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On inductive limit of commutative triples

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Abstract. In this paper, we extend Olshanski's work on Gelfand pairs to commutative triples. We introduce the notion of spherical triples as a generalization of commutative triples. We prove that inductive limit of an increasing sequence of commutative triples is a spherical triple which shows that the former is also a generalization of spherical pairs. Furthermore, we define spherical functions associated with these spherical triples. Finally, we characterize these spherical functions by a functional equation and we give some of its properties.

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1. Introduction

The notion of Gelfand pairs has been introduced by I. Gelfand in 1950 and developed by many authors. An extension of Gelfand pairs is the notion of commutative triples. The notion of commutative triples has been enough studied by many authors such as: R. Camporesi [8], J. Faraut [1], F. Ricci[9], I. Toure [11], etc.... It has permitted to establish a connection between harmonic analysis on non commutative locally compact groups and the theory of commutative Banach algebra. Indeed the spectrum of a commutative Banach subalgebra in the algebra(for the convolution product) of integrable functions is identified with functions defined on G, called δ-spherical functions which play the same role as exponential functions. This identification has allowed to define in the general case, the δ-spherical Fourier transform and to establish the majority of harmonic analysis results on $\mathbb{R}^n$. In 1980's, Olshanski ([4], [5], [6] has studied infinite dimensional unitary representations for pairs (G_∞, K_∞) which are inductive limit of Gelfand pairs (G_n, K_n), where $G_\infty = \cup_{n=1}^{\infty} G_n$ and $K_\infty = \cup_{n=1}^{\infty} K_n$. Olshanski has proved that inductive limit of increasing sequence of Gelfand pairs is a spherical pair and has given characterizations of spherical functions associated with these pairs. Some authors have also obtained some

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results about inductive limits of Gelfand pairs. We can mention Vershik[5], S. Kerov[2], J. Faraut [1], R. Marouane[3] etc .... For example in 2007, Rabaoui has proved a Bochner type theorem for pairs (G_∞, K_∞).
In this paper, we extend to commutative triples some Olshanski's results([4], [5], [6]. In second section, we give some definitions and notations which will be useful for well understanding of this paper. In the last section, we first extend to commutative triples the notion of spherical pairs namely spherical triples and we define spherical function for these triples. Then, we prove that inductive limit of a sequence of commutative triples is a spherical triple. Finally, we characterize spherical functions for these triples by a functional equation and we give some properties of these spherical functions.

2. Preliminaries

In this section, we give some notations and definitions for the well-understanding of this paper. Let G be a locally compact group and let K be a compact subgroup of G. G is equipped with a left Haar measure dx and K is equipped with its normalized Haar measure α. Let δ be a unitary irreducible representation of K and let us denote by E_δ the realization space of the representation δ. We put $End(E_\delta)$, the space of endomorphisms of E_δ and denote by $\mathcal{C}_c(G, End(E_\delta))$ the space of compactly supported continuous functions of G with values in $End(E_\delta)$. $\mathcal{C}_c(G, End(E_\delta))$ is a convolution algebra where the convolution is defined by: for $F, H \in \mathcal{C}_c(G, End(E_\delta))$ and $x \in G$,

$$F * H(x) = \int_G F(y^{-1}x)H(y)dy$$

We set,

$$\mathcal{C}_c(G, K, \delta, \delta) := \left\{F \in \mathcal{C}_c(G, End(E_\delta)) : F(kxk') = \delta(k'^{-1})F(x)\delta(k^{-1}) \forall k, k' \in K, \forall x \in G\right\},$$

the space of continuous δ-radial functions of G with compact support. $\mathcal{C}_c(G, K, \delta, \delta)$ is a subalgebra of the convolution algebra $\mathcal{C}_c(G, End(E_\delta))$. We say that (G, K, δ) is a commutative triple if the convolution algebra $\mathcal{C}_c(G, K, \delta, \delta)$ is commutative. If δ is the one dimensional trivial representation then we obtain the classical notion of Gelfand pairs. Let us put $\chi_\delta := d(\delta)\xi_\delta$, where $d(\delta)$ is the degree of δ and ξ_δ the character of δ. Let us denote by $\hat{G}$(resp.$\hat{K}$) the unitary dual of G(resp. K). For $U \in \hat{G}$, we denote by $mtp(\delta, U)$ the multiplicity of δ in $U_{|K}$. We know by ([9], theorem 1.1, page 4) that (G, K, δ) is commutative if and only if $mtp(\delta, U) \leq 1$ for all $U \in \hat{G}$. Let $\hat{G}(\delta)$ be the subset of $\hat{G}$ consisting of those $U \in \hat{G}$ that contains δ upon restriction to K. For $U \in \hat{G}(\delta)$ and $\mathcal{H}$ its realization space, we designate by $\mathcal{H}(\delta)$ the isotypic component of δ that is the subspace of vectors which transform under K according to δ. The projection P from $\mathcal{H}$ onto $\mathcal{H}(\delta)$ is defined by:

