So

$$\begin{aligned} v_p(t) &= \sum_{m=1}^3 u_m(t) \int_a^t \frac{f(s)W_m^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\ &= \exp\left(-\frac{1}{\alpha} t^\alpha\right) \int_a^t \frac{f(s)W_1^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds + \cos \frac{1}{\alpha} t^\alpha \int_a^t \frac{f(s)W_2^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds + \sin \frac{1}{\alpha} t^\alpha \int_a^t \frac{f(s)W_3^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \end{aligned}$$

where $W^\alpha = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1^{(\alpha)} & u_2^{(\alpha)} & u_3^{(\alpha)} \\ u_1^{(2\alpha)} & u_2^{(2\alpha)} & u_3^{(2\alpha)} \end{vmatrix}$ and W_m^α is the determinant obtained from

W^α by replacing m^{th} column by the column $(0, 0, 1)$, $m = 1, 2, 3$. Thus

$$v_p(t) = \exp\left(-\frac{1}{\alpha} t^\alpha\right) \int_a^t \frac{f(s)}{2 \exp\left(-\frac{1}{\alpha} s^\alpha\right) s^{1-\alpha}} ds - \cos \frac{1}{\alpha} t^\alpha \int_a^t \frac{f(s) (\cos s + \sin s)}{2s^{1-\alpha}} ds$$

$$\begin{aligned}
 & + \sin \frac{1}{\alpha} t^\alpha \int_a^t \frac{f(s) (\cos s - \sin s)}{2s^{1-\alpha}} ds \\
 = & \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) - \cos \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) \\
 & + \sin \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos s - \sin t)}{2} \right)
 \end{aligned}$$

Hence

$$\begin{aligned}
 v(t) & = \mathbf{v}_h(\mathbf{t}) + \mathbf{v}_p(t) \\
 & = c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) + c_2 \cos \frac{1}{\alpha} t^\alpha + c_3 \sin \frac{1}{\alpha} t^\alpha \\
 & \quad + \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) \\
 & \quad - \cos \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) \\
 & \quad + \sin \frac{1}{\alpha} t^\alpha I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 v^{(\alpha)}(t) & = -c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) - c_2 \sin \frac{1}{\alpha} t^\alpha + c_3 \cos \frac{1}{\alpha} t^\alpha \\
 & \quad + \exp \left(-\frac{1}{\alpha} t^\alpha \right) \left[-I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) + \frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right] \\
 & \quad + \sin \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) + \frac{f(t) (\cos t - \sin t)}{2} \right] \\
 & \quad + \cos \frac{1}{\alpha} t^\alpha \left[I_\alpha^a \left(\frac{f(t) (\cos t - \sin t)}{2} \right) - \frac{f(t) (\cos t + \sin t)}{2} \right]
 \end{aligned}$$

$$\begin{aligned}
 v^{(2\alpha)}(t) & = c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) - c_2 \cos \frac{1}{\alpha} t^\alpha - c_3 \sin \frac{1}{\alpha} t^\alpha \\
 & \quad + \exp \left(-\frac{1}{\alpha} t^\alpha \right) \left[\begin{aligned} & I_\alpha^a \left(\frac{f(t)}{2 \exp \left(-\frac{1}{\alpha} t^\alpha \right)} \right) \\ & + \frac{f^{(\alpha)}(t) \exp \left(-\frac{1}{\alpha} t^\alpha \right) + f(t) \exp \left(-\frac{1}{\alpha} t^\alpha \right)}{2 \exp \left(-\frac{2}{\alpha} t^\alpha \right)} \end{aligned} \right] \\
 & \quad - f(t) + \cos \frac{1}{\alpha} t^\alpha \left[\begin{aligned} & I_\alpha^a \left(\frac{f(t) (\cos t + \sin t)}{2} \right) \\ & + \frac{f(t) (\cos t - \sin t)}{2} - \frac{f^{(\alpha)}(t) (\cos t + \sin t)}{2} \end{aligned} \right]
 \end{aligned}$$

$$-\sin \frac{1}{\alpha} t^\alpha \left[\begin{array}{c} I_\alpha^a \left(\frac{f(t)(\cos t - \sin t)}{2} \right) \\ - \frac{f(t)(\cos t + \sin t)}{2} - \frac{f^{(\alpha)}(t)(\cos t - \sin t)}{2} \end{array} \right]$$

