EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 1, 2022, 169-198
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Rough Pythagorean Fuzzy Sets in UP-Algebras

Akarachai Satirad[1], Ronnason Chinram[2], Pongpun Julatha[3], Aiyared Iampan[1,*]

[1] *Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[2] *Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*
[3] *Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand*

Abstract. This paper aims to apply the concept of rough sets to Pythagorean fuzzy sets in UP-algebras. Then we introduce fifteen types of rough Pythagorean fuzzy sets in UP-algebras and study their generalization. In addition, we will also discuss t-level subsets of rough Pythagorean fuzzy sets in UP-algebras to study the relationships between rough Pythagorean fuzzy sets and rough sets in UP-algebras.

2020 Mathematics Subject Classifications: 03G25, 03E72, 08A72

Key Words and Phrases: UP-algebra, Pythagorean fuzzy set, Upper approximation, Lower approximation, Rough set, Rough Pyhagorean fuzzy set, t-level subset

1. Introduction and Preliminaries

The concept of fuzzy sets (FSs) was first considered by Zadeh [29] in 1965. Zadeh's and others' FS concepts have found numerous applications in mathematics and other fields. Following the introduction of the concept of FSs, various researchers were interviewed about generalizations of the concept of FSs, including: Atanassov [5] defined a new concept called an intuitionistic fuzzy set (IFS) which is a generalization of a FS, Yager [27] introduced a new class of non-standard fuzzy subsets called a Pythagorean fuzzy set (PFS) and the related idea of Pythagorean membership grades.

The concept of rough sets (RSs) was first considered by Pawlak [18] in 1982. After the introduction of the concept of RSs, several authors have applied the concept of RSs to the generalizations of the concept of FSs in many algebraic structures such as: Chen and Wang [6] combined RSs and fuzzy subalgebras (fuzzy ideals) fruitfully by defining rough fuzzy subalgebras (rough fuzzy ideals) of BCI-algebras, Moradiana et al. [17] presented a

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i1.4254
Email addresses: akarachai.sa@gmail.com (A. Satirad), ronnason.c@psu.ac.th (R. Chinram), pongpun.j@psru.ac.th (P. Julatha), aiyared.ia@up.ac.th (A. Iampan)

definition of the lower and upper approximation of subsets of BCK-algebras concerning a fuzzy ideal. Ahn and Kim [1] introduced the concept of rough fuzzy filters in BE-algebras, Ahn and Ko [2] introduced the concept of rough ideals and rough fuzzy ideals in BCK/BCI-algebras, Hussain et al. [10] introduced the concept of rough Pythagorean fuzzy ideals in semigroups, Chinram and Panityakul [7] introduced rough Pythagorean fuzzy ideals in ternary semigroups and gave some remarkable properties. Jun et al. [15] studied the concept of a (strong) set-valued BCK/BCI-morphism and introduced the concept of a generalized rough subalgebra (ideal) in BCK/BCI-algebras.

In this study, we extend the RS concept to PFSs in UP-algebras and establish fifteen different types of rough Pythagorean fuzzy sets (RPFSs) in UP-algebras: upper rough Pythagorean fuzzy UP-subalgebras (UpRPFUPSs), upper rough Pythagorean fuzzy near UP-filters (UpRPFNUPFs), upper rough Pythagorean fuzzy UP-filters (UpRPFUPFs), upper rough Pythagorean fuzzy UP-ideals (UpRPFUPIs), upper rough Pythagorean fuzzy strong UP-ideals (UpRPFSUPIs), lower rough Pythagorean fuzzy UP-subalgebras (LoRP-FUPSs), lower rough Pythagorean fuzzy near UP-filters (LoRPFNUPFs), lower rough Pythagorean fuzzy UP-filters (LoRPFUPFs), lower rough Pythagorean fuzzy UP-ideals (LoRPFUPIs), lower rough Pythagorean fuzzy strong UP-ideals (LoRPFSUPIs), rough Pythagorean fuzzy UP-subalgebras (RPFUPSs), rough Pythagorean fuzzy near UP-filters (RPFNUPFs), rough Pythagorean fuzzy UP-filters (RPFUPFs), rough Pythagorean fuzzy UP-ideals (RPFUPIs), and rough Pythagorean fuzzy strong UP-ideals (RPFSUPIs). Moreover, we verify their generalization of theirs. Then, to investigate the relationships between PFSs and special subsets of UP-algebras, we explore t-level subsets of PFSs. Finally, we study the relationships between RPFSs and RSs in UP-algebras by analyzing t-level subsets of RPFSs.

Let's go through the definition of UP-algebras first.

Definition 1. *[11] A UP-algebra is one that has the algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$ of type $(2, 0)$, where $\mathcal{U}$ is a nonempty set, $\star$ is a binary operation on $\mathcal{U}$, and 0 is a fixed element of $\mathcal{U}$ if it meets the following axioms:*

(UP-1) $(\forall a, b, c \in \mathcal{U})((b \star c) \star ((a \star b) \star (a \star c)) = 0)$,

(UP-2) $(\forall a \in \mathcal{U})(0 \star a = a)$,

(UP-3) $(\forall a \in \mathcal{U})(a \star 0 = 0)$,

(UP-4) $(\forall a, b \in \mathcal{U})(a \star b = 0, b \star a = 0 \Rightarrow a = b)$.

For more examples of UP-algebras, see [3, 4, 8, 12, 14, 22–25]. According to [11], we know that the concept of UP-algebras is a generalization of KU-algebras (see [19]).

Unless otherwise indicated, we will assume that $\mathcal{U}$ is a UP-algebra $(\mathcal{U}, \star, 0)$.