$$P = \int_K \chi_\delta(k^{-1})U(k)dk.$$

A function $\Phi : G \longrightarrow End(E_\delta)$ is said to be unitary if $\forall g \in G, \Phi(g)^* = \Phi(g^{-1})$, where $\Phi(g)^*$ designates the adjoint of $\Phi(g)$.

3. Spherical triples

In this section, we extend to commutative triples some of Olshanski's results. That will permit us to introduce the notion of spherical triples. We recall first the definition of an admissible representation for the pair (G, K). In fact, if G is a topological Hausdorff group (not necessary localement compact) and K a closed subgroup of G, a unitary representation of G is an admissible representation of the pair (G, K) if its restriction to K is a discrete direct sum of irreducible representations.

Definition 1. *Let G be a Hausdorff topological group, K be a closed subgroup of G and (δ, E_δ) a unitary irreducible representation of K.*
(G, K, δ) is a spherical triple if for any unitary irreducible admissible representation $(U, \mathcal{H})$ of G, $mtp(\delta, U_{|K}) \leq 1$.
If $mtp(\delta, U_{|K}) = 1$, U is called a δ-spherical representation.

Remark 1. *If G is a locally compact group and K is a compact subgroup of G, (G, K, δ) is a spherical triple if and only if (G, K, δ) is a commutative triple.*

Remark 2. *Let us also mention that the notion of spherical triples is a generalization of spherical pairs which corresponds to the case when δ is the one dimensional trivial representation. In fact, if $mtp(1_K, U_{|K}) \leq 1$ then $dim\mathcal{H}_K \leq 1$, where $\mathcal{H}_K$ is the space of K-invariant vectors of $\mathcal{H}$.*

In what follows, we define δ-spherical functions for spherical triples.
Let us mention that if $mtp(\delta, U_{|K}) = 1$ then $\mathcal{H}(\delta)$ is isomorphic to E_δ.

Definition 2. *Let (G, K, δ) be a spherical triple. A function $\Phi : G \longrightarrow End(E_\delta)$ is a δ-spherical if there exists a δ-spherical representation $(U, \mathcal{H})$ of G such that*

$$\Phi(g)u = PU(g^{-1})u, \forall g \in G, \forall u \in E_\delta,$$

where P is the orthogonal projection of $\mathcal{H}$ onto E_δ.

The following theorem gives some properties of δ-spherical functions.

Theorem 1. *Let $\Phi : G \longrightarrow End(E_\delta)$ be a δ-spherical function. Then*

i) *$\Phi(e) = I$, where I is the identity operator of E_δ.*

ii) *Φ is δ-radial.*

Proof. Let $\Phi : G \longrightarrow End(E_\delta)$ be a δ-spherical function.

i) Then there exits a δ-spherical representation $(U, \mathcal{H})$ of G such that $\Phi(g)u = PU(g^{-1})u$, $\forall g \in G, \forall u \in E_\delta$. $\forall v \in E_\delta$, $\Phi(e)v = PU(e)v = Pv = v$. Hence $\Phi(e) = I$.

ii) $\forall k_1, k_2 \in K, \forall x \in G$ and $\forall u \in E_\delta$,

$$\begin{aligned}\Phi(k_1 x k_2)u &= PU(k_2^{-1}x^{-1}k_1^{-1})u \\ &= PU(k_2^{-1})U(x^{-1})U(k_1^{-1})u \\ &= U(k_2^{-1})PU(x^{-1})U(k_1^{-1})u \\ &= \delta(k_2^{-1})PU(x^{-1})\delta(k_1^{-1})u \\ &= \delta(k_2^{-1})\Phi(x)\delta(k_1^{-1})u\end{aligned}$$

Hence Φ is a δ-radial function.