Using conditions in (**) we get

$$\begin{cases} c_1 + c_2 = 1 \\ c_3 - c_1 = 1 \\ c_1 - c_2 + I_\alpha^a(f(0)) = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 - \frac{1}{2} I_\alpha^a(f(0)) \\ c_2 = \frac{1}{2} I_\alpha^a(f(0)) \\ c_3 = 2 - \frac{1}{2} I_\alpha^a(f(0)) \end{cases}$$

Hence

$$\begin{aligned} v(t) &= \left(1 - \frac{1}{2} I_\alpha^a(f(0)) \right) \exp\left(-\frac{1}{\alpha} t^\alpha\right) + \frac{1}{2} I_\alpha^a(f(0)) \cos \frac{1}{\alpha} t^\alpha \\ &+ \left(2 - \frac{1}{2} I_\alpha^a(f(0)) \right) \sin \frac{1}{\alpha} t^\alpha + \\ &\exp\left(-\frac{1}{\alpha} t^\alpha\right) I_\alpha^a\left(\frac{f(t)}{2 \exp\left(-\frac{1}{\alpha} t^\alpha\right)}\right) \\ &- \cos \frac{1}{\alpha} t^\alpha I_\alpha^a\left(\frac{f(t)(\cos t + \sin t)}{2}\right) \\ &+ \sin \frac{1}{\alpha} t^\alpha I_\alpha^a\left(\frac{f(t)(\cos t - \sin t)}{2}\right) \end{aligned}$$

and the atomic solution that satisfies the conditions is $u(t) = v(t) \otimes z$.

Now, if $v^{(3\alpha)} + v^{(2\alpha)} = v^{(\alpha)} + v = f$, then $(I + B)x = z$

We want to solve $v^{(3\alpha)} + v^{(2\alpha)} = f$ and $v^{(\alpha)} + v = f$

For $v^{(3\alpha)} + v^{(2\alpha)} = f$

Using [12], we get the general solution $v_g = v_h + v_p$, the sum of the homogenous solution and the particular solution.

For homogenous solution v_h , the associated characteristic equation is $r^2(r + 1) = 0$, which has the roots 0, 0 and -1.

$$v_h(t) = c_1 + c_2 \frac{t^\alpha}{\alpha} + c_3 e^{-\frac{t^\alpha}{\alpha}}$$

For the particular solution, we use variation of parameters introduced in [12]. Let $u_1 = 1, u_2 = \frac{1}{\alpha} t^\alpha$ and $u_3 = e^{-\frac{t^\alpha}{\alpha}}$

$$v_p(t) = \sum_{m=1}^3 u_m(t) \int_a^t \frac{f(s)W_m^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds$$

$$\begin{aligned}
 &= \int_a^t \frac{f(s)W_1^\alpha(s)}{\overline{W}^\alpha(s)s^{1-\alpha}} ds + \frac{1}{\alpha} t^\alpha \int_a^t \frac{f(s)W_2^\alpha(s)}{\overline{W}^\alpha(s)s^{1-\alpha}} ds + e^{\frac{-t^\alpha}{\alpha}} \int_a^t \frac{f(s)W_3^\alpha(s)}{\overline{W}^\alpha(s)s^{1-\alpha}} ds \\
 &= - \int_a^t \frac{f(s) \left(\frac{s^\alpha}{\alpha} + 1\right)}{s^{1-\alpha}} ds + \frac{t^\alpha}{\alpha} \int_a^t \frac{f(s)}{s^{1-\alpha}} ds + e^{\frac{-t^\alpha}{\alpha}} \int_a^t \frac{f(s)}{e^{\frac{-s^\alpha}{\alpha}} s^{1-\alpha}} ds \\
 &= -I_\alpha^a \left(f(t) \left(\frac{t^\alpha}{\alpha} + 1 \right) \right) + \frac{t^\alpha}{\alpha} I_\alpha^a (f(t)) + e^{\frac{-t^\alpha}{\alpha}} I_\alpha^a \left(\frac{f(t)}{e^{\frac{-t^\alpha}{\alpha}}} \right)
 \end{aligned}$$