In $\mathcal{U}$, the following assertions are valid (see [11, 12]).

$$(\forall a \in \mathcal{U})(a \star a = 0), \tag{1.1}$$

$$(\forall a, b, c \in \mathcal{U})(a \star b = 0, b \star c = 0 \Rightarrow a \star c = 0), \tag{1.2}$$

$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (c \star a) \star (c \star b) = 0), \tag{1.3}$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (b \star c) \star (a \star c) = 0), \tag{1.4}$$
$$(\forall a, b \in \mathcal{U})(a \star (b \star a) = 0), \tag{1.5}$$
$$(\forall a, b \in \mathcal{U})((b \star a) \star a = 0 \Leftrightarrow a = b \star a), \tag{1.6}$$
$$(\forall a, b \in \mathcal{U})(a \star (b \star b) = 0), \tag{1.7}$$
$$(\forall u, a, b, c \in \mathcal{U})((a \star (b \star c)) \star (a \star ((u \star b) \star (u \star c))) = 0), \tag{1.8}$$
$$(\forall u, a, b, c \in \mathcal{U})((((u \star a) \star (u \star b)) \star c) \star ((a \star b) \star c) = 0), \tag{1.9}$$
$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star c) = 0), \tag{1.10}$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow a \star (c \star b) = 0), \tag{1.11}$$
$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (a \star (b \star c)) = 0), \tag{1.12}$$
$$(\forall u, a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star (u \star c)) = 0). \tag{1.13}$$

According to [11], the binary relation $\leq$ on $\mathcal{U}$ is defined as follows:

$$(\forall a, b \in \mathcal{U})(a \leq b \Leftrightarrow a \star b = 0).$$

Definition 2. *[9, 11, 26] A nonempty subset S of $\mathcal{U}$ is called*

(1) *a UP-subalgebra (UPS) of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in S)(a \star b \in S), \tag{1.14}$$

(2) *a near UP-filter (NUPF) of $\mathcal{U}$ if it satisfies the following condition:*

$$(\forall a, b \in \mathcal{U})(b \in S \Rightarrow a \star b \in S), \tag{1.15}$$

(3) *a UP-filter (UPF) of $\mathcal{U}$ if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } \mathcal{U} \text{ is in } S, \tag{1.16}$$

$$(\forall a, b \in \mathcal{U})(a \star b \in S, a \in S \Rightarrow b \in S), \tag{1.17}$$

(4) *a UP-ideal (UPI) of $\mathcal{U}$ if it satisfies the condition* (1.16) *and the following condition:*

$$(\forall a, b, c \in \mathcal{U})(a \star (b \star c) \in S, b \in S \Rightarrow a \star c \in S), \tag{1.18}$$

(5) *a strong UP-ideal (SUPI) of $\mathcal{U}$ if it satisfies the condition* (1.16) *and the following condition:*

$$(\forall a, b, c \in \mathcal{U})((c \star b) \star (c \star a) \in S, b \in S \Rightarrow a \in S). \tag{1.19}$$

Guntasow et al. [9] and Iampan [13] proved that the concept of UPSs is a generalization of NUPFs, NUPFs is a generalization of UPFs, UPFs is a generalization of UPIs, and UPIs is a generalization of SUPIs. They also proved that $\mathcal{U}$ is the only SUPI.

Definition 3. *[29] A fuzzy set (FS)* F *in a nonempty set* $\mathcal{U}$ *is described by its membership function* μ_{F}*. To every point* $a \in \mathcal{U}$*, this function associates a real number* $\mu_{\mathrm{F}}(a)$ *in the closed interval* $[0,1]$*. The real number* $\mu_{\mathrm{F}}(a)$ *is interpreted for the point as a degree of membership of an object* $a \in \mathcal{U}$ *to the FS* F*, that is,* $\mathrm{F} := \{(a, \mu_{\mathrm{F}}(a)) \mid a \in \mathcal{U}\}$*. We say that a FS* F *in* $\mathcal{U}$ *is constant fuzzy set if its membership function* μ_{F} *is constant.*

In 2013, Yager [27] and Yager and Abbasov [28] introduced the concept of PFSs for the first time.

Definition 4. *[27, 28] A Pythagorean fuzzy set (PFS)* P *in a nonempty set* $\mathcal{U}$ *is described by their membership function* μ_{P} *and non-membership function* ν_{P}*. To every point* $a \in \mathcal{U}$*, these functions associate real numbers* $\mu_{\mathrm{P}}(a)$ *and* $\nu_{\mathrm{P}}(a)$ *in the closed interval* $[0,1]$*, with the following condition:*

$$(\forall a \in \mathcal{U})(0 \leq \mu_{\mathrm{P}}(a)^2 + \nu_{\mathrm{P}}(a)^2 \leq 1). \tag{1.20}$$

The real numbers $\mu_{\mathrm{P}}(a)$ *and* $\nu_{\mathrm{P}}(a)$ *are interpreted for the point as a degree of membership and non-membership of an object* $a \in \mathcal{U}$*, respectively, to the PFS* P*, that is,* $\mathrm{P} := \{(a, \mu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(a)) \mid a \in \mathcal{U}\}$*. For the sake of simplicity, a PFS* P *is denoted by* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$*. We say that a PFS* P *in* $\mathcal{U}$ *is constant Pythagorean fuzzy set if their membership function* μ_{P} *and non-membership function* ν_{P} *are constant.*

Definition 5. *[20, 21] A PFS* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* $\mathcal{U}$ *is called*

(1) *a Pythagorean fuzzy UP-subalgebra (PFUPS) of* $\mathcal{U}$ *if it satisfies the following conditions:*

$$(\forall a, b \in \mathcal{U})(\mu_{\mathrm{P}}(a \star b) \geq \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(b)\}), \tag{1.21}$$

$$(\forall a, b \in \mathcal{U})(\nu_{\mathrm{P}}(a \star b) \leq \max\{\nu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(b)\}), \tag{1.22}$$

