EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 1, 2022, 144-157
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Periodic Solution of Caputo-Fabrizio Fractional Integro–differential Equation with Periodic and Integral Boundary Conditions

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Abstract. In this paper, we study a new approach of investigation of existence, uniqueness and stability of the periodic solution of the nonlinear fractional integro-differential equation of type Caputo-Fabrizio fractional derivative with the initial condition, periodic boundary conditions, and integral boundary conditions by using successive approximations method and Banach fixed point theorem. Finally, some examples are present to illustrate the theorems.

2020 Mathematics Subject Classifications: 34A08, 26A33, 34G20, 34C25, 45J05

Key Words and Phrases: Caputo- Fabrizio fractional derivative, integro-differential equation, periodic and integral boundary conditions, Periodic solutions, successive approximation method, Banach fixed point theorem

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1. Introduction

Fractional differential equations have been recognized in the last decade as important tools to describe the mathematical modeling of processes in the fields of physics, chemistry, engineering, statistics, aerodynamics, control theory, signal and image processing, etc.[10, 11, 13]. On the other hand, we observe periodic motions in every field of science and everywhere in real life [6]. The theory and applications of the fractional differential equations have recently been addressed by several researchers for a variety of problems, which we refer the reader to [1, 2, 4]. We mention here some of these definitions, such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Weyl, Riesz, Erdélyi-Kober, and Caputo. Compared with an integer order, a significant feature of a fractional order differential operator appeared in its hereditary property. In other words, when we describe a process by a fractional operator, we predict the future state by its current as well as its past states [14, 16].

However, the new definition suggested by Caputo and Fabrizio [5], which has all the characteristics of the old definitions, assumes two different representations for the temporal and spatial variables.They claimed that the classical definition given by Caputo

DOI: https://doi.org/10.29020/nybg.ejpam.v15i1.4247

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appears to be particularly convenient for mechanical phenomena, related to plasticity, fatigue, damage, and with electromagnetic hysteresis. The main advantage of the Caputo-Fabrizio approach is that the boundary conditions of the fractional differential equations with Caputo-Fabrizio derivatives admit the same form as for the integer-order differential equations. On the other hand, the Caputo-Fabrizio fractional derivative has many significant properties, such as its ability in describing matter heterogeneities and configurations with different scales [12, 20].

In [21], we have the analytic solutions of a viscous fluid with the Caputo and Caputo-Fabrizio fractional derivatives. In [8], the authors used the fractional derivative with a nonsingular kernel to model a Maxwell fluid and found semianalytical solutions. In [22], we found a comparison approach of two latest fractional derivatives models, namely, Atangana-Baleanu and Caputo- Fabrizio, for a generalized Casson fluid and obtained exact solutions. Due to the abovementioned applications, the existence of solutions for nonlinear differential equations is an attractive research topic and has been studied using different techniques of nonlinear analysis [9, 18]. One of the most important theorems in ordinary differential equations is Picard's existence and uniqueness theorem. This theorem, which is applied on first-order ordinary differential equations, can be generalized to establish existence and uniqueness results for both higher-order ordinary differential equations and systems of differential equations [3, 7, 15, 17].

In this paper, we investigate the existence and approximate periodic solution of the following nonlinear fractional integro-differential equation:

$$ {}_{0}^{CF}D_t^{\alpha}(u(t)) = h\left(t, u(t), \int_0^{a(t)} \mathrm{g}(\mathrm{s}, u(s))ds\right) \quad (1.1) $$

such that $t \in J = [0, T]$, with the initial condition $\mathrm{u}(0) = \mathrm{u}_0$, where ${}_{0}^{\mathrm{CF}}D_t^{\alpha}$ denotes the Caputo-Fabrizio fractional derivative ($\alpha \in (0, 1]$). We extend Picard's theorem to this problem, and by the successive approximation method, an iterative process is provided to obtain the periodic solution.

2. Preliminaries

In this section, we recall some notations and definitions which are needed throughout this paper. Further, some lemmas and theorems are stated as preparations for the main results. First, in the following, we provide some basic concepts and definitions in connection with the new Caputo-Fabrizio derivative.
Let $H^1(a, b) = \{g | g \in L^2(a, b), g' \in L^2(a, b)\}$, where $L^2(a, b)$ is the space of square integrable functions on the interval (a, b).

Definition 1. *[10] For a function $g : (0, \infty) \to \mathbb{R}$, the Caputo derivative of order $\alpha > 0$ of g is defined by*

$$ {}_{0}^{t}D^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds \quad (2.1) $$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α, *and* $\Gamma(.)$ *denotes the Gamma function, i.e.,* $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

Definition 2. *[10] Let* g *be a function which is defined almost everywhere a.e on* $[a, b]$, *for* $\alpha > 0$, *we define*

$$ {}_a^b D^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{a-1} g(t) dt \quad (2.2) $$

provided that the integral (Lebesgue)exists.