Hence

$$\begin{aligned}
 v(t) &= \mathbf{v}_h(\mathbf{t}) + \mathbf{v}_p(t) \\
 &= c_1 + c_2 \frac{t^\alpha}{\alpha} + c_3 e^{\frac{-t^\alpha}{\alpha}} - I_\alpha^a \left(f(t) \left(\frac{t^\alpha}{\alpha} + 1 \right) \right) + \frac{t^\alpha}{\alpha} I_\alpha^a (f(t)) + e^{\frac{-t^\alpha}{\alpha}} I_\alpha^a \left(\frac{f(t)}{e^{\frac{-t^\alpha}{\alpha}}} \right)
 \end{aligned}$$

For $v^{(\alpha)} + v = f$

By result in [12] the solution of equation (7) is given by

$$\begin{aligned}
 v(t) &= e^{-\int_0^t x^{\alpha-1} dx} + \int_0^t \left(e^{-\int_s^t x^{\alpha-1} dx} f(s) s^{\alpha-1} \right) ds \\
 &= e^{-\frac{1}{\alpha} t^\alpha} + \int_0^t \left(e^{-\frac{1}{\alpha}(t^\alpha-s^\alpha)} f(s) s^{\alpha-1} \right) ds
 \end{aligned}$$

But the two solution must be equal, then

$$c_3 + I_\alpha^a \left(\frac{f(t)}{e^{\frac{-t^\alpha}{\alpha}}} \right) = 1 \text{ and } \int_0^t \left(e^{-\frac{1}{\alpha}(t^\alpha-s^\alpha)} f(s) s^{\alpha-1} \right) ds = c_1 + (c_2 + I_\alpha^a (f(t))) \frac{t^\alpha}{\alpha} - I_\alpha^a (f(t) \left(\frac{t^\alpha}{\alpha} + 1 \right))$$

Then

$$v(t) = e^{-\frac{1}{\alpha} t^\alpha} + \int_0^t \left(e^{-\frac{1}{\alpha}(t^\alpha-s^\alpha)} f(s) s^{\alpha-1} \right) ds$$

C. The Third Case.

$$x = Ax \text{ and } Bx = Cx \tag{a}$$

In this case

$$\left[v^{(3\alpha)}(t) + v^{(2\alpha)}(t) \right] \otimes x + \left[v^{(\alpha)}(t) + v(t) \right] \otimes Bx = f(t) \otimes z$$

If

$$x = Bx = z, \tag{9}$$

then

$$v^{(3\alpha)}(t) + v^{(2\alpha)}(t) + v^{(\alpha)}(t) + v(t) = f(t).$$

And this can be solved as before in case1.

$$v^{(3\alpha)}(t) + v^{(2\alpha)}(t) = v^{(\alpha)}(t) + v(t) = f(t). \tag{b}$$

Now $v^{(3\alpha)}(t) + v^{(2\alpha)}(t) = v^{(\alpha)}(t) + v(t)$, the associated characteristic equation is $r^3 + r^2 - r - 1 = 0$ which has the roots $-1, -1$ and 1 .

Then

$$v(t) = c_1 e^{\frac{1}{\alpha}t^\alpha} + c_2 \frac{1}{\alpha} t^\alpha e^{-\frac{1}{\alpha}t^\alpha} + c_3 e^{-\frac{1}{\alpha}t^\alpha}.$$

$v(0) = v^{(\alpha)}(0) = v^{(2\alpha)}(0) = 1$ gives

$$\begin{cases} c_1 + c_3 = 1 \\ c_1 + c_2 - c_3 = 1 \\ c_1 - 2c_2 + c_3 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

Hence

$$v(t) = e^{\frac{1}{\alpha}t^\alpha}.$$

This forces $f(t) = 2e^{\frac{1}{\alpha}t^\alpha}$, if not then there is no atomic solution.

Now in case of (9) and from (a) we get $x = Ax = Bx = Cx$, so x is an eigenvector for A, B, C and $x = z$.

In case (b), we get

$$2e^{\frac{1}{\alpha}t^\alpha} + 2e^{\frac{1}{\alpha}t^\alpha} = 2e^{\frac{1}{\alpha}t^\alpha} z$$

So $x + Bx = z$. Hence $(I + B)x = z$.