(2) *a Pythagorean fuzzy near UP-filter (PFNUPF) of* $\mathcal{U}$ *if it satisfies the following conditions:*

$$(\forall a, b \in \mathcal{U})(\mu_{\mathrm{P}}(a \star b) \geq \mu_{\mathrm{P}}(b)), \tag{1.23}$$

$$(\forall a, b \in \mathcal{U})(\nu_{\mathrm{P}}(a \star b) \leq \nu_{\mathrm{P}}(b)), \tag{1.24}$$

(3) *a Pythagorean fuzzy UP-filter (PFUPF) of* $\mathcal{U}$ *if it satisfies the following conditions:*

$$(\forall a \in \mathcal{U})(\mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a)), \tag{1.25}$$

$$(\forall a \in \mathcal{U})(\nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a)), \tag{1.26}$$

$$(\forall a, b \in \mathcal{U})(\mu_{\mathrm{P}}(b) \geq \min\{\mu_{\mathrm{P}}(a \star b), \mu_{\mathrm{P}}(a)\}), \tag{1.27}$$

$$(\forall a, b \in \mathcal{U})(\nu_{\mathrm{P}}(b) \leq \max\{\nu_{\mathrm{P}}(a \star b), \nu_{\mathrm{P}}(a)\}), \tag{1.28}$$

(4) *a Pythagorean fuzzy UP-ideal (PFUPI) of* $\mathcal{U}$ *if it satisfies the conditions* (1.25) *and* (1.26) *and the following conditions:*

$$(\forall a, b, c \in \mathcal{U})(\mu_{\mathrm{P}}(a \star c) \geq \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\}), \tag{1.29}$$

$$(\forall a, b, c \in \mathcal{U})(\nu_{\mathrm{P}}(a \star c) \leq \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\}), \tag{1.30}$$

(5) *a Pythagorean fuzzy strong UP-ideal (PFSUPI) of* $\mathcal{U}$ *if it satisfies the conditions* (1.25) *and* (1.26) *and the following conditions:*

$$(\forall a, b, c \in \mathcal{U})(\mu_{\mathrm{P}}(a) \geq \min\{\mu_{\mathrm{P}}((c \star b) \star (c \star a)), \mu_{\mathrm{P}}(b)\}), \tag{1.31}$$

$$(\forall a, b, c \in \mathcal{U})(\nu_{\mathrm{P}}(a) \leq \max\{\nu_{\mathrm{P}}((c \star b) \star (c \star a)), \nu_{\mathrm{P}}(b)\}). \tag{1.32}$$

Satirad et al. [20] proved that the concept of PFUPSs is a generalization of PFNUPFs, PFNUPFs is a generalization of PFUPFs, PFUPFs is a generalization of PFUPIs, and PFUPIs is a generalization of PFSUPIs. Furthermore, they proved that PFSUPIs and constant PFSs coincide in $\mathcal{U}$.

Let ρ be an equivalence relation (ER) on a set $\mathcal{U}$. If $a \in \mathcal{U}$, then the ρ-class of a is the set $(a)_\rho$ defined as follows:

$$(a)_\rho = \{b \in \mathcal{U} \mid (a, b) \in \rho\}.$$

An ER ρ on $\mathcal{U}$ is called a *congruence relation* (CR) if

$$(\forall a, b, c \in \mathcal{U})((a, b) \in \rho \Rightarrow (a \star c, b \star c) \in \rho, (c \star a, c \star b) \in \rho).$$

Definition 6. *For nonempty subsets A and B of* $\mathcal{U}$*, we denote*

$$AB = A \star B = \{u \star v \mid u \in A \text{ and } v \in B\}.$$

If ρ is a CR on $\mathcal{U}$, then

$$(\forall a, b \in \mathcal{U})((a)_\rho (b)_\rho \subseteq (a \star b)_\rho). \qquad \text{(see [16])}$$

Definition 7. *Let ρ be an ER on a nonempty set $\mathcal{U}$ and $S \in \mathcal{P}(\mathcal{U})$. The upper approximation of S is defined by*

$$\rho^+(S) = \{a \in \mathcal{U} \mid (a)_\rho \subseteq S\},$$

the lower approximation of S is defined by

$$\rho^-(S) = \{a \in \mathcal{U} \mid (a)_\rho \cap S \neq \emptyset\}.$$

We know that $\rho^+(S)$ and $\rho^-(S)$ are subset of $\mathcal{U}$. Then we call S that a *rough set* (RS) of $\mathcal{U}$.

Definition 8. *[16] Let ρ be an ER on $\mathcal{U}$. Then a nonempty subset S of $\mathcal{U}$ is called*

(1) *an upper rough UP-subalgebra (UpRUPS) of $\mathcal{U}$ if $\rho^+(S)$ is a UPS of $\mathcal{U}$,*

(2) *an upper rough near UP-filter (UpRNUPF) of* $\mathcal{U}$ *if* $\rho^+(S)$ *is a NUPF of* $\mathcal{U}$*,*

(3) *an upper rough UP-filter (UpRUPF) of* $\mathcal{U}$ *if* $\rho^+(\mathrm{P})$ *is a UPF of* $\mathcal{U}$*,*

(4) *an upper rough UP-ideal (UpRUPI) of* $\mathcal{U}$ *if* $\rho^+(S)$ *is a UPI of* $\mathcal{U}$*,*

(5) *an upper rough strong UP-ideal (UpRSUPI) of* $\mathcal{U}$ *if* $\rho^+(S)$ *is a SUPI of* $\mathcal{U}$*,*