Now, let $G_1 \subseteq G_2 \subseteq ... \subseteq G_n \subseteq ...$ be an increasing sequence of locally compact groups such that for each n, G_n is a closed subgroup of G_{n+1}. We consider again $K_1 \subseteq K_2 \subseteq ... \subseteq K_n \subseteq ...$ an increasing sequence of compact groups such that for each n, K_n is a compact subgroup of G_n and $K_n = G_n \cap K_{n+1}$.
The family of pairs $(G_n, K_n)_{n\geq 1}$ that we consider, equipped with the system of canonical continuous embeddings $t_{m,n} : G_n \longrightarrow G_m, n, m \in \mathbb{N}^*$, $n \leq m$, constitutes an inductive countable system of topological groups. Hence, we can define the following inductive limit groups :

$$G_\infty = \bigcup_{n\geq 1} G_n \qquad K_\infty = \bigcup_{n\geq 1} K_n.$$

The topology defined on G_∞ is the inductive limit topology.
Let (E_{δ_n}) be an increasing sequence of Hilbert spaces and for any $n \geq 1$, let us consider a unitary representation (δ_n, E_{δ_n}) of K_n. For each $n \geq 1$, we consider an isometric embedding $i_n : E_{\delta_n} \longrightarrow E_{\delta_{n+1}}$ commuting with the action of K_n. Let us put

$$E_{\delta_\infty} = \overline{\bigcup_{n\in\mathbb{N}^*} E_{\delta_n}}$$

the Hilbert completion of $\bigcup_{n\in\mathbb{N}^*} E_{\delta_n}$. Then there exists a unique representation δ_∞ of G_∞ such that $\delta_\infty(k)u = \delta_n(k)u, \forall k \in K_n, \forall u \in E_{\delta_n}$. δ_∞ is the inductive limit of sequence of representations $(\delta_n)_{n\in\mathbb{N}^*}$.
$(G_\infty, K_\infty, \delta_\infty)$ will be called the inductive limit of $(G_n, K_n, \delta_n)_{n\in\mathbb{N}^*}$. In what follows, we assume that (G_n, K_n, δ_n) is a commutative triple for each $n \in \mathbb{N}^*$. We recall the notion of approximation of irreducible representations for inductive limits. Let us consider $(T, \mathcal{H})$ a unitary representation of G_∞ and $(T_n, \mathcal{H}_n)$ a sequence of unitary representations of groups G_n. Let us put

$$\Sigma = \{\xi_1, ..., \xi_s\} \subset \mathcal{H}$$

and

$$\Sigma_n = \{\xi_{1n}, ..., \xi_{sn}\} \subset \mathcal{H}_n, n = 1, 2, ...$$

We shall write $(T_n, \Sigma_n) \longrightarrow (T, \Sigma)$ if $(T_n(g)\xi_{in}, \xi_{jn})$ converges to $(T(g)\xi_i, \xi_j)$ uniformly on compact sets in G_∞ $(1 \leq i, j \leq s, g \in G_\infty)$.(The reader can refer to [6] for more details)
We say that the sequence (T_n) of unitary representations of groups G_n approximates the

unitary representation T of group G_∞ if for any finite subset $\Sigma \subset \mathcal{H}$, it is possible to select finite subsets $\Sigma_n \subset \mathcal{H}_n$ of the same cardinalitiy such that

$$(T_n, \Sigma_n) \longrightarrow (T, \Sigma).$$

We know by ([6], theorem 22.9, page 434) that, for any irreducible unitary representation T of the group G_∞, there exists a sequence (T_n) of irreducible unitary representations of groups G_n approximating T.

In the following theorem, we prove that $(G_\infty, K_\infty, \delta_\infty)$ is a spherical triple and we characterize δ_∞-spherical functions for $(G_\infty, K_\infty, \delta_\infty)$.

Theorem 2. *i) The inductive limit $(G_\infty, K_\infty, \delta_\infty)$ of an increasing sequence of commutative triples (G_n, K_n, δ_n) is a spherical triple.*

ii) A δ_∞-radial unitary function of positive type $\Phi : G_\infty \longrightarrow End(E_{\delta_\infty})$ is δ_∞-spherical if and only if

$$\Phi(e) = I_\infty$$

and

$$\forall x, y \in G_\infty, \Phi(y)\Phi(x) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xky)d\alpha_n(k),$$

where I_∞ is the identity operator of E_{δ_∞}.