D. The Forth Case

$$\left\{ \begin{array}{l} \text{(i)} \quad v^{(3\alpha)}(t) \otimes x + v(t) \otimes Cx = v_1(t) \otimes y_1 \\ \text{and} \\ \text{(ii)} \quad v^{(2\alpha)}(t) \otimes Ax + v^{(\alpha)}(t) \otimes Bx = v_2(t) \otimes y_2 \end{array} \right.$$

Which gives

Situation (1) $\begin{cases} v^{(3\alpha)} = v = v_1(t) \\ \text{and} \\ v^{(2\alpha)} = v^{(\alpha)} = v_2(t) \end{cases}$

For $v^{(3\alpha)} = v = v_1(t)$, the associated characteristic equation is $r^3 - 1 = 0$. Which has the roots $1, \frac{-1 \pm \sqrt{3}i}{2}$.

Hence using [12], we get

$$v(t) = c_1 \exp \frac{1}{\alpha} t^\alpha + \exp \left(-\frac{1}{2\alpha} t^\alpha \right) \left(c_2 \cos \frac{\sqrt{3}}{2\alpha} t^\alpha + c_3 \sin \frac{\sqrt{3}}{2\alpha} t^\alpha \right).$$

Using conditions (**) to get

$$v(t) = \exp \frac{1}{\alpha} t^\alpha.$$

Now for $v^{(2\alpha)} = v^{(\alpha)}$, the associated characteristic equation is $r^2 - r = 0$. Hence using [12], we get

$$v(t) = c_1 + c_2 \exp \frac{1}{\alpha} t^\alpha.$$

By the conditions (**) we get

$$v(t) = \exp \frac{1}{\alpha} t^\alpha.$$

Substitute in the main equation

$$\exp \frac{1}{\alpha} t^\alpha \otimes [x + Ax + Bx + Cx] = f(t) \otimes z.$$

So, f must be equal to $\exp \frac{1}{\alpha} t^\alpha$ for the atomic solution to exist and the image of x under $[I + A + B + C]x = z$.

$$\textbf{Situation (2)} \quad \left\{ \begin{array}{l} v^{(3\alpha)} = v = v_1(t) \\ \text{and} \\ Ax = Bx = y_2 \end{array} \right.$$

Now for $v^{(3\alpha)} = v$, we previously found

$$v(t) = \exp \frac{1}{\alpha} t^\alpha.$$

Substitute in the main equation we get

$$\exp \frac{1}{\alpha} t^\alpha \otimes x + \exp \frac{1}{\alpha} t^\alpha \otimes Ax + \exp \frac{1}{\alpha} t^\alpha \otimes Bx + \exp \frac{1}{\alpha} t^\alpha \otimes Cx = f(t) \otimes z$$

But $Ax = Bx$, so

$$\exp \frac{1}{\alpha} t^\alpha \otimes (x + 2Ax + Cx) = f(t) \otimes z.$$

Then for the atomic solution to exist, f must be equal to $\exp \frac{1}{\alpha} t^\alpha$, and $[I + 2A + C]x = z$.

$$\textbf{Situation (3)} \quad \left\{ \begin{array}{l} x = Cx = y_1 \\ \text{and} \\ v^{(2\alpha)} = v^{(\alpha)} = v_2 \end{array} \right.$$

Now for $v^{(2\alpha)} = v^{(\alpha)}$, we already have

$$v(t) = \exp \frac{1}{\alpha} t^\alpha .$$

Substitute in the main equation we get

$$\exp \frac{1}{\alpha} t^\alpha \otimes x + \exp \frac{1}{\alpha} t^\alpha \otimes Ax + \exp \frac{1}{\alpha} t^\alpha \otimes Bx + \exp \frac{1}{\alpha} t^\alpha \otimes Cx = f(t) \otimes z$$

Since $x = Cx$, then

$$\exp \frac{1}{\alpha} t^\alpha \otimes (2x + Ax + Bx) = f(t) \otimes z .$$

Then for the atomic solution to exist, f must be equal to $\exp \frac{1}{\alpha} t^\alpha$, and $[2I + A + B] x = z$.