Definition 3. *[5] Let* g *be a given function in* $H^1(a, b)$. *The Caputo-Fabrizio derivative of fractional order* $\alpha \in (0, 1)$ *is defined as*

$$ {}_a^{CF} D_{\mathrm{t}}^{\alpha}(g(t)) = \left(\frac{N(\alpha)}{1-\alpha}\right) \int_a^t g'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \quad (2.3) $$

where $N(\alpha)$ *is a normalization function. Also, if a certain function* g *does not satisfy in the restriction* $g \in H^1(a, b)$, *then its fractional derivative is redefined as*

$$ {}_a^{CF} D_t^{\alpha}(g(t)) = \frac{\alpha N(\alpha)}{1-\alpha} \int_a^t (g(t) - g(x)) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \quad (2.4) $$

Clearly, if one sets $\sigma = (1-\alpha)/\alpha \in (0, \infty)$ *and* $\alpha = 1/(1+\sigma) \in (0, 1)$, *then the Caputo-Fabrizio definition becomes*

$$ {}_a^{CF} D_t^{e}(g(t)) = \frac{N(\sigma)}{\sigma} \int_a^t g'(x) \exp\left[-\frac{t-x}{\sigma}\right] dx \quad (2.5) $$

where $N(0) = N(\infty) = 1$, *and*

$$ \lim_{\sigma \to 0} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \quad (2.6) $$

Also, the fractional derivative of order $(n+\alpha)$ *when* $n \geq 1$ *and* $\alpha \in [0, 1]$ *is defined by the following*

$$ {}_a^{CF} D_t^{(a+n)}(g(t)) = \alpha^{CF} D_t^{(a)} \left(D_t^{(n)} g(t)\right) \quad (2.7) $$

Definition 4. *[5] Let* $g \in H^1(a, b)$, *then its fractional integral of an arbitrary order is defined as follows:*

$$ {}_\alpha C_t^{x}(g(t)) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t) + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_a^t g(s) ds, t \geq 0 \quad (2.8) $$

It is dear, in view of the abowe definition, that the α *th Caputo-Fabrizio derivative of function* g *is average between* g *and its first-order integral. Therefore,*

$$ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)} = 1 \quad (2.9) $$

So, we arrive at the following

$$ N(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1 \quad (2.10) $$

Definition 5. *The periodic solution of the fractional integro-differential equation (1.1), with initial condition $u(0) = u_0$ and periodic boundary condition $u(0) = u(T)$ are defining the following integral equation*

$$u\left(t,u_0\right) = \mathrm{u}_0 + \tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h(t,u(t),\textstyle\int_0^{a(t)} \mathrm{g}(\mathrm{s},u(s))ds)$$
$$-\,(\tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \tfrac{1}{\mathrm{T}} \textstyle\int_0^T h(s,u(s),\int_0^{a(s)} \mathrm{g}(\tau,u(\tau))d\tau)ds + \tfrac{2\alpha}{(2-\alpha)N(\alpha)}$$
$$\textstyle\int_0^t (h(s,u(s),\int_0^{a(s)} \mathrm{g}(\tau,u(\tau))d\tau) - \tfrac{1}{\mathrm{T}} \int_0^T h(s,u(s),\int_0^{a(s)} \mathrm{g}(\tau,u(\tau))d\tau)ds)ds \quad (2.11)$$

For all $t \in J$.

Lemma 1. *[19] Let $g(t)$ be a vector function which is defined in the interval $0 \le t \le T$, then:*

$$\left|\int_0^t \left(g(s) - \frac{1}{T}\int_0^T g(s)ds\right)ds\right| \le \beta(t)M, \quad (2.12)$$

where $M = \max_{t\in[0,T]} |g(t)|$ and $\beta(t) = 2t\left(1 - \frac{t}{T}\right), \max_{t\in[0,T]} |\beta(t)| \le \frac{\mathrm{T}}{2}$.
The proof follows directly from the estimate:

$$\left|\int_0^t \left(g(s) - \frac{1}{T}\int_0^T g(s)ds\right)ds\right| \le \left(1 - \frac{t}{T}\right)\int_0^t |g(s)|ds + \frac{t}{T}\int_t^T |g(s)|ds$$
$$\le \beta(t)M$$

Theorem 1. *[10] (Banach fixed point theorem). Let $(E, \|.\|)$ be a Banach space and $P: E \to E$ be a contraction mapping i.e. Lipchitz continuous with Lipchitz constant $L \in [0,1)$. Then $\varphi \in E$ has a unique fixed point.*

3. Conditions for Convergence of Successive Approximation

Some conditions are needed for investigate of the successive approximation for periodic solution of the problem (1.1) with $\mathrm{u}(0) = \mathrm{u}_0$, suppose that the functions $h \in C([0,T] \times D_1 \times D_2, \mathbb{R})$,$g \in C([0,T] \times D_1, \mathbb{R})$, D_1 and D_2 are compact subset of $\mathbb{R}$, $a(t)$ is continuous functions on $[0,T]$, moreover define $|.| = \max_{t\in[0,T]} |.|$, and satisfies the following hypothesis.