(6) *a lower rough UP-subalgebra (LoRUPS) of* $\mathcal{U}$ *if* $\emptyset \neq \rho^-(S)$ *is a UPS of* $\mathcal{U}$*,*

(7) *a lower rough near UP-filter (LoRNUPF) of* $\mathcal{U}$ *if* $\emptyset \neq \rho^-(S)$ *is a NUPF of* $\mathcal{U}$*,*

(8) *a lower rough UP-filter (LoRUPF) of* $\mathcal{U}$ *if* $\emptyset \neq \rho^-(S)$ *is a UPF of* $\mathcal{U}$*,*

(9) *a lower rough UP-ideal (LoRUPI) of* $\mathcal{U}$ *if* $\emptyset \neq \rho^-(S)$ *is a UPI of* $\mathcal{U}$*,*

(10) *a lower rough strong UP-ideal (LoRSUPI) of* $\mathcal{U}$ *if* $\emptyset \neq \rho^-(S)$ *is a SUPI of* $\mathcal{U}$*,*

(11) *a rough UP-subalgebra (RUPS) of* $\mathcal{U}$ *if it is both an UpRUPS and a LoRUPS of* $\mathcal{U}$*,*

(12) *a rough near UP-filter (RNUPF) of* $\mathcal{U}$ *if it is both an UpRNUPF and a LoRNUPF of* $\mathcal{U}$*,*

(13) *a rough UP-filter (RUPF) of* $\mathcal{U}$ *if it is both an UpRUPF and a LoRUPF of* $\mathcal{U}$*,*

(14) *a rough UP-ideal (RUPI) of* $\mathcal{U}$ *if it is both an UpRUPI and a LoRUPI of* $\mathcal{U}$*, and*

(15) *a rough strong UP-ideal (RSUPI) of* $\mathcal{U}$ *if it is both an UpRSUPI and a LoRSUPI of* $\mathcal{U}$*.*

2. RPFSs in UP-algebras

Definition 9. *Let* ρ *be an ER on a nonempty set* $\mathcal{U}$ *and* $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ *a PFS in* $\mathcal{U}$*. The upper approximation of* P *is defined by*

$$\rho^+(\mathrm{P}) = \{(a, \overline{\mu}_\mathrm{P}(a), \overline{\nu}_\mathrm{P}(a)) \mid a \in \mathcal{U}\},$$

where $\overline{\mu}_\mathrm{P}(a) = \sup_{u\in(a)_\rho} \{\mu_\mathrm{P}(u)\}$ *and* $\overline{\nu}_\mathrm{P}(a) = \inf_{u\in(a)_\rho} \{\nu_\mathrm{P}(u)\}$*. The lower approximation of* P *is defined by*

$$\rho^-(\mathrm{P}) = \{(a, \underline{\mu}_\mathrm{P}(a), \underline{\nu}_\mathrm{P}(a)) \mid a \in \mathcal{U}\},$$

where $\underline{\mu}_\mathrm{P}(a) = \inf_{u\in(a)_\rho} \{\mu_\mathrm{P}(u)\}$ *and* $\underline{\nu}_\mathrm{P}(a) = \sup_{u\in(a)_\rho} \{\nu_\mathrm{P}(u)\}$*.*

It is easy to proof that $\rho^+(\mathrm{P})$ and $\rho^-(\mathrm{P})$ are PFSs in $\mathcal{U}$. Then we call P that a *rough Pythagorean fuzzy set* (RPFS) in $\mathcal{U}$. Thus we can denote the upper approximation and the lower approximation by $\rho^+(\mathrm{P}) = (\overline{\mu}_\mathrm{P}, \overline{\nu}_\mathrm{P})$ and $\rho^-(\mathrm{P}) = (\underline{\mu}_\mathrm{P}, \underline{\nu}_\mathrm{P})$, respectively.

Definition 10. *Let ρ be an ER on $\mathcal{U}$ and $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ a PFS in $\mathcal{U}$. Then a RPFS P in $\mathcal{U}$ is called* constant rough Pythagorean fuzzy set *in $\mathcal{U}$ if their membership functions $\overline{\mu}_{\mathrm{P}}, \underline{\mu}_{\mathrm{P}}$ and non-membership functions $\overline{\nu}_{\mathrm{P}}, \underline{\nu}_{\mathrm{P}}$ are constant.*

Next, we apply the concept of RPFSs to UP-algebras and introduce the fifteen types of RPFSs in UP-algebras.

Definition 11. *Let ρ be an ER on $\mathcal{U}$. Then a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ in $\mathcal{U}$ is called*

(1) *an upper rough Pythagorean fuzzy UP-subalgebra (UpRPFUPS) of $\mathcal{U}$ if $\rho^+(\mathrm{P})$ is a PFUPS of $\mathcal{U}$,*

(2) *an upper rough Pythagorean fuzzy near UP-filter (UpRPFNUPF) of $\mathcal{U}$ if $\rho^+(\mathrm{P})$ is a PFNUPF of $\mathcal{U}$,*

(3) *an upper rough Pythagorean fuzzy UP-filter (UpRPFUPF) of $\mathcal{U}$ if $\rho^+(\mathrm{P})$ is a PFUPF of $\mathcal{U}$,*

(4) *an upper rough Pythagorean fuzzy UP-ideal (UpRPFUPI) of $\mathcal{U}$ if $\rho^+(\mathrm{P})$ is a PFUPI of $\mathcal{U}$,*

(5) *an upper rough Pythagorean fuzzy strong UP-ideal (UpRPFSUPI) of $\mathcal{U}$ if $\rho^+(\mathrm{P})$ is a PFSUPI of $\mathcal{U}$,*