Situation (4) $\left\{ \begin{array}{l} x = Cx = y_1 \\ \text{and} \\ Ax = Bx = y_2 \end{array} \right.$

So $[v^{(3\alpha)} + v] \otimes x + [v^{(2\alpha)} + v^{(\alpha)}] \otimes Ax = f \otimes z$.

Then we have two cases $v^{(3\alpha)} + v = v^{(2\alpha)} + v^{(\alpha)} = f$ or $x = Ax = z$

If $x = Ax$, then

$$[v^{(3\alpha)} + v^{(2\alpha)} + v^{(\alpha)} + v] \otimes x = f \otimes z .$$

Then

$$v^{(3\alpha)} + v^{(2\alpha)} + v^{(\alpha)} + v = f$$

which already solved in Case 2.

If $v^{(3\alpha)} + v = v^{(2\alpha)} + v^{(\alpha)} = f$, then $(I + A)x = z$.

Now for $v^{(3\alpha)} + v = f$, the general solution is given by $v_g = v_h + v_p$, the sum of the homogenous solution and the particular solution.

For the homogenous solution v_h , the associated characteristic equation is $r^3 + 1 = 0$, which has the roots $-1, \frac{1 \pm \sqrt{3}i}{2}$. Then

$$v_h(t) = c_1 \exp \left(-\frac{1}{\alpha} t^\alpha \right) + \exp \left(\frac{1}{2\alpha} t^\alpha \right) \left(c_2 \cos \frac{\sqrt{3}}{2\alpha} t^\alpha + c_3 \sin \frac{\sqrt{3}}{2\alpha} t^\alpha \right) .$$

For the particular solution, we use variation of parameters. Let $u_1 = \exp \left(-\frac{1}{\alpha} t^\alpha \right)$,

$u_2 = \exp \left(\frac{1}{2\alpha} t^\alpha \right) \cos \frac{\sqrt{3}}{2\alpha} t^\alpha$ and $u_3 = \exp \left(\frac{1}{2\alpha} t^\alpha \right) \sin \frac{\sqrt{3}}{2\alpha} t^\alpha$.

$$v_p(t) = \sum_{m=1}^3 u_m(t) \int_a^t \frac{f(s)W_m^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds$$

$$\begin{aligned}
&= \exp\left(-\frac{1}{\alpha}t^\alpha\right) \int_a^t \frac{f(s)W_1^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\
&\quad + \exp\left(\frac{1}{2\alpha}t^\alpha\right) \cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \int_a^t \frac{f(s)W_2^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\
&\quad + \exp\left(\frac{1}{2\alpha}t^\alpha\right) \sin\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \int_a^t \frac{f(s)W_3^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\
&= \exp\left(-\frac{1}{\alpha}t^\alpha\right) \int_a^t \frac{f(s)}{3s^{1-\alpha}} ds \\
&\quad - \exp\left(\frac{1}{2\alpha}t^\alpha\right) \cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \times \\
&\quad \quad \int_a^t \frac{f(s) \exp\left(-\frac{1}{2\alpha}s^\alpha\right) \left[\cos\left(\frac{\sqrt{3}}{2\alpha}s^\alpha\right) + \sin\left(\frac{\sqrt{3}}{2\alpha}s^\alpha\right)\right]}{3\sqrt{3}s^{1-\alpha}} ds \\
&\quad + \exp\left(\frac{1}{2\alpha}t^\alpha\right) \sin\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \times \\
&\quad \quad \int_a^t \frac{f(s) \exp\left(-\frac{1}{2\alpha}s^\alpha\right) \left[\cos\left(\frac{\sqrt{3}}{2\alpha}s^\alpha\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2\alpha}s^\alpha\right)\right]}{3\sqrt{3}s^{1-\alpha}} ds \\
&= \exp\left(-\frac{1}{\alpha}t^\alpha\right) I_\alpha^a\left(\frac{1}{3}f(t)\right) - \exp\left(\frac{1}{2\alpha}t^\alpha\right) \cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \times \\
&\quad I_\alpha^a\left(\frac{1}{3}f(t) \exp\left(-\frac{1}{2\alpha}t^\alpha\right) \left[\cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) + \sin\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right)\right]\right) \\
&\quad + \exp\left(\frac{1}{2\alpha}t^\alpha\right) \sin\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) I_\alpha^a\left(\frac{1}{3}f(t) \exp\left(-\frac{1}{2\alpha}t^\alpha\right) \times \right. \\
&\quad \quad \left. \left[\cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right)\right]\right)
\end{aligned}$$