The following lemmas are useful to prove this theorem.

Lemma 1. *Let $(\mathcal{H}_n)$ be an increasing sequence of subspaces of a Hilbert space $\mathcal{H}$. We put:*

$$\mathcal{H}_\infty = \overline{\bigcup_{n\geq 1} \mathcal{H}_n}$$

the Hilbert completion of $\bigcup_{n\geq 1} \mathcal{H}_n$ and $P : \mathcal{H} \longrightarrow \mathcal{H}_\infty$ the projection of $\mathcal{H}$ onto $\mathcal{H}_\infty$. For any n, we denote by $P_n : \mathcal{H} \longrightarrow \mathcal{H}_n$ the projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Then P_n strongly converges to P.

Proof. Let $v \in \mathcal{H}$, then $Pv \in \mathcal{H}_\infty$. Since $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is dense in $\mathcal{H}_\infty$, there exists a sequence $(v_m)_m \subset \bigcup_{n=1}^{\infty} \mathcal{H}_n$ such that $\lim_{m\to+\infty} ||v_m - Pv||_{\mathcal{H}} = 0$. Since $(\mathcal{H}_n)$ is an increasing sequence then for any $m \in \mathbb{N}^*$, there exists $n_m \in \mathbb{N}^*$ such that $\forall n \geq n_m, v_m \in \mathcal{H}_n$.
Let us fix n such that $n_m \leq n \leq m$. Since $v_m \in \mathcal{H}_n$, we have: $P_n v_m = v_m$. Hence

$$\begin{aligned} ||P_nPv - Pv||_{\mathcal{H}} &= ||P_nPv - P_nv_m + v_m - Pv||_{\mathcal{H}} \\ &\leq ||P_nPv - P_nv_m||_{\mathcal{H}} + ||v_m - Pv||_{\mathcal{H}} \\ &\leq (||P_n|| + 1)||Pv - v_m||_{\mathcal{H}} = 2||Pv - v_m||_{\mathcal{H}}. \end{aligned}$$

If $n \longrightarrow +\infty$ then $m \longrightarrow +\infty$. Since $\lim_{m\to+\infty} ||v_m - Pv||_{\mathcal{H}} = 0$ then $\lim_{n\to+\infty} ||P_nPv - Pv||_{\mathcal{H}} = 0$. Hence P_nP converges strongly to P. Consequently P_n converges strongly to P because for any $n \geq 1, P_nP = P_n$.

Lemma 2. *Let* $\Phi : G_\infty \longrightarrow End(E_{\delta_\infty})$ *be a function verifying* $\Phi(e) = I_\infty$ *and*

$$\forall x, y \in G_\infty, \Phi(y)\Phi(x) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xky)d\alpha_n(k).$$

Then

i) $I_\infty = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(k)d\alpha_n(k)$, *where* I_∞ *is the operator identity of* E_{δ_∞}.

ii) $\forall x \in G_\infty$,

$$\Phi(x) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(kx)d\alpha_n(k) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xk)d\alpha_n(k)$$

iii) $\forall x \in G_\infty$, $\forall k \in K_\infty$, $\Phi(xk) = \Phi(k)\Phi(x)$ *and* $\Phi(kx) = \Phi(x)\Phi(k)$.

Proof.

i) $I_\infty = \Phi(e)\Phi(e) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(k)d\alpha_n(k)$

ii) $\forall x \in G_\infty$, we have:

$$\Phi(x) = \Phi(e)\Phi(x) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xk)d\alpha_n(k).$$

In the same way

$$\forall x \in G_\infty, \Phi(x) = \Phi(x)\Phi(e) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(kx)d\alpha_n(k).$$

iii) $\forall k, k' \in K_n, \chi_{\delta_n}(kk') = \chi_{\delta_n}(k'k)$. Consequently $\forall x \in G_\infty$ and $\forall k \in K_\infty$, we have:

$$\begin{aligned}
\Phi(kx) &= \Phi(kx)\Phi(e) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1)\Phi(k_1kx)d\alpha_n(k_1) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1k^{-1})\Phi(k_1x)d\alpha_n(k_1) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k^{-1}k_1)\Phi(k_1x)d\alpha_n(k_1) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1)\Phi(kk_1x)d\alpha_n(k_1) \\
&= \Phi(x)\Phi(k)
\end{aligned}$$

$$\begin{aligned}
\Phi(xk) &= \Phi(e)\Phi(xk) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1)\Phi(xkk_1)d\alpha_n(k_1)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k^{-1}k_1)\Phi(xk_1)d\alpha_n(k_1) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1k^{-1})\Phi(xk_1)d\alpha_n(k_1) \\
&= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k_1)\Phi(xk_1k)d\alpha_n(k_1) \\
&= \Phi(k)\Phi(x)
\end{aligned}
$$