(6) *a lower rough Pythagorean fuzzy UP-subalgebra (LoRPFUPS) of $\mathcal{U}$ if $\rho^-(\mathrm{P})$ is a PFUPS of $\mathcal{U}$,*

(7) *a lower rough Pythagorean fuzzy near UP-filter (LoRPFNUPF) of $\mathcal{U}$ if $\rho^-(\mathrm{P})$ is a PFNUPF of $\mathcal{U}$,*

(8) *a lower rough Pythagorean fuzzy UP-filter (LoRPFUPF) of $\mathcal{U}$ if $\rho^-(\mathrm{P})$ is a PFUPF of $\mathcal{U}$,*

(9) *a lower rough Pythagorean fuzzy UP-ideal (LoRPFUPI) of $\mathcal{U}$ if $\rho^-(\mathrm{P})$ is a PFUPI of $\mathcal{U}$,*

(10) *a lower rough Pythagorean fuzzy strong UP-ideal (LoRPFSUPI) of $\mathcal{U}$ if $\rho^-(\mathrm{P})$ is a PFSUPI of $\mathcal{U}$,*

(11) *a rough Pythagorean fuzzy UP-subalgebra (RPFUPS) of $\mathcal{U}$ if it is both an UpRPFUPS and a LoRPFUPS of $\mathcal{U}$,*

(12) *a rough Pythagorean fuzzy near UP-filter (RPFNUPF) of $\mathcal{U}$ if it is both an UpRPFNUPF and a LoRPFNUPF of $\mathcal{U}$,*

(13) *a rough Pythagorean fuzzy UP-filter (RPFUPF) of $\mathcal{U}$ if it is both an UpRPFUPF and a LoRPFUPF of $\mathcal{U}$,*

(14) *a rough Pythagorean fuzzy UP-ideal (RPFUPI) of* $\mathcal{U}$ *if it is both an UpRPFUPI and a LoRPFUPI of* $\mathcal{U}$*, and*

(15) *a rough Pythagorean fuzzy strong UP-ideal (RPFSUPI) of* $\mathcal{U}$ *if it is both an UpRPF-SUPI and a LoRPFSUPI of* $\mathcal{U}$*.*

It is simple to verify the generalizations of RPFSs in UP-algebras. As a result, we obtain the diagram of the generalization of RPFSs in UP-algebras, which is shown in Figure 1.

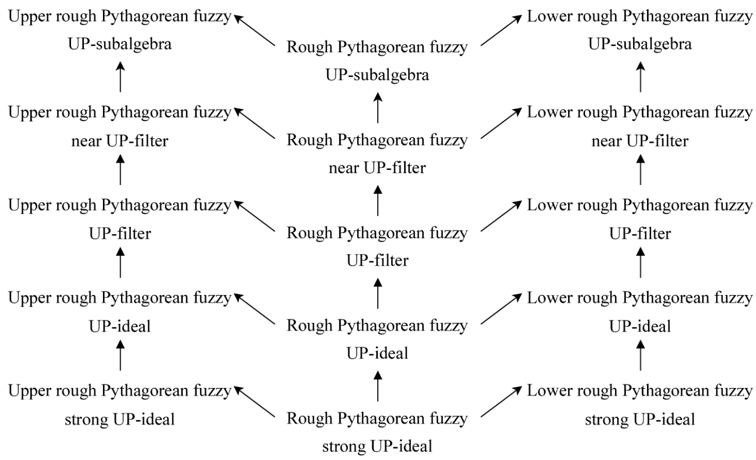

Figure 1: Rough Pythagorean fuzzy sets in UP-algebras

Theorem 1. *Let* ρ *be an ER (CR) on* $\mathcal{U}$ *and* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *a PFS in* $\mathcal{U}$*. If* P *is a PFSUPI of* $\mathcal{U}$*, then* P *is a RPFSUPI of* $\mathcal{U}$*.*

Proof. Let P be a PFSUPI of $\mathcal{U}$. Then it is constant. For all $a, b \in \mathcal{U}$, $\mu_{\mathrm{P}}(a) = \mu_{\mathrm{P}}(b)$ and $\nu_{\mathrm{P}}(a) = \nu_{\mathrm{P}}(b)$. Let $u, v \in \mathcal{U}$. Then

$$\overline{\mu}_{\mathrm{P}}(u) = \sup_{a \in (u)_\rho} \{\mu_{\mathrm{P}}(a)\} = \sup_{b \in (v)_\rho} \{\mu_{\mathrm{P}}(b)\} = \overline{\mu}_{\mathrm{P}}(v),$$

$$\overline{\nu}_{\mathrm{P}}(u) = \inf_{a \in (u)_\rho} \{\nu_{\mathrm{P}}(a)\} = \inf_{b \in (v)_\rho} \{\nu_{\mathrm{P}}(b)\} = \overline{\nu}_{\mathrm{P}}(v),$$

$$\underline{\mu}_{\mathrm{P}}(u) = \inf_{a \in (u)_\rho} \{\mu_{\mathrm{P}}(a)\} = \inf_{b \in (v)_\rho} \{\mu_{\mathrm{P}}(b)\} = \underline{\mu}_{\mathrm{P}}(v), \text{ and}$$

$$\underline{\nu}_{\mathrm{P}}(u) = \sup_{a \in (u)_\rho} \{\nu_{\mathrm{P}}(a)\} = \sup_{b \in (v)_\rho} \{\nu_{\mathrm{P}}(b)\} = \underline{\nu}_{\mathrm{P}}(v).$$

So $\rho^+(\mathrm{P})$ and $\rho^-(\mathrm{P})$ are constant. This means that $\rho^+(\mathrm{P})$ and $\rho^-(\mathrm{P})$ are PFSUPIs of $\mathcal{U}$. Therefore, P is a RPFSUPI of $\mathcal{U}$.

