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A quadruple integral involving the Legendre function $P_n(x)$ of the first kind: derivation and evaluation

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Abstract. A closed form expression of a quadruple integral involving the Legerndre polynomial $P_n(x)$ is derived. Special cases are expressed in terms of special functions and fundamental constants. All the results in this work are new.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

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1. Significance Statement

The Legendre functions are the most well known particular cases of the hypergeometric function. They have been discovered by Laplace and Legendre as early as in the 18th century. Later on their importance has grown substantially due to their connections with many problems of mathematical physics [3]. In this present work we investigate the quadruple integral involving the Legendre polynomial $P_n(x)$ and the invariance of the parameter n with respect to the Hurwitz-Lerch zeta function.

2. Introduction

In this paper we derive the quadruple definite integral given by

$$\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}P_v(x)\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{1}{2}(m-v-1)}\left(\frac{1}{y}\right)\log^{\frac{m+v}{2}}\left(\frac{1}{z}\right) \log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt \quad (1)$$

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where the parameters k, a, v, m are general complex numbers and $Re(v) < Re(m) < 1/2$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [6]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, z and t, then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of x, y, z and t and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [6]. The variable of integration in the contour integral is $u = w + m$. The cut and contour are in the first quadrant of the complex u-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing y by $\log\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)$ and multiplying by $x^{m-1}P_v(x)\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{1}{2}(m-v-1)}\left(\frac{1}{y}\right)\log^{\frac{m+v}{2}}\left(\frac{1}{z}\right)$ then taking the definite integral with respect to $x \in [0,1]$, $y \in 0,1]$, $z \in 0,1]$ and $t \in 0,1]$ to obtain

$$\frac{1}{\Gamma(k+1)}\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}P_v(x)\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{1}{2}(m-v-1)}\left(\frac{1}{y}\right)\log^{\frac{m+v}{2}}\left(\frac{1}{z}\right)$$
$$\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt$$

$$= \frac{1}{2\pi i}\int_0^1\int_0^1\int_0^1\int_0^1\int_C a^w w^{-k-1}P_v(x)x^{m+w-1}\log^{-m-w}\left(\frac{1}{t}\right)$$
$$\log^{\frac{1}{2}(m-v+w-1)}\left(\frac{1}{y}\right)\log^{\frac{1}{2}(m+v+w)}\left(\frac{1}{z}\right)dwdxdydzdt$$

$$= \frac{1}{2\pi i}\int_C\int_0^1\int_0^1\int_0^1\int_0^1 a^w w^{-k-1}P_v(x)x^{m+w-1}\log^{-m-w}\left(\frac{1}{t}\right)$$

$$\log^{\frac{1}{2}(m-v+w-1)}\left(\frac{1}{y}\right)\log^{\frac{1}{2}(m+v+w)}\left(\frac{1}{z}\right)dxdydzdtdw$$

$$=\frac{1}{2\pi i}\int_C \pi^{3/2}a^w w^{-k-1}2^{-m-w}\csc(\pi(m+w))dw \quad (3)$$

from equation (1.8.8.1) in [4] and equation (4.215.1) in [2] where $Re(\pi(m+w))>0$ and using the reflection formula (8.334.3) in [2] for the Gamma function. We are able to switch the order of integration over x, y, z and t using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,1]\times[0,1]\times[0,1]\times[0,1]$.

4. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [1] has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (4)$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \quad (5)$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

4.2. Infinite sum of the Contour Integral

Using equation (2) and replacing y by $\log(a)+i\pi(2y+1)-\log(2)$ then multiplying both sides by
$-i\pi^{3/2}2^{1-m}e^{i\pi m(2y+1)}$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of

the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)} i^{k-1}\pi^{k+\frac{3}{2}}e^{i\pi m}2^{k-m+1}\Phi\left(e^{2im\pi},-k,\frac{-i\log(a)+i\log(2)+\pi}{2\pi}\right)\\
&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C i\pi^{3/2}a^w w^{-k-1}2^{-m-w+1}e^{i\pi(2y+1)(m+w)}dw\\
&=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty} i\pi^{3/2}a^w w^{-k-1}2^{-m-w+1}e^{i\pi(2y+1)(m+w)}dw\\
&=\frac{1}{2\pi i}\int_C \pi^{3/2}a^w w^{-k-1}2^{-m-w}\csc(\pi(m+w))dw
\end{aligned} \tag{6}$$

from equation (1.232.3) in [2] where $Im(\pi(m+w))>0$ in order for the sum to converge.

5. Definite Integral in terms of the Lerch Function and invariant index forms

Theorem 1. *For all $k,a,v,m\in\mathbb{C}, Re(v)<Re(m)\leq 1/2$,*

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}P_v(x)\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{1}{2}(m-v-1)}\left(\frac{1}{y}\right)\log^{\frac{m+v}{2}}\left(\frac{1}{z}\right)\\
&\qquad\log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt\\
&\qquad= i^{k-1}\pi^{k+\frac{3}{2}}e^{i\pi m}2^{k-m+1}\Phi\left(e^{2im\pi},-k,\frac{-i\log(a)+i\log(2)+\pi}{2\pi}\right)
\end{aligned} \tag{7}$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

Example 1. *The degenerate case.*

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 x^{m-1}P_v(x)\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{1}{2}(m-v-1)}\left(\frac{1}{y}\right)\log^{\frac{m+v}{2}}\left(\frac{1}{z}\right)dxdydzdt\\
&\qquad=\pi^{3/2}2^{-m}\csc(\pi m)
\end{aligned} \tag{8}$$

Proof. Use equation (7) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [5].