We come back to the proof of theorem 2.
Proof.

i) Let $(G_\infty, K_\infty, \delta_\infty)$ be the inductive limit of an increasing sequence of commutative triples (G_n, K_n, δ_n). Let us consider $(\pi, \mathcal{H})$ a unitary irreducible admissible representation of G_∞ and $\mathcal{H}(\delta_\infty)$ the isotypic component of δ_∞ in $\mathcal{H}$. Let us assume that $mtp(\delta_\infty, \pi_{|K_\infty}) > 1$. Let ξ_1, ξ_2 be two non-zero vectors of $\mathcal{H}(\delta_\infty)$. Let us denote by

$$E_{\xi_1} = \overline{\langle \delta_\infty(k)\xi_1, k \in K_\infty \rangle},$$

the Hilbert completion of vector subspace generated by the set $\{\delta_\infty(k)\xi_1, k \in K_\infty\}$ and

$$E_{\xi_2} = \overline{\langle \delta_\infty(k)\xi_2, k \in K_\infty \rangle}$$

the Hilbert completion of vector subspace generated by the set $\{\delta_\infty(k)\xi_2, k \in K_\infty\}$. E_{ξ_1} and E_{ξ_2} are two distinct copies of E_{δ_∞} in $\mathcal{H}$. Since $(\pi, \mathcal{H})$ is a unitary irreducible representation of G_∞ then by ([6], theorem 22.9, page 434), there exists a sequence $(\pi_n, \mathcal{H}_n)$ of unitary irreducible representations of groups G_n approximating $(\pi, \mathcal{H})$. Since $\{\xi_1, \xi_2\} \subset \mathcal{H}$ then there exists a sequence $\{\xi_1^n, \xi_2^n\} \subset \mathcal{H}_n$ such that

$$(\pi_n(k)\xi_i^n, \xi_j^n) \underset{n\to\infty}{\longrightarrow} (\pi(k)\xi_i, \xi_j), \forall k \in K_\infty, \forall i, j \in \{1; 2\}$$

uniformly on compact sets. Hence

$$(\pi_n(k)\xi_i^n, \xi_j^n) \underset{n\to\infty}{\longrightarrow} (\delta_\infty(k)\xi_i, \xi_j), \forall k \in K_\infty, \forall i, j \in \{1; 2\}.$$

In particular, if $i = j$, we have:

$$(\pi_n(k)\xi_i^n, \xi_i^n) \underset{n\to\infty}{\longrightarrow} (\delta_\infty(k)\xi_i, \xi_i).$$

Let us put $h_i^n = \xi_i^n - P_{\mathcal{H}_n(\delta_n)}(\xi_i^n)$, where $\mathcal{H}_n(\delta_n)$ is the isotypic component of δ_n in $\mathcal{H}_n$ and $P_{\mathcal{H}_n(\delta_n)}$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_n(\delta_n)$. Since for any $n, (G_n, K_n, \delta_n)$ is a commutative triple then $\forall n, E_{\delta_n} \simeq \mathcal{H}_n(\delta_n)$. Since (E_{δ_n}) is an increasing sequence of vector spaces then the sequence $(\mathcal{H}_n(\delta_n))$ is also increasing. Then by the lemma 1, $P_{\mathcal{H}_n(\delta_n)}$ converges strongly to $P_{\mathcal{H}_\infty}$, where $P_{\mathcal{H}_\infty}$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_\infty$

and $\mathcal{H}_\infty$ is the Hilbert completion of $\bigcup_{n\geq 1}\mathcal{H}_n(\delta_n)$.
For n sufficiently large, we have

$$(\pi_n(k)h_i^n, h_i^n) = (\pi_n(k)\xi_i^n, \xi_i^n) - (\pi_n(k)\xi_i^n, P_{\mathcal{H}_n(\delta_n)}(\xi_i^n)) - \\ (\pi_n(k)P_{\mathcal{H}_n(\delta_n)}(\xi_i^n), \xi_i^n) + (\pi_n(k)P_{\mathcal{H}_n(\delta_n)}(\xi_i^n), P_{\mathcal{H}_n(\delta_n)}(\xi_i^n)).$$