H_1 There exist positive constants M, L, k_1, k_2,and L_1, such that

$$|h(t,u,z)| \le M \quad (3.1)$$
$$|g(t,u)| \le L \quad (3.2)$$
$$|h(t,u_1,z_1) - h(t,u_2,z_2)| \le k_1|u_1 - u_2| + k_2|z_1 - z_2| \quad (3.3)$$
$$|g(t,u_1) - g(t,u_2)| \le L_1|u_1 - u_2| \quad (3.4)$$

where $z_i = \int_0^{a(t)} \mathrm{g}(\mathrm{s},u_i(s))\,ds$ and for all $t \in [0,T]$, $u, u_1, u_2 \in D_1$ and $z_i \in D_2$, $i = 1,2$
H_2 : There exist positive constants a_T, such that for $t \in [0,T]$,

$$|a(t)| \le a_T \quad (3.5)$$

Define the non-empty set

$$\mathrm{D_h} = \mathrm{D_1} - M_1 \quad (3.6)$$

where

$$M_1 = \left(2(1-\alpha) + \frac{\alpha T}{2}\right) M$$

Furthermore, we suppose that the following condition is valid:

$$\Lambda = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_1 k_2) < 1 \quad (3.7)$$

4. Main Results

Our main results separate to the following parts:

4.1. Approximation of Periodic Solution of (1.1)

In this section, we study the periodic approximation solutions of nonlinear fractional integro-differential equations (1.1) with $\mathrm{u}(0) = \mathrm{u_0}$. In the beginning, we define the following sequence of functions $\{\mathrm{u_{m+1}}\}_{m=0}^{\infty}$ given by the iterative formulas

$$u_{m+1}(t, u_0) = u_0 + \tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h(t, u_m(t), \textstyle\int_0^{a(t)} g(s, u_m(s))\, ds)$$
$$-\tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \tfrac{1}{T} \textstyle\int_0^T h(s, u_m(s), \int_0^{a(s)} g(\tau, u_m(\tau)) d\tau)\, ds + \tfrac{2\alpha}{(2-\alpha)N(\alpha)}$$
$$\textstyle\int_0^t (h(s, u_m(s), \int_0^{a(s)} \mathrm{g}(\tau, u_m(\tau))\, d\tau) - \tfrac{1}{\mathrm{T}} \int_0^T h(s, u_m(s), \int_0^{a(s)} \mathrm{g}(\tau, u_m(\tau))\, d\tau) ds) ds \quad (4.1)$$

For all $t \in J, u_0(t) = u_0, m = 0, 1, 2, \ldots,$
then will be introduced by the following theorems.

Theorem 2. *If the nonlinear fractional integro-differential equation* (1.1) *with* $\mathrm{u}(0) = \mathrm{u_0}$ *satisfy the conditions H_1, and H_2, then the sequence of functions* (4.1), *which are periodic in t of period T, converges uniformly as $m \to \infty$ on the domain:-*

$$(\mathrm{t}, \mathrm{u_0}) \in [0, T] \times D_1 \quad (4.2)$$

to the limit functions u_θ defined on the domain (4.2) which is periodic in t of period T and satisfies the following integral equations:

$$u(t, u_0) = \mathrm{u_0} + \tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h(t, u(t), \textstyle\int_0^{a(t)} \mathrm{g}(\mathrm{s}, u(s)) ds)$$
$$-\,(\tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \tfrac{1}{\mathrm{T}} \textstyle\int_0^T h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau)) d\tau) ds + \tfrac{2\alpha}{(2-\alpha)N(\alpha)}$$
$$\textstyle\int_0^t (h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau)) d\tau) - \tfrac{1}{\mathrm{T}} \int_0^T h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau)) d\tau) ds) ds \quad (4.3)$$

on the domain (4.2), provided that

$$|u(t, u_0) - u_{m+1}(t, u_0)| \le \Lambda^{\mathrm{m}} (\mathrm{E} - \Lambda)^{-1} \mathrm{M_1} \quad (4.4)$$

for all $\mathrm{m} \geq 0, \mathrm{u}_0 \in \mathrm{D}$, and $\mathrm{t} \in \mathrm{J}$
Proof. Setting $\mathrm{m}=0$ in the sequence of functions (4.1) and by using Lemma 1, we have

$$\begin{aligned}\left|u_1\left(t, u_0\right)-u_0\right| & \leq\left(\frac{4(1-\alpha)}{(2-\alpha) N(\alpha)}+\frac{2 \alpha}{(2-\alpha) N(\alpha)} \beta(t)\right) M \\ & \leq\left(2(1-\alpha)+\frac{\alpha T}{2}\right) M=M_1\end{aligned}$$