Hence

$$\begin{aligned}
v(t) &= v_h + v_p \\
&= c_1 \exp\left(-\frac{1}{\alpha}t^\alpha\right) + \exp\left(\frac{1}{2\alpha}t^\alpha\right) \left(c_2 \cos\frac{\sqrt{3}}{2\alpha}t^\alpha + c_3 \sin\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \\
&\quad + \exp\left(-\frac{1}{\alpha}t^\alpha\right) I_\alpha^a\left(\frac{1}{3}f(t)\right) - \exp\left(\frac{1}{2\alpha}t^\alpha\right) \cos\left(\frac{\sqrt{3}}{2\alpha}t^\alpha\right) \times
\end{aligned}$$

$$\begin{aligned}
 & I_\alpha^a \left(\frac{1}{3} f(t) \exp \left(-\frac{1}{2\alpha} t^\alpha \right) \left[\cos \left(\frac{\sqrt{3}}{2\alpha} t^\alpha \right) + \sin \left(\frac{\sqrt{3}}{2\alpha} t^\alpha \right) \right] \right) \\
 & + \exp \left(\frac{1}{2\alpha} t^\alpha \right) \sin \left(\frac{\sqrt{3}}{2\alpha} t^\alpha \right) \times \\
 & I_\alpha^a \left(\frac{1}{3} f(t) \exp \left(-\frac{1}{2\alpha} t^\alpha \right) \left[\cos \left(\frac{\sqrt{3}}{2\alpha} t^\alpha \right) - \sqrt{3} \sin \left(\frac{\sqrt{3}}{2\alpha} t^\alpha \right) \right] \right) .
 \end{aligned}$$

For $v^{(2\alpha)} + v^{(\alpha)} = f$, the associated characteristic equation is $r^2 + r = 0$, which has the roots 0 and -1 .

Then

$$v_h(t) = c_4 + c_5 \exp \left(-\frac{1}{\alpha} t^\alpha \right) .$$

For the particular solution, we use variation of parameters. Let $u_1 = 1, u_2 = \exp \left(-\frac{1}{\alpha} t^\alpha \right)$

So

$$\begin{aligned}
 v_p(t) &= \int_a^t \frac{f(s)W_1^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds + \exp \left(-\frac{1}{\alpha} t^\alpha \right) \int_a^t \frac{f(s)W_2^\alpha(s)}{W^\alpha(s)s^{1-\alpha}} ds \\
 &= \int_a^t \frac{f(s)}{s^{1-\alpha}} ds - \exp \left(-\frac{1}{\alpha} t^\alpha \right) \int_a^t \frac{f(s) \exp \left(\frac{1}{\alpha} s^\alpha \right)}{(s)s^{1-\alpha}} ds \\
 &= I_\alpha^a (f(t)) - \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(f(t) \exp \left(\frac{1}{\alpha} t^\alpha \right) \right)
 \end{aligned}$$

Hence

$$\begin{aligned}
 v(t) &= v_h + v_p \\
 &= c_4 + c_5 \exp \left(-\frac{1}{\alpha} t^\alpha \right) + I_\alpha^a (f(t)) - \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(f(t) \exp \left(\frac{1}{\alpha} t^\alpha \right) \right)
 \end{aligned}$$

By the condition $(**)$ we get

$$\begin{cases} c_4 = 2 - I_\alpha^a (f(0)) \\ c_5 = -1 + I_\alpha^a (f(0)) \end{cases} .$$

But the two solutions must be equal, then

$$\begin{cases} c_1 = c_5 \\ c_2 = c_3 = 0 \end{cases} .$$

Hence

$$\begin{aligned}
 v(t) &= (-1 + I_\alpha^a (f(0))) \exp \left(-\frac{1}{\alpha} t^\alpha \right) + I_\alpha^a (f(t)) \\
 &\quad - \exp \left(-\frac{1}{\alpha} t^\alpha \right) I_\alpha^a \left(f(t) \exp \left(\frac{1}{\alpha} t^\alpha \right) \right) + 2 - I_\alpha^a (f(0))
 \end{aligned}$$

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