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A Triple Integral involving the Struve function $\boldsymbol{H}_v(t)$ expressed in terms of the Hurwitz-Lerch zeta Function

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Abstract. The main focus of the present paper is to establish a triple integral involving the Struve function in terms of the Hurwitz-Lerch zeta function by using our contour integral method. We also consider some useful examples as special cases of this integral involving the Struve function to give the applications of our main results.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

Key Words and Phrases: Struve function, Triple integral, Hurwitz-Lerch zeta function

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1. Significance Statement

The Struve function $\boldsymbol{H}_v(t)$ was introduced by Hermann von Struve [7] and today this function is carrying his name. This function is related to the non-homogeneous Bessel type ordinary differential equation of special type called Struve differential equation [1]. The Struve function has a variety of applications, such as describing the vibrations of thin disks and in the theory of electromagnetism, see section (57:1) in [3]. Other applications of Struve functions are detailed in section (11.12) in [1]. In this work we use our contour integral method to derive a triple integral involving the Struve function and expressed in terms of the Hurwitz-Lerch zeta function. Multiple integrals involving special functions are used to solve ordinary differential equations [9] and our goal is to provide an expansion of such work.

2. Introduction

In this paper we derive the triple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty y^{1-m} t^{m-v-1} x^{-m+2v+1} \boldsymbol{H}_v(t) e^{-bx^2-cy^2} \log^k\left(\frac{at}{xy}\right) dxdydt \tag{1}$$

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where the parameters k, a, b, c are general complex numbers and $Re(m) < Re(v) + 3/2$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [4]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y and t, then take a definite triple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of y and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [4]. The variable of integration in the contour integral is $\beta = w + m$. The cut and contour are in the second quadrant of the complex *beta*-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing y by $\log\left(\frac{at}{xy}\right)$ and multiplying by $y^{1-m}t^{m-v-1}x^{-m+2v+1}\boldsymbol{H}_v(t)e^{-bx^2-cy^2}$ then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$ and $t \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty\int_0^\infty\int_0^\infty y^{1-m}t^{m-v-1}x^{-m+2v+1}\boldsymbol{H}_v(t)e^{-bx^2-cy^2}\log^k\left(\frac{at}{xy}\right)dxdydt\\
&=\frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_0^\infty\int_C a^w w^{-k-1}y^{-m-w+1}\boldsymbol{H}_v(t)\\
&e^{-bx^2-cy^2}t^{m-v+w-1}x^{-m+2v-w+1}dwdxdydt\\
&=\frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty\int_0^\infty a^w w^{-k-1}y^{-m-w+1}\boldsymbol{H}_v(t)\\
&e^{-bx^2-cy^2}t^{m-v+w-1}x^{-m+2v-w+1}dxdydtdw\\
&=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}c^{\frac{m+w}{2}-1}2^{m-v+w-3}\sec\left(\frac{1}{2}\pi(m+w)\right)b^{\frac{m+w}{2}-v-1}dw
\end{aligned} \tag{3}$$

from equation (13.2) on page 392 in [8] and equation (3.326.2) in [2] where $-1 < Re(w+m) < 0$ and $Re(w+m) < Re(v)+3/2, Re(b) > 0, Re(c) > 0$ and using the reflection formula (8.334.3) in [2] for the Gamma function. We are able to switch the order of integration over w, x, y and t using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)$

4. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [1] and [5, 6] has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^{n} \tag{4}$$

where $|z|< 1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{5}$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

4.2. Infinite sum of the Contour Integral

Using equation (2) and replacing y by $\log(a)+\frac{\log(b)}{2}+\frac{\log(c)}{2}+\frac{1}{2}i\pi(2y+1)+\log(2)$ then multiplying both sides by $\pi c^{\frac{m}{2}-1}2^{m-v-2}b^{\frac{m}{2}-v-1}$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\pi^{k+1}c^{\frac{m}{2}-1}e^{\frac{1}{2}i\pi(k+m)}2^{m-v-2}b^{\frac{m}{2}-v-1}\\
&\quad\Phi\left(-e^{im\pi},-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+\pi}{2\pi}\right)\\
&\quad=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\pi(-1)^y a^w w^{-k-1}c^{\frac{1}{2}(m+w-2)}2^{m-v+w-2}\\
&\quad e^{\frac{1}{2}i\pi(2y+1)(m+w)}b^{\frac{1}{2}(m-2v+w-2)}dw\\
&\quad=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\pi(-1)^y a^w w^{-k-1}c^{\frac{1}{2}(m+w-2)}2^{m-v+w-2}\\
&\quad e^{\frac{1}{2}i\pi(2y+1)(m+w)}b^{\frac{1}{2}(m-2v+w-2)}dw\\
&\quad=\frac{1}{2\pi i}\int_C\pi a^w w^{-k-1}c^{\frac{1}{2}(m+w-2)}2^{m-v+w-3}\sec\left(\frac{1}{2}\pi(m+w)\right)b^{\frac{1}{2}(m-2v+w-2)}dw
\end{aligned} \tag{6}$$

from equation (1.232.2) in [2] where $Im(w+m)>0$ in order for the sum to converge.