for all $t \in[0, T], u_0 \in D_h$ we get $u_1\left(t, u_0\right) \in D_1$. Thus by mathematical induction, we find that

$$\left|u_m\left(t, u_0\right)-u_0\right| \leq \mathrm{M}_1 \quad (4.5)$$

mean that for all $t \in[0, T], u_0 \in D_h$ we get $u_m\left(t, u_0\right) \in D_1, m=0,1,2, \ldots$
Now, we claim that the sequences of functions (4.1) are uniformly convergent on the domain (4.2). By the inequalities (3.3)-(3.5), we obtain

$$\begin{aligned}\left|u_{m+1}\left(t, u_0\right)-u_m\left(t, u_0\right)\right| \leq & \left(2(1-\alpha)+\frac{\alpha T}{2}\right)\left(k_1+a_T L_1 k_2\right) \\ & \left|u_m\left(t,, u_0\right)-u_{m-1}\left(t,, u_0\right)\right| \\ & =\Lambda\left|u_m\left(t,, u_0\right)-u_{m-1}\left(t_r, u_0\right)\right| \quad (4.6)\end{aligned}$$

By mathematical induction, we obtain that

$$\left|u_{m+1}\left(t, \mathrm{u}_0\right)-u_m\left(t, \mathrm{u}_0\right)\right| \leq \Lambda^{\mathrm{m}}\left|u_1\left(t, \mathrm{u}_0\right)-\mathrm{u}_0\right| \quad (4.7)$$

Now from $\mathrm{m}=1,2, \ldots$ and $\mathrm{p} \geq 1$, we find that

$$\begin{aligned}\left|u_{m+p}\left(t, \mathrm{u}_0\right)-u_m\left(t, \mathrm{u}_0\right)\right| & \leq \Lambda^{\mathrm{m}}(1-\Lambda)^{-1}\left(\left(2(1-\alpha)+\frac{\alpha T}{2}\right) M\right) \\ & \leq \Lambda^{\mathrm{m}}(1-\Lambda)^{-1} M_1, \quad (4.8)\end{aligned}$$

for all $\mathrm{t} \in[0,\ \mathrm{T}], u_0 \in \mathrm{D_h}$.
Since $\Lambda=\left(2(1-\alpha)+\frac{\alpha T}{2}\right)\left(k_1+a_T L_1 k_2\right)<1$ and $\lim_{m \rightarrow \infty} \Lambda^{\mathrm{m}}=0$, so that the right side of (4.8) tends to zero. Therefore the sequence of functions $u_m\left(t, u_0\right), m=1,2,3, \ldots$ is converges uniformly on the domain (4.2) to the limit function $\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0\right)$ which is defined on the same domain. Let

$$\lim_{m \rightarrow \infty} u_m\left(t, u_0\right)=\mathrm{u}_\theta\left(t, u_0\right) \quad (4.9)$$

Since the sequence of functions (4.1) are periodic in t of period T, then the limiting function $\mathrm{u}_\theta\left(\mathrm{t}, \mathrm{u}_0\ \right)$ is also periodic in t of period T. By using the relation (4.9) and proceeding in (4.1) to limit, when $\mathrm{m} \rightarrow \infty$, it is converging that the limiting function $u\left(t, u_0\right)$ is the periodic solution of the integral equation (4.3).

Theorem 3. *If all assumptions of the Theorem* 2 *are satisfy, then* $\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0\ \right)$ *is a unique solution of the problem (* 1.1 *) with* $\mathrm{u}(0)=\mathrm{u}_0$.

Proof. Assume that $\hat{u}\,(t, u_0)$ is another solution of the problem (1.1) with $u(0) = u_0$, as follows

$$\hat{u}\,(t, u_0) = \mathrm{u}_0 + \tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} h(t, \hat{u}(t), \textstyle\int_0^{a(t)} \mathrm{g}(\mathrm{s}, \hat{u}(s))ds)$$
$$- (\tfrac{2(1-\alpha)}{(2-\alpha)N(\alpha)} \tfrac{1}{\mathrm{T}} \textstyle\int_0^T h(s, \hat{u}(s), \int_0^{a(s)} \mathrm{g}(\tau, \hat{u}(\tau))d\tau)ds + \tfrac{2\alpha}{(2-\alpha)N(\alpha)}$$
$$\textstyle\int_0^t (h(s, \hat{u}(s), \int_0^{a(s)} \mathrm{g}(\tau, \hat{u}(\tau))d\tau) - \frac{1}{\mathrm{T}} \int_0^T h(s, \hat{u}(s), \int_0^{a(s)} \mathrm{g}(\tau, \hat{u}(\tau))d\tau)ds)ds \quad (4.10)$$