Then

$$(\pi_n(k)h_i^n, h_i^n)\underset{n\to\infty}{\longrightarrow}(\delta_\infty(k)\xi_i, \xi_i) - (\pi(k)\xi_i, P_{\mathcal{H}(\delta_\infty)}(\xi_i)) - \\ (\pi(k)P_{\mathcal{H}(\delta_\infty)}(\xi_i), \xi_i) + (\pi(k)P_{\mathcal{H}(\delta_\infty}(\xi_i), P_{\mathcal{H}(\delta_\infty)}(\xi_i)).$$

Since $P_{\mathcal{H}(\delta_\infty)}(\xi_i) = \xi_i$ then $(\delta_\infty(k)\xi_i, \xi_i) - (\pi(k)\xi_i, P_{\mathcal{H}(\delta_\infty)}(\xi_i)) - (\pi(k)P_{\mathcal{H}(\delta_\infty)}(\xi_i), \xi_i) + (\pi(k)P_{\mathcal{H}(\delta_\infty}(\xi_i), P_{\mathcal{H}(\delta_\infty)}(\xi_i)) = 0$. Hence $(\pi_n(k)h_i^n, h_i^n)\underset{n\to\infty}{\longrightarrow}0$. In particular, if $k = e$, we have $||h_i^n||\underset{n\to\infty}{\longrightarrow}0$. Therefore for n sufficiently large, ξ_i^n is arbitrarily close to $P_{\mathcal{H}_n(\delta_n)}(\xi_i^n)$. So that we can assume for n sufficiently large, $\xi_i^n \in \mathcal{H}_n(\delta_n)$. Let us put

$$E_i^n = \overline{\langle \pi_n(k)\xi_i^n, k \in K_n\rangle}, i = 1, 2,$$

the Hilbert completion of the vector subspace generated by the family $\{\pi_n(k)\xi_i^n, k \in K_n\}$. For any i, E_i^n is copy of E_{δ_n} in $\mathcal{H}_n$. Let us suppose that there exists $k, k_1 \in K_n$ such that

$$\xi^n = \pi_n(k)\xi_1^n = \pi_n(k_1)\xi_2^n.$$

$$\begin{aligned}(\xi^n, \xi^n) &= (\pi_n(k)\xi_1^n, \pi_n(k_1)\xi_2^n) \\ &= (\pi_n(k_1^{-1}k)\xi_1^n, \xi_2^n)\underset{n\to\infty}{\longrightarrow}(\delta_\infty(k)\xi_1, \xi_2) = 0.\end{aligned}$$

Consequently $||\xi^n||\underset{n\to\infty}{\longrightarrow}0$. So E_1^n and E_2^n are distinct. It follows that $\mathcal{H}_n$ contains two distinct copies of E_{δ_n}, which is absurd because (G_n, K_n, δ_n) is a commutative triple for any n and so the multiplicity of δ_n in $\pi_{|K_n}$ is at most 1.

ii) Let us assume that Φ is a δ_∞-spherical function. Then there exists a δ_∞-spherical representation $(U, \mathcal{H})$ of G_∞ such that $\forall g \in G, \forall u \in E_{\delta_\infty}$,

$$\Phi(g)u = PU(g^{-1})u,$$

where P is the projection of $\mathcal{H}$ onto E_{δ_∞}. For any $n \geq 1$, $P_n = \int_{K_n}\chi_{\delta_n}(k^{-1})U(k)d\alpha_n(k)$ is the projection of $\mathcal{H}$ onto E_{δ_n}, where α_n is the normalized Haar measure on K_n. $\forall x, y \in G$, and $\forall v \in E_\delta$,

$$\begin{aligned}\Phi(y)\Phi(x)v &= PU(y^{-1})PU(x^{-1})v \\ &= \lim_{n\to\infty} PU(y^{-1})P_nU(x^{-1})v \\ &= \lim_{n\to\infty} P\int_{K_n}\chi_{\delta_n}(k^{-1})U(y^{-1}kx^{-1})d\alpha_n(k)v\end{aligned}$$

$$= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k^{-1}) PU(y^{-1}kx^{-1}) d\alpha_n(k) v$$
$$= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k^{-1}) \Phi(xk^{-1}y) d\alpha_n(k) v$$
$$= \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k) \Phi(xky) d\alpha_n(k) v$$