The following examples show the relationships between PFSs in $\mathcal{U}$ and RPFSs in $\mathcal{U}$ with ρ is an ER on $\mathcal{U}$.

Example 1. *Consider a UP-algebra* $\mathcal{U} = (\mathcal{U}, \star, 0)$*, where* $\mathcal{U} = \{0, 1, 2, 3\}$ *is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* $\mathcal{U}$ *as follows:*

$\mathcal{U}$	0	1	2	3
μ_{P}	0.7	0.3	0.6	0.6
ν_{P}	0.1	0.8	0.4	0.4

Then P *is a PFUPI (resp., PFUPF, PFNUPF, and PFUPS) of* $\mathcal{U}$*. Let*

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (0,3), (3,0), (1,3), (3,1)\}.$$

Then ρ *is an ER on* $\mathcal{U}$*. But* $\rho^+(\mathrm{P})$ *and* $\rho^-(\mathrm{P})$ *are not PFUPIs (resp., PFUPFs, PFNUPFs, and PFUPSs) of* $\mathcal{U}$*.*

From Example 1, we get the results that if P is a PFUPS (resp., PFNUPF, PFUPF, and PFUPI), then it may not be a RPFUPS (resp., RPFNUPF, RPFUPF, and RPFUPI).

Example 2. *Consider a UP-algebra* $\mathcal{U} = (\mathcal{U}, \star, 0)$*, where* $\mathcal{U} = \{0, 1, 2, 3\}$ *is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* $\mathcal{U}$ *as follows:*

$\mathcal{U}$	0	1	2	3
μ_{P}	0.8	0.5	0.4	0.5
ν_{P}	0.2	0.4	0.7	0.4

Then P *is not a PFUPS (resp., PFNUPF, PFUPF, and PFUPI) of* $\mathcal{U}$*. Let*

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}.$$

Then ρ *is an ER on* $\mathcal{U}$*. But* $\rho^+(\mathrm{P})$ *and* $\rho^-(\mathrm{P})$ *are PFUPSs (resp., PFNUPFs, PFUPFs, and PFUPIs) of* $\mathcal{U}$*.*

Example 3. *Consider a UP-algebra* $\mathcal{U} = (\mathcal{U}, \star, 0)$*, where* $\mathcal{U} = \{0, 1, 2, 3\}$ *is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	0
3	0	1	2	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* $\mathcal{U}$ *as follows:*

$\mathcal{U}$	0	1	2	3
μ_{P}	0.5	0.4	0.3	0.2
ν_{P}	0.1	0.2	0.3	0.4

Then P *is not a PFSUPI of* $\mathcal{U}$*. Let*

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(0,2),(2,0),(0,3),(3,0),$$

$$(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}.$$

Then ρ *is an ER on* $\mathcal{U}$*. But* $\rho^{+}(\mathrm{P})$ *and* $\rho^{-}(\mathrm{P})$ *are PFSUPIs of* $\mathcal{U}$*.*

From Examples 2 and 3, we get the results that if P is a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI), then it may not be a PFUPS (resp., PFNUPF, PFUPF, PFUPI, and PFSUPI).

Example 4. *Consider a UP-algebra* $\mathcal{U} = (\mathcal{U}, \star, 0)$*, where* $\mathcal{U} = \{0, 1, 2, 3\}$ *is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	0
3	0	1	2	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* $\mathcal{U}$ *as follows:*

$\mathcal{U}$	0	1	2	3
μ_{P}	1	0.2	0.1	0.2
ν_{P}	0	0.6	0.9	0.6

Then P *is a PFUPI (resp., PFUPF, PFNUPF, and PFUPS) of* $\mathcal{U}$*. Let*

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1)\}.$$

Then ρ *is an ER on* $\mathcal{U}$*. Thus* $\rho^{+}(\mathrm{P})$ *and* $\rho^{-}(\mathrm{P})$ *are PFUPIs (resp., PFUPFs, PFNUPFs, and PFUPSs) of* $\mathcal{U}$*.*

From Example 4 and Theorem 1, we get the results that P can be a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI) and a PFUPS (resp., PFNUPF, PFUPF, PFUPI, and PFSUPI) in the same time.

The following examples show the relationships between PFSs in $\mathcal{U}$ and RPFSs in $\mathcal{U}$ with ρ is a CR on $\mathcal{U}$.

Example 5. *Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_{P}	0.8	0.3	0.5	0.5
ν_{P}	0.2	0.8	0.3	0.3

Then P *is a PFUPI (resp., PFUPF, PFNUPF, and PFUPS) of $\mathcal{U}$. Let*

$$\rho = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0)\}.$$

Then ρ is a CR on $\mathcal{U}$. But $\rho^{-}(\mathrm{P})$ is not a PFUPI (resp., PFUPF, PFNUPF, and PFUPS) of $\mathcal{U}$.

From Example 5, we get the results that if P is a PFUPS (resp., PFNUPF, PFUPF, and PFUPI), then it may not be a RPFUPS (resp., RPFNUPF, RPFUPF, and RPFUPI).

Example 6. *Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	1	2	0

We define a PFS $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_{P}	0.5	0.4	0.3	0.2
ν_{P}	0.1	0.2	0.3	0.4

Then P *is not a PFUPS (resp., PFNUPF, PFUPI, and PFSUPI) of $\mathcal{U}$. Let*

$$\rho = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (0, 2), (2, 0), (0, 3), (3, 0),$$

$$(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}.$$

Then ρ is a CR on $\mathcal{U}$. But $\rho^+(\mathrm{P})$ and $\rho^-(\mathrm{P})$ are PFUPSs (resp., PFNUPFs, PFUPFs, PFUPIs, and PFSUPIs) of $\mathcal{U}$.