Now, the difference between the two solutions $\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0)$ and $\hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)$,for all $\mathrm{t} \in [0,\ \mathrm{T}]$ and $\mathrm{u}_0 \in \mathrm{D_h}$, hence, by the inequalities $(3, 3) - (3.5)$, we get

$$\begin{aligned} |\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0) - \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)| &\leq \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_Y L_1 k_2)\,|\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0) - \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)| \\ &\leq \Lambda\,|\mathrm{u}\,(\mathrm{t}_r \mathrm{u}_0) - \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)| \quad (4.11) \end{aligned}$$

By mathematical induction, we find that

$$|\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0) - \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)| \leq \Lambda^{\mathrm{m}}\,|\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0) - \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)| \quad (4.12)$$

From the condition (3.7), shows that the solution $\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0) = \hat{\mathrm{u}}\,(\mathrm{t}, \mathrm{u}_0)$, thus $\mathrm{u}\,(\mathrm{t}, \mathrm{u}_0\ )$ is a unique periodic solution on the domain (4.2).

4.2. Existence of Periodic Solutions of (1.1)

The problem of the existence of the periodic solution for the problem (1.1) with $\mathrm{u}(0) = \mathrm{u}_0$ is uniquely connected with the existence of the zeros of the functions:-

$$\mu\,(0, \mathrm{u}_0) = \frac{1}{\mathrm{T}} \int_0^T h\left(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau))d\tau\right) ds \quad (4.13)$$

Also, we define the sequences of functions $\mu_m\,(0, u_0)$ are approximately determined by the following:

$$\mu_{\mathrm{m}}\,(0, \mathrm{u}_0) = \frac{1}{\mathrm{T}} \int_0^T h\left(s, u_m(s), \int_0^{a(s)} \mathrm{g}\,(\tau, u_m(\tau))\,d\tau\right) ds \quad (4.14)$$

Theorem 4. *If the hypotheses and all the conditions of the theorem* 2 *are given, the following inequalities are satisfied:-*

$$|\mu\,(0, \mathrm{u}_0) - \mu_{\mathrm{m}}\,(0, \mathrm{u}_0)| \leq (k_1 + a_T L_1 k_2)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1} M_1 \quad (4.15)$$

holds for all $\mathrm{m} \geq 0$

Proof. From equations (4.13) to (4.14), we obtain that

$$\begin{aligned} |\mu\,(0, \mathrm{u}_0) - \mu_m\,(0, \mathrm{u}_0)| &\leq (k_1 + a_T L_1 k_2)\,|u\,(t, \mathrm{u}_0) - u_m\,(t, \mathrm{u}_0)| \\ &\leq (k_1 + a_r L_1 k_2)\,\Lambda^{\mathrm{m}}(1-\Lambda)^{-1} M_1 \quad (4.16) \end{aligned}$$

The inequality (4.15) is hold for all $\mathrm{m} \geq 0$.

Theorem 5. *Let the function $h(s,u(s),z(t))$ be defined on the intervals $[c,d]$ on R and periodic in t of period T, suppose that for all $\mathrm{m} \geq 0$, then the sequences of the functions $\mu_{\mathrm{m}}\left(0,u_0\right)$ which are defined in (4.14) satisfy the inequalities:-*

$$\left.\begin{array}{l} \min_{u_0\in[c,d]} \mu_{\mathrm{m}}\left(0,\mathrm{u}_0\right) \leq -\left(k_1+a_\tau L_1 k_2\right)\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1 \\ \max_{u_0\in[c,d]} \mu_{\mathrm{m}}\left(0,\mathrm{u}_0\right) \geq \left(k_1+a_r L_1 k_2\right)\Lambda^{\mathrm{m}}(1-\Lambda)^{-1}M_1 \end{array}\right\} \quad (4.17)$$

Then the problem (1.1) has a periodic solution $\mathrm{u}\left(\mathrm{t},\mathrm{u}_0\right.$ *) such that* $\mathrm{u}_0 \in [\mathrm{c},\mathrm{d}] = [\mathrm{c}+M_1,\ \mathrm{d}-M_1]$

Proof. Let u_1 and u_2 be any points belonging to the intervals $[c,d]$, such that

$$\left.\begin{array}{l} \mu_{\mathrm{m}}\left(0,\mathrm{u}_1\right) = \min_{u_0\in[c,d]} \mu_{\mathrm{m}}\left(0,\mathrm{u}_0\right) \\ \mu_{\mathrm{m}}\left(0,\mathrm{u}_2\right) = \max_{u_0\in[c,d]} \mu_{\mathrm{m}}\left(0,\mathrm{u}_0\right) \end{array}\right\} \quad (4.18)$$