Hence $\forall x, y \in G_\infty$,

$$\Phi(y)\Phi(x) = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xky)d\alpha_n(k).$$

Conversely, let us assume that Φ is a δ_∞-radial unitary function of positive type verifying $\Phi(e) = I_\infty$ and $\lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(xky)d\alpha_n(k) = \Phi(y)\Phi(x)$. Let $v \in E_{\delta_\infty}$. Let us put $l(x) = (\Phi(x)v, v), \forall x \in G_\infty$.
Since Φ is of positive type, l is also of positive type. Hence there exists a unitary representation $(U^l, \mathcal{H}^l)$ with a cylic vector ξ^l such that

$$\forall x \in G_\infty, (\Phi(x)v, v) = l(x) = (\xi^l, U^l(x)\xi^l).$$

We know by the lemma 2 *iii)*, $\Phi(k_1xk_2) = \Phi(k_2)\Phi(x)\Phi(k_1), \forall k_1, k_2 \in K_\infty, \forall x \in G_\infty$ and by the lemma 2 *i)*,

$$I_\infty = \lim_{n\to\infty} \int_{K_n} \chi_{\delta_n}(k)\Phi(k)d\alpha_n(k).$$

So we have:

$$\begin{aligned}
(\Phi(x)v, v) &= \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{K_n}\int_{K_m} \chi_{\delta_n}(k_1)\chi_{\delta_m}(k_2^{-1})(\Phi(x)\Phi(k_1)v, \Phi(k_2)v)d\alpha_n(k_1)d\alpha_m(k_2) \\
&= \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{K_n}\int_{K_m} \chi_{\delta_n}(k_1)\chi_{\delta_m}(k_2^{-1})(\Phi(k_2^{-1})\Phi(x)\Phi(k_1)v, v)d\alpha_n(k_1)d\alpha_m(k_2) \\
&= \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{K_n}\int_{K_m} \chi_{\delta_n}(k_1)\chi_{\delta_m}(k_2^{-1})(\Phi(k_1xk_2^{-1})v, v)d\alpha_n(k_1)d\alpha_m(k_2) \\
&= \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{K_n}\int_{K_m} \chi_{\delta_n}(k_1)\chi_{\delta_m}(k_2^{-1})(\xi^l, U^l(k_1xk_2^{-1})\xi^l)d\alpha_n(k_1)d\alpha_m(k_2) \\
&= \lim_{\substack{n\to\infty \\ m\to\infty}} \int_{K_n}\int_{K_m} \chi_{\delta_n}(k_1)\chi_{\delta_m}(k_2^{-1})(U^l(k_1^{-1}), U^l(x)U^l(k_2^{-1})\xi^l)d\alpha_n(k_1)d\alpha_m(k_2) \\
&= \lim_{\substack{n\to\infty \\ m\to\infty}} (U^l(x^{-1})P_n\xi^l, P_m\xi^l) \\
&= (U^l(x^{-1})P\xi^l, P\xi^l) = (PU^l(x^{-1})P\xi^l, \xi^l)
\end{aligned}$$

Therefore we can assume $\xi^l \in \mathcal{H}^l(\delta_\infty)$ by changing $\mathcal{H}^l$ by the subspace generated by $P\xi^l$. Let us put

$$\Phi^{U^l}(x) = PU^l(x^{-1})P, \forall x \in G_\infty.$$

Hence

$$(\Phi(x)v, v) = (\Phi^{U^l}(x)\xi^l, \xi^l), \forall x \in G_\infty \tag{3.1}$$