5. Definite Integral in terms of the Lerch function

Theorem 1. *For all* $k, a \in \mathbb{C}, Re(v) > Re(m) > 0, Re(b) > 0, Re(c) > 0,$

$$\int_0^\infty \int_0^\infty \int_0^\infty y^{1-m} t^{m-v-1} x^{-m+2v+1} \boldsymbol{H}_v(t) e^{-bx^2-cy^2} \log^k\left(\frac{at}{xy}\right) dxdydt$$
$$= \pi^{k+1} c^{\frac{m}{2}-1} e^{\frac{1}{2}i\pi(k+m)} 2^{m-v-2} b^{\frac{m}{2}-v-1} \tag{7}$$
$$\Phi\left(-e^{im\pi}, -k, \frac{-2i\log(2a) - i\log(b) - i\log(c) + \pi}{2\pi}\right)$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

Example 1. *The degenerate case.*

$$\begin{aligned}&\int_0^\infty \int_0^\infty \int_0^\infty y^{1-m} t^{m-v-1} x^{-m+2v+1} \boldsymbol{H}_v(t) e^{-bx^2-cy^2} dxdydt \\ &\quad = \pi c^{\frac{m}{2}-1} 2^{m-v-3} \sec\left(\frac{\pi m}{2}\right) b^{\frac{m}{2}-v-1}\end{aligned} \tag{8}$$

Proof. Use equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [3].

Example 2. *An example in terms of the Riemann zeta function* $\zeta(k)$.

$$\int_0^\infty \int_0^\infty \int_0^\infty y t^{-v-1} x^{2v+1} e^{-x^2-y^2} \boldsymbol{H}_v(t) \log^k\left(\frac{it}{2xy}\right) dxdydt$$
$$= \left(2^{k+1} - 1\right) e^{\frac{i\pi k}{2}} \pi^{k+1} \left(-2^{-v-2}\right) \zeta(-k) \tag{9}$$

Proof. Use equation (7) and set $m = 0$ and simplify in terms of the Hurwitz zeta function $\zeta(k, a)$ using entry (4) in Table below (64:12:7) in [3]. Next set $a = i/2, b = c = 1$ and simplify in terms of the Riemann zeta function using entry (2) in Table below (64:7) in [3].

Example 3.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{y t^{-v-1} x^{2v+1} e^{-x^2-y^2} \boldsymbol{H}_v(t)}{\log\left(\frac{it}{2xy}\right)} dxdydt = -i 2^{-v-2} \log(2) \tag{10}$$

Proof. Use equation (9) and apply l'Hopital's rule to the right-hand side as $k \to -1$ and simplify using equation (25.6.11) in [1].

Example 4.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^2 y(1-\cos(t))e^{-x^2-y^2}}{t^2\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = \frac{\log(2)}{16\sqrt{\pi}} \tag{11}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^2 y(\cos(t)+1)e^{-x^2-y^2}\log\left(\frac{t}{2xy}\right)}{t^2\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = 0 \tag{12}$$

Proof. Use equation (10) and set $v = 1/2$ and rationalize the denominator and equate real and imaginary parts and simplify.

Example 5.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^7 y \boldsymbol{H}_3(t)e^{-x^2-y^2}}{t^4\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = \frac{\log(2)}{64\pi} \tag{13}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^7 y \boldsymbol{H}_3(t)e^{-x^2-y^2}\log\left(\frac{t}{2xy}\right)}{t^4\left(\log^2\left(\frac{t}{2xy}\right)+\frac{\pi^2}{4}\right)} dxdydt = 0 \tag{14}$$

Proof. Use equation (10) and set $v = 3$ and rationalize the denominator and equate real and imaginary parts and simplify.

Example 6.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{xy \boldsymbol{H}_0(t)e^{-x^2-y^2}}{t\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = \frac{\log(2)}{8\pi} \tag{15}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{xy \boldsymbol{H}_0(t)e^{-x^2-y^2}\log\left(\frac{t}{2xy}\right)}{t\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = 0 \tag{16}$$

Proof. Use equation (10) and set $v = 0$ and rationalize the denominator and equate real and imaginary parts and simplify.

Example 7.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^3 y \boldsymbol{H}_1(t)e^{-x^2-y^2}}{t^2\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = \frac{\log(2)}{16\pi} \tag{17}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{4x^3 y \boldsymbol{H}_1(t)e^{-x^2-y^2}\log\left(\frac{t}{2xy}\right)}{t^2\left(4\log^2\left(\frac{t}{2xy}\right)+\pi^2\right)} dxdydt = 0 \tag{18}$$

Proof. Use equation (10) and set $v = 1$ and rationalize the denominator and equate real and imaginary parts and simplify.

6. Discussion

In this paper, we have presented a novel method for deriving a new integral transform in terms of the Struve function along with some interesting definite integrals with many more possible, using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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