By using inequalities (4.15) to (4.18),, the following are obtained:-

$$\left.\begin{array}{l} \mu\left(0,\mathrm{u}_1\right) = \mu_{\mathrm{m}}\left(0,\mathrm{u}_1\right) + \left(\mu\left(0,\mathrm{u}_1\right) - \mu_{\mathrm{m}}\left(0,\mathrm{u}_1\right)\right) < 0 \\ \mu\left(0,\mathrm{u}_2\right) = \mu_{\mathrm{m}}\left(0,\mathrm{u}_2\right) + \left(\mu\left(0,\mathrm{u}_2\right) - \mu_m\left(0,\mathrm{u}_2\right)\right) > 0 \end{array}\right\} \quad (4.19)$$

and from the continuity of the functions $\mu\left(0,u_1\right), \mu\left(0,u_2\right)$ and the inequalities (4.19), then the isolated singular points $u^0 \in [c,d]$ exist such that $\mu\left(0,u^0\right) = 0$. This means that (1.1) has a periodic solution $\mathrm{u}\left(\mathrm{t},\mathrm{u}_0\right)$.

4.3. Stability of Periodic Solution of (1.1)

In this section, we investigate the stability or periodic solution of (1.1).

Theorem 6. *Let the function $\mu\left(0,\mathrm{u}_0\right)$ be defined by the equation* (4.13) *where* $\mathrm{u}\left(\mathrm{t},\mathrm{u}_0\right)$ *is a limit of the sequence of the function (4.1), then the following inequalities yield:-*

$$\left|\mu\left(0,u_0\right)\right| \leq M \quad (4.20)$$

and

$$\left|\mu\left(0,\mathrm{u}_0^1\right) - \mu\left(0,\mathrm{u}_0^2\right)\right| \leq F_2F_3\left|u_0^1-u_0^2\right| \quad (4.21)$$

where

$$F_1 = 2(1-\alpha)+\frac{\alpha T}{2}, \quad F_2 = k_1+a_T L_1 k_2, \quad F_3 = \left(1-F_1F_2\right)^{-1}$$

Proof From the properties of the function $u\left(t,u_0\right)$ as in the Theorem 2, the function $\mu\left(0,\mathrm{u}_0\right), u_0 \in D$ is continuous and bounded by $\frac{1-\alpha}{\alpha T}g^T + M$ in the domain (4.2). From (4.13), we obtained that

$$\left|\mu\left(0,\mathrm{u}_0\right)\right| \leq \frac{1}{\mathrm{T}}\int_0^T \left|h\left(s,u(s),\int_0^{a(s)} \mathrm{g}(\tau,u(\tau))d\tau\right)\right| ds \leq M \quad (4.22)$$

Next, from inequality (4.13), we get

$$\left|\mu\left(0, \mathrm{u}_0^1\right)-\mu\left(0, \mathrm{u}_0^2\right)\right| \leq\left(k_1+a_T L_1 k_2\right)\left|u\left(t, u_0^1\right)-u\left(t, u_0^2\right)\right|$$

$$\leq F_2\left|u\left(t, u_0^1\right)-u\left(t, u_0^2\right)\right| \qquad (4.23)$$

where the functions $\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)$ and $\mathrm{u}\left(\mathrm{t}, u_0^2\right)$ are solutions of the integral equation:-

$$u\left(t, \mathrm{u}_0^{\mathrm{k}}\right)=\mathrm{u}_0^{\mathrm{k}}+\frac{2(1-\alpha)}{(2-\alpha) N(\alpha)} h\left(t, u\left(t, \mathrm{u}_0^{\mathrm{k}}\right), \int_0^{a(t)} \mathrm{g}\left(\mathrm{s}, u\left(s, \mathrm{u}_0^{\mathrm{k}}\right)\right) d s\right)-$$
$$\frac{2(1-\alpha)}{(2-\alpha) N(\alpha)} \frac{1}{\mathrm{T}} \int_0^T h\left(s, u\left(s, \mathrm{u}_0^{\mathrm{k}}\right), \int_0^{a(s)} \mathrm{g}\left(\tau, u\left(\tau, \mathrm{u}_0^{\mathrm{k}}\right)\right) d \tau\right) d s+$$
$$\frac{2 \alpha}{(2-\alpha) N(\alpha)} \int_0^t(h(s, u\left(s, \mathrm{u}_0^{\mathrm{k}}\right), \int_0^{a(s)} \mathrm{g}\left(\tau, u\left(\tau, \mathrm{u}_0^{\mathrm{k}}\right)\right) d \tau)-$$
$$\frac{1}{\mathrm{T}} \int_0^T h(s, u\left(s, \mathrm{u}_0^{\mathrm{k}}\right), \int_0^{a(s)} \mathrm{g}\left(\tau, u\left(\tau, \mathrm{u}_0^{\mathrm{k}}\right)\right) d \tau) d s) d s \qquad (4.24)$$

where $\mathrm{k}=1,2$, from (4.24), we get

$$\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right| \leq\left|\, \mathrm{u}_0^1-u_0^2\right]+\frac{4(1-\alpha)}{(2-\alpha) N(a)}\left(k_1+a_T L_1 k_2\right)\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right|+$$

$$\frac{\alpha T}{(2-\alpha) N(\alpha)}\left(k_1+a_T L_1 k_2\right)\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right| \qquad (4.25)$$