Example 2. *The Hurwitz zeta function $\zeta(k,a)$, where the right-hand side is invariant with respect to v,*

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{\frac{1}{2}\left(-v-\frac{1}{2}\right)}\left(\frac{1}{y}\right)\log^{\frac{1}{2}\left(v+\frac{1}{2}\right)}\left(\frac{1}{z}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}} \log^k\left(\frac{ax\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt \tag{9}$$

$$= i^k 2^{k+\frac{1}{2}}\pi^{k+\frac{3}{2}}\left(2^k\zeta\left(-k,\frac{-i\log(a)+i\log(2)+\pi}{4\pi}\right) - 2^k\zeta\left(-k,\frac{1}{2}\left(\frac{-i\log(a)+i\log(2)+\pi}{2\pi}+1\right)\right)\right)$$

Proof. Use equation (7) and set $m=1/2$ and simplify using entry (4) in Table below (64:12:7) in [5].

Example 3. *The zeta function of Riemann $\zeta(k)$,*

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{-\frac{v}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)\log^{\frac{v}{2}+\frac{1}{4}}\left(\frac{1}{z}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}}\log^k\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)dxdydzdt = -i^k 2^{k+\frac{1}{2}}\left(2^{k+1}-1\right)\pi^{k+\frac{3}{2}}\zeta(-k) \tag{10}$$

Proof. Use equation 9 and set $a=-2$ and simplify using entry (2) in Table below (64:7) in [5].

Example 4. *The fundamental constant* $\log(2)$,

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{-\frac{v}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)\log^{\frac{v}{2}+\frac{1}{4}}\left(\frac{1}{z}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}\log\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}dxdydzdt = -i\sqrt{\frac{\pi}{2}}\log(2) \tag{11}$$

Proof. Use equation (10) and apply l'Hopital's rule as $k\to -1$ and simplify using equation (25.6.11) in [1].

Example 5. *Apéry's constant* $\zeta(3)$,

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{-\frac{v}{2}-\frac{1}{4}}\left(\frac{1}{y}\right)\log^{\frac{v}{2}+\frac{1}{4}}\left(\frac{1}{z}\right)}{\sqrt{x}\sqrt{\log\left(\frac{1}{t}\right)}\log^3\left(-\frac{2x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}dxdydzdt$$
$$= \frac{3i\zeta(3)}{16\sqrt{2}\pi^{3/2}} \tag{12}$$

Proof. Use equation (10) and set $k=-3$ and simplify.

Example 6.

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{-\frac{v}{2}-\frac{1}{2}}\left(\frac{1}{y}\right)\log^{\frac{v}{2}}\left(\frac{1}{z}\right)}{x\log\left(\frac{x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}$$
$$\left(x^m\log^{-m}\left(\frac{1}{t}\right)\log^{\frac{m}{2}}\left(\frac{1}{y}\right)\log^{\frac{m}{2}}\left(\frac{1}{z}\right)-x^p\log^{-p}\left(\frac{1}{t}\right)\log^{\frac{p}{2}}\left(\frac{1}{y}\right)\log^{\frac{p}{2}}\left(\frac{1}{z}\right)\right)dxdydzdt$$

$$= \sqrt{\pi}\left(2^{-p}e^{i\pi p}\Phi\left(e^{2ip\pi},1,\frac{\pi+i\log(2)}{2\pi}\right)-2^{-m}e^{i\pi m}\Phi\left(e^{2im\pi},1,\frac{\pi+i\log(2)}{2\pi}\right)\right) \tag{13}$$

Proof. Use equation (7) and form a second equation by replacing $m\to p$ and taking their difference and setting $k=-1, a=1$ and simplify.

Example 7.

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{P_v(x)\log^{-\frac{v}{2}-\frac{3}{8}}\left(\frac{1}{y}\right)\log^{\frac{v}{2}+\frac{1}{8}}\left(\frac{1}{z}\right)}{x^{3/4}\sqrt{\log\left(\frac{1}{t}\right)}\log\left(\frac{x\sqrt{\log\left(\frac{1}{y}\right)}\sqrt{\log\left(\frac{1}{z}\right)}}{\log\left(\frac{1}{t}\right)}\right)}$$
$$\left(\sqrt[4]{x}\sqrt[8]{\log\left(\frac{1}{y}\right)}\sqrt[8]{\log\left(\frac{1}{z}\right)}-\sqrt[4]{\log\left(\frac{1}{t}\right)}\right)dxdydzdt$$

$$= \sqrt{\frac{\pi}{2}}\left(\sqrt[4]{-2}\Phi\left(i,1,\frac{\pi+i\log(2)}{2\pi}\right)-i\Phi\left(-1,1,\frac{\pi+i\log(2)}{2\pi}\right)\right) \tag{14}$$

Proof. Use equation (13) and set $p=1/2, m=1/4$ and simplify.

6. Discussion

In this paper, we have presented a novel method for deriving a new integral involving the Legendre polynomial $P_n(x)$ along with some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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