Thanks to Lemma 2 *ii*), we have $\Phi(k) = \delta(k^{-1}), \forall k \in K_\infty$. Thus any vector u of E_∞ can be expressed as a linear combination $\sum_i \Phi(k_i)v$, where $k_i \in K_\infty, \forall i$.
Let $T : E_{\delta_\infty} \longrightarrow \mathcal{H}^l(\delta_\infty)$ be the function defined by:

$$T\left(\sum_i a_i\Phi(k_i)v\right) = \sum_i a_i\Phi^{U^l}(k_i)\xi^l,$$

where $a_i \in \mathbb{C}$, and $k_i \in K_\infty, \forall i$. Let us show that T is well-defined. For that we assume that $\sum_i a_i\Phi(k_i)v = 0$. By the equation (3.1), we have

$$\chi_{\delta_n}(k)(\Phi(ykx)v, v) = \chi_{\delta_n}(k)(\Phi^{U^l}(ykx)\xi^l, \xi^l), \forall x, y \in G_\infty, k \in K_n. \tag{3.2}$$

By applying integral and limit on both sides of the equation (3.2), we get

$$(\Phi(x)\Phi(y)v, v) = (\Phi^{U^l}(x)\Phi^{U^l}(y)\xi^l, \xi^l), \forall x, y \in G_\infty.$$

Hence

$$(\Phi(y)v, \Phi(x^{-1})v) = (\Phi^{U^l}(y)\xi^l, \Phi^{U^l}(x^{-1})\xi^l), \forall x, y \in G_\infty. \tag{3.3}$$

We deduce that

$$0 = (\sum_i a_i\Phi(k_i)v, \sum_j b_j\Phi(c_j^{-1})v) = (\sum_i a_i\Phi^{U^l}(k_i)\xi^l, \sum_j b_j\Phi^{U^l}(c_j^{-1})\xi^l),$$

where $c_j, k_i \in K_\infty$ and $a_i, b_j \in \mathbb{C}; \forall i, \forall j$. Hence $T\left(\sum_i a_i\Phi(k_i)v\right) = \sum_i a_i\Phi^{U^l}(k_i)\xi^l = 0$. Thus T is well defined. T is obviously linear. Now, let's show that T is bijective. For that we assume that $T\left(\sum_i a_i\Phi(k_i)v\right) = \sum_i a_i\Phi^{U^l}(k_i)\xi^l = 0$. Then we claim that $\sum_i a_i\Phi(k_i)v = 0$. By the equation (3.3), we have

$$(\Phi(y)v, \Phi(x^{-1})v) = (\Phi^{U^l}(y)\xi^l, \Phi^{U^l}(x^{-1})\xi^l), \forall x, y \in G_\infty.$$

Hence

$$0 = (\sum_i a_i\Phi(k_i)v, \sum_j b_j\Phi(c_j^{-1})v) = (\sum_i a_i\Phi^{U^l}(k_i)\xi^l, \sum_j b_j\Phi^{U^l}(c_j^{-1})\xi^l),$$

where $c_j, k_i \in K_\infty, \forall i, \forall j$. Then $\sum_i a_i\Phi(k_i)v = 0$. Consequently T is bijective.
In other hand, we have

$$(T\sum_i a_i\Phi(k_i)v, T\sum_j b_j\Phi(c_j)v) = (\sum_i a_i\Phi^{U^l}(k_i)\xi^l, \sum_j b_j\Phi^{U^l}(c_j)\xi^l)$$

$$= \sum_i \sum_j a_i \overline{b_j} (\Phi^{U^l}(c_j^{-1}k_i)\xi^l, \xi^l)$$
$$= \sum_i \sum_j a_i \overline{b_j} (\Phi(c_j^{-1}k_i)v, v) = (\sum_i a_i \Phi(k_i)v, \sum_j b_j \Phi(c_j)v)$$

Then T is a unitary linear isomorphism.
For any $g \in G_\infty$, we have

$$\begin{aligned} T\Phi(g)\sum_i a_i\Phi(k_i)v &= \lim_{n\to\infty}\sum_i a_i \int_{K_n} \chi_{\delta_n}(k)T\Phi(k_ikg)v d\alpha_n(k) \\ &= \lim_{n\to\infty}\sum_i a_i \int_{K_n} \chi_{\delta_n}(k)\Phi^{U^l}(k_ikg)\xi^l d\alpha_n(k) \\ &= \Phi^{U^l}(g)\sum_i a_i\Phi^{U^l}(k_i)\xi^l \\ &= \Phi^{U^l}(g)T\sum_i a_i\Phi(k_i)v, \end{aligned}$$

Consequently Φ is unitarily equivalent to Φ^{U^l} which is a δ_∞-spherical function. Therefore Φ is a δ_∞-spherical function.

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