Therefore, we obtain that

$$\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right| \leq\left|\mathrm{u}_0^1-u_0^2\right|+\left(2(1-\alpha)+\frac{\alpha T}{2}\right)\left(k_1+a_T L_1 k_2\right)\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right|$$

$$\leq\left|\mathrm{u}_0^1-u_0^2\right|+F_1 F_2\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right| \qquad (4.26)$$

From equations (4.26), we have

$$\left|\mathrm{u}\left(\mathrm{t}, \mathrm{u}_0^1\right)-\mathrm{u}\left(\mathrm{t}, u_0^2\right)\right| \leq F_3\left|\mathrm{u}_0^1-u_0^2\right| \qquad (4.27)$$

Substitutes (4.27) in (4.23), we get that (4.21)

Remark 1. *[22]. Theorem 6 confirms the stability of the solution of the problem (1.1), when a slight change happens in the points u_0, then a slight change will happen in the function $\mu\left(0, u_0\right)$.*

4.4. Existence and uniqueness of periodic Solution of (1.1) with integral boundary condition

In this section, we investigate the periodic solution of the problem (1.1) with integral boundary conditions:

$$u(0)-u(T)=\int_0^T H(u(s)) d s \qquad (4.28)$$

Where the function $H(u(s))$ defined and continuous on are compact subset of R and periodic on t of periodic T.

Theorem 7. *All assumptions of the Theorem 2 are satisfy, and the function $H(u(s))$ satisfies*

$$|H(u_1) - H(u_2)| \leq L_2 |u_1 - u_2| \quad (4.29)$$

then the problem (1.1) and integral boundary condition(4.28) has unique solution if

$$Q = \left(2(1-\alpha) + \frac{\alpha T}{2}\right)(k_1 + a_T L_1 k_2) + \frac{(1-\alpha+\alpha T)}{\alpha} L_2 < 1 \quad (4.30)$$

Proof. We define an operator $P : C[0,T] \to C[0,T]$

$P(u(t)) = \mathrm{u}_0 - \frac{(1-\alpha+\alpha t)}{\alpha T} \int_0^T H(u(s))dt +$
$\frac{2(1-\alpha)}{(2-\alpha)N(a)} [h(t, u(t), \int_0^{a(t)} \mathrm{g}(\mathrm{s}, u(s))ds) - \frac{1}{\mathrm{T}} \int_0^T h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau))d\tau)ds] +$
$\frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t (h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau))d\tau) - \frac{1}{\mathrm{T}} \int_0^T h(s, u(s), \int_0^{a(s)} \mathrm{g}(\tau, u(\tau))d\tau)ds)ds$

Therefore, we get

$$|P(u(t)) - P(w(t))| = ((2(1-\alpha) + \frac{\alpha T}{2})(k_1 + a_T L_1 k_2) + \frac{(1-\alpha+\alpha T)}{\alpha} L_2)|u(t) - w(t)|$$

From (4.30), the operator P satisfies contraction mapping, hence the problem (1.1) and (4.28) has unique solution.

Theorem 8. *If the hypotheses and all the conditions of the theorem 2 and the inequality (4.29) are given, the following inequalities are satisfied:-*

$$|\sigma(0, \mathrm{u}_0) - \sigma_{\mathrm{m}}(0, \mathrm{u}_0)| \leq \left(k_1 + a_\Upsilon L_1 k_2 + \frac{L_1}{\alpha}\right) Q^{\mathrm{m}} (1-Q)^{-1} M_3 \quad (4.31)$$

where

$$\sigma_{\mathrm{m}}(0, \mathrm{u}_0) = \frac{1}{\mathrm{T}} \int_0^T h\left(s, u_m(s), \int_0^{a(s)} \mathrm{g}(\tau, u_m(\tau))\, d\tau\right) ds$$
$$+ \frac{(2-\alpha)\mathrm{N}(\alpha)}{2\alpha\mathrm{T}} \int_0^T H(u_m(s))\, ds \quad (4.32)$$

holds for all $\mathrm{m} \geq 0$, *here*

$$M_3 = M_1 + \frac{(1-\alpha+\alpha T)}{a} M_2 \quad (4.33)$$

and

$$M_2 \geq |H(u(t))| \quad (4.34)$$

The proof of this theorem is direct.

Theorem 9. *Let the functions $h(s,u(s),z(t))$ and $H(u(t))$ be defined on the intervals $[c_1,d_1]$ on R and periodic in t of period* T, *suppose that for all* $\mathrm{m} \geq 0$, *then the sequences of the functions $\sigma_{\mathrm{m}}(0,\mathrm{u}_0)$ which are defined in (4.32) satisfy the inequalities:-*

$$\left.\begin{array}{l} \min_{u_0\in[c_1,d_1]} \sigma_m(0,u_0) \leq -\left(k_1+a_T L_1 k_2+\frac{L_2}{\alpha}\right) Q^m(1-Q)^{-1}M_3 \\ \max_{u_0\in[c_1,d_1 1} \sigma_m(0,u_0) \geq \left(k_1+a_T L_1 k_2+\frac{L_2}{\alpha}\right) Q^m(1-Q)^{-1}M_3 \end{array}\right\} \quad (4.35)$$

Then the problem (1.1) with (4.28) *has a periodic solution such that $u_0 \in [c_1+M_3, d_1-M_3]$ where M_3 defined in (4.33).*

This theorem's proof was similar to that of theorem 5.

Theorem 10. *Let the function $\sigma(0,\mathrm{u}_0)$ be defined by the equations (4.32), then the following inequalities yield:-*

$$|\sigma(0,\mathrm{u}_0)| \leq M+\frac{M_2}{\alpha} \quad (4.36)$$

and

$$\left|\sigma\left(0,\mathrm{u}_0^1\right)-\sigma\left(0,\mathrm{u}_0^2\right)\right| \leq E_2E_3\left|u_0^1-u_0^2\right| \quad (4.37)$$

where

$$E_1=2(1-\alpha)+\frac{\alpha T}{2}, \quad E_2=k_1+a_T L_1 k_2+\frac{L_1}{\alpha}, \quad E_3=(1-\mathrm{E}_1E_2)^{-1}$$

The proof of this theorem was similar to the proof of theorem 6.

5. Examples

In this section contains two example to illustrate the previous theorems.
Example 5.1. Consider the following fractional integro-differential equation

$${}_0^{CF}D_t^{0.7}(u(t))=\frac{1}{e^t+5}u(t)+\int_0^{t^2}\frac{1}{2(s+2)^3}\sin(u(s))ds \quad (5.1)$$

such that $t \in J=[0,2]$, with the initial condition $\mathrm{u}(0)=1$, where ${}_0^{CF}D_t^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative ($a=0.7 \in (0,1]$). Here $T=2,,a(t)=t^2$,

$$h(t,u(t),z(t))=\frac{1}{e^t+5}u(t)+\int_0^{t^2}\frac{1}{2(s+2)^3}\sin(u(s))ds$$
$$\mathrm{g}(\mathrm{t},u(t))=\frac{1}{2(s+2)^3}\sin(u(s))$$

We obtain that $k_1=0.2, k_2=1, a_T=4, L_1=0.0625$,
so that $\Lambda=\left(2(1-\alpha)+\frac{\alpha r}{2}\right)(k_1+a_T L_1 k_2)=0.585<1$.
Therefore, by Theorem 2 and Theorem 3, the fractional differential equation (5.1) has

exactly one periodic solution.

Example 5.2. Consider the fractional integro-differential equation (5.1) with integral boundary conditions

$$u(0) - u(1) = \int_0^1 \frac{1}{2} \cos(u(t))dt \quad (5.2)$$

such that $t \in (0, 1]$, where ${}_0^{CF}D_t^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative ($\alpha = 0.7 \in [0, 1]$). Here $T = 1,\ , a(t), h(t, u(t), \mathrm{z(t)}), \mathrm{g(t}, u(t))$ are defined in previous example and

$$H(u(t)) = \frac{1}{2} \cos(u(t))$$

we obtain that $k_1 = 0.2, k_2 = 1, a_T = 1, L_1 = 0.0625$, and $L_2 = 0.5$,
so that

$$\mathrm{Q} = \left(2(1-\alpha) + \frac{aT}{2}\right)(k_1 + a_T L_1 k_2) + \frac{(1-\alpha+\alpha T)}{a} L_2 = 0.9637 < 1$$

Therefore, by Theorem 7, the boundary value problem (5.1) and (5.2) has exactly one periodic solution.

6. Conclusion

In this paper,we studied the existence, uniqueness, and stability of periodic solutions of nonlinear fractional integro-differential equation (1.1) where ${}_0^{CF}D_t^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative with the initial condition, periodic boundary conditions, and integral boundary conditions by using technique successive approximations method and Banach fixed point theorem. Here conclude that we could investigate the existence, uniqueness, and stability of periodic solution of Caputo-Fabrizio fractional differential equation with integral boundary condition

$$Au(0) - Bu(T) = \sum_{i=1}^{m} C_i \int_0^T H_i(u(s))ds,$$

where A, B and $C_i,\ i = 1, 2, ..., m$ are constants, and $H_i, i = 1, 2, ..., m$ are defined and continuous functions on $[0, T]$. Finally, some examples are presented to illustrate the previous theorems.

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