From Example 6, we get the results that if P is a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI), then it may not be a PFUPS (resp., PFNUPF, PFUPF, PFUPI, and PFSUPI).

Example 7. *Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.*

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

We define a PFS $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.9	0.2	0.3	0.3
ν_P	0.2	0.6	0.5	0.5

Then P is a PFUPI (resp., PFUPF, PFNUPF, and PFUPS) of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,3),(3,0)\}.$$

Then ρ is a CR on $\mathcal{U}$. Thus $\rho^+(\mathrm{P})$ and $\rho^-(\mathrm{P})$ are PFUPIs (resp., PFUPFs, PFNUPFs, and PFUPSs) of $\mathcal{U}$.

From Example 7, we get the results that P can be a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI) and a PFUPS (resp., PFNUPF, PFUPF, PFUPI, and PFSUPI) in the same time.

Hence, we get the diagram of the relationships between RPFSs and PFSs in UP-algebras, which is shown with Figure 2.

3. t-Level Subsets of a PFS

In this section, we shall let P be a PFS $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ in $\mathcal{U}$. We shall discuss the relationships between PFUPSs (resp., PFNUPFs, PFUPFs, PFUPIs, PFSUPIs, RPFUPSs, RPFNUPFs, RPFUPFs, RPFUPIs, and RPFSUPIs) of UP-algebras and their t-level subsets.

Definition 12. *[26] Let F be a FS with the membership function μ_F in $\mathcal{U}$. The sets*

$$U(\mu_\mathrm{F}, t) = \{a \in \mathcal{U} \mid \mu_\mathrm{F}(a) \geq t\},$$
$$U^+(\mu_\mathrm{F}, t) = \{a \in \mathcal{U} \mid \mu_\mathrm{F}(a) > t\},$$

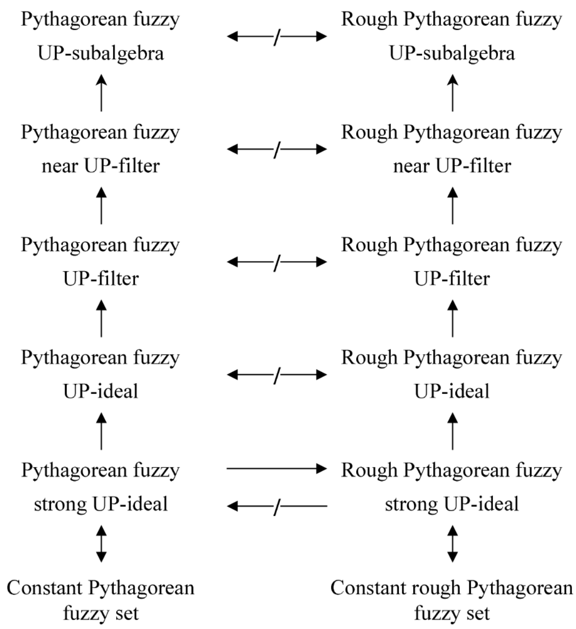

Figure 2: Relationships between rough Pythagorean fuzzy sets and Pythagorean fuzzy sets in UP-algebras

$$\begin{aligned} L(\mu_{\mathrm{F}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathrm{F}}(a) \leq t\}, \\ L^{-}(\mu_{\mathrm{F}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathrm{F}}(a) < t\}, \\ E(\mu_{\mathrm{F}}, t) &= \{a \in \mathcal{U} \mid \mu_{\mathrm{F}}(a) = t\} \end{aligned}$$

are referred to as an upper t-level subset, an upper t-strong level subset, a lower t-level subset, a lower t-strong level subset, and an equal t-level subset of F*, respectively, for any* $t \in [0,1]$.

Theorem 2. P *is a PFUPS of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPSs of* $\mathcal{U}$ *for every* $t \in [0,1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFUPS of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U(\mu_{\mathrm{P}}, t)$, $L(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a, b \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a) \geq t, \mu_{\mathrm{P}}(b) \geq t \\ &\Rightarrow \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(b)\} \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(a \star b) \geq \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(b)\} \geq t && ((1.21)) \\ &\Rightarrow a \star b \in U(\mu_{\mathrm{P}}, t) \end{aligned}$$

and

$$\begin{aligned} a, b \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a) \leq t, \nu_{\mathrm{P}}(b) \leq t \\ &\Rightarrow \max\{\mu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(b)\} \leq t \end{aligned}$$

$$\Rightarrow \nu_{\rm P}(a \star b) \leq \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\} \leq t \qquad ((1.22))$$
$$\Rightarrow a \star b \in L(\nu_{\rm P}, t).$$

Hence, $U(\mu_{\rm P}, t)$ and $L(\nu_{\rm P}, t)$ are UPSs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U(\mu_{\rm P}, t)$ and $L(\nu_{\rm P}, t)$ are UPSs of $\mathcal{U}$ if the sets are nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\} \in [0,1]$. Then $\mu_{\rm P}(a) \geq t$ and $\mu_{\rm P}(b) \geq t$. Thus $a, b \in U(\mu_{\rm P}, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_{\rm P}, t)$ is a UPS of $\mathcal{U}$ and so $a \star b \in U(\mu_{\rm P}, t)$. Thus $\mu_{\rm P}(a \star b) \geq t = \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\}$.

Choose $t = \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\} \in [0,1]$. Then $\nu_{\rm P}(a) \leq t$ and $\nu_{\rm P}(b) \leq t$. Thus $a, b \in L(\nu_{\rm P}, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_{\rm P}, t)$ is a UPS of $\mathcal{U}$ and so $a \star b \in U(\nu_{\rm P}, t)$. Thus $\nu_{\rm P}(a \star b) \leq t = \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\}$.

Hence, P is a PFUPS of $\mathcal{U}$.

Theorem 3. P *is a PFUPS of* $\mathcal{U}$ *if and only if* $U^+(\mu_{\rm P}, t)$ *and* $L^-(\nu_{\rm P}, t)$ *are, if the sets are nonempty, UPSs of* $\mathcal{U}$ *for every* $t \in [0,1]$.

Proof. Assume $\mathrm{P} = (\mu_{\rm P}, \nu_{\rm P})$ is a PFUPS of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U^+(\mu_{\rm P}, t)$, $L^-(\nu_{\rm P}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a, b \in U^+(\mu_{\rm P}, t) &\Rightarrow \mu_{\rm P}(a) > t, \mu_{\rm P}(b) > t \\ &\Rightarrow \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\} > t \\ &\Rightarrow \mu_{\rm P}(a \star b) \geq \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\} > t \qquad ((1.21)) \\ &\Rightarrow a \star b \in U^+(\mu_{\rm P}, t) \end{aligned}$$

and

$$\begin{aligned} a, b \in L^-(\nu_{\rm P}, t) &\Rightarrow \nu_{\rm P}(a) < t, \nu_{\rm P}(b) < t \\ &\Rightarrow \max\{\mu_{\rm P}(a), \nu_{\rm P}(b)\} < t \\ &\Rightarrow \nu_{\rm P}(a \star b) \leq \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\} < t \qquad ((1.22)) \\ &\Rightarrow a \star b \in L^-(\nu_{\rm P}, t). \end{aligned}$$

Hence, $U^+(\mu_{\rm P}, t)$ and $L^-(\nu_{\rm P}, t)$ are UPSs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U^+(\mu_{\rm P}, t)$ and $L^-(\nu_{\rm P}, t)$ are UPSs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\rm P}(a \star b) < \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\}$. Choose $t = \mu_{\rm P}(a \star b) \in [0,1]$. Then $\mu_{\rm P}(a) > t$ and $\mu_{\rm P}(b) > t$. Thus $a, b \in U^+(\mu_{\rm P}, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_{\rm P}, t)$ is a UPS of $\mathcal{U}$ and so $a \star b \in U^+(\mu_{\rm P}, t)$. Thus $\mu_{\rm P}(a \star b) > t = \mu_{\rm P}(a \star b)$, a contradiction. Hence, $\mu_{\rm P}(a \star b) \geq \min\{\mu_{\rm P}(a), \mu_{\rm P}(b)\}$ for all $a, b \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\nu_{\rm P}(a \star b) > \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\}$. Choose $t = \nu_{\rm P}(a \star b) \in [0,1]$. Then $\nu_{\rm P}(a) < t$ and $\nu_{\rm P}(b) < t$. Thus $a, b \in L^-(\nu_{\rm P}, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_{\rm P}, t)$ is a UPS of $\mathcal{U}$ and so $a \star b \in L^-(\nu_{\rm P}, t)$. Thus $\nu_{\rm P}(a \star b) < t = \nu_{\rm P}(a \star b)$, a contradiction. Hence, $\nu_{\rm P}(a \star b) \leq \max\{\nu_{\rm P}(a), \nu_{\rm P}(b)\}$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFUPS of $\mathcal{U}$.

Theorem 4. P *is a PFNUPF of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, NUPFs for every* $t \in [0, 1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFNUPF of $\mathcal{U}$. Let $t \in [0, 1]$ be such that $U(\mu_{\mathrm{P}}, t), L(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} b \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(b) \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(a \star b) \geq \mu_{\mathrm{P}}(b) \geq t \qquad ((1.23)) \\ &\Rightarrow a \star b \in U(\mu_{\mathrm{P}}, t) \end{aligned}$$

and

$$\begin{aligned} a, b \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(b) \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(a \star b) \leq \nu_{\mathrm{P}}(b) \leq t \qquad ((1.24)) \\ &\Rightarrow a \star b \in L(\nu_{\mathrm{P}}, t). \end{aligned}$$

Hence, $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are NUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0, 1], U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are NUPFs of $\mathcal{U}$ if the sets are nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_{\mathrm{P}}(b) \in [0, 1]$. Then $\mu_{\mathrm{P}}(b) \geq t$. Thus $b \in U(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}}, t)$ is a NUPF of $\mathcal{U}$ and so $a \star b \in U(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(a \star b) \geq t = \mu_{\mathrm{P}}(b)$.

Choose $t = \nu_{\mathrm{P}}(b) \in [0, 1]$. The $\nu_{\mathrm{P}}(b) \leq t$. Thus $b \in L(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_{\mathrm{P}}, t)$ is a NUPF of $\mathcal{U}$ and so $a \star b \in U(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(a \star b) \leq t = \nu_{\mathrm{P}}(b)$.

Hence, P is a PFNUPF of $\mathcal{U}$.

Theorem 5. P *is a PFNUPF of* $\mathcal{U}$ *if and only if* $U^+(\mu_{\mathrm{P}}, t)$ *and* $L^-(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, NUPFs of* $\mathcal{U}$ *for every* $t \in [0, 1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFNUPF of $\mathcal{U}$. Let $t \in [0, 1]$ be such that $U^+(\mu_{\mathrm{P}}, t), L^-(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} b \in U^+(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(b) > t \\ &\Rightarrow \mu_{\mathrm{P}}(a \star b) \geq \mu_{\mathrm{P}}(b) > t \qquad ((1.23)) \\ &\Rightarrow a \star b \in U^+(\mu_{\mathrm{P}}, t) \end{aligned}$$

and

$$\begin{aligned} b \in L^-(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(b) < t \\ &\Rightarrow \nu_{\mathrm{P}}(a \star b) \leq \nu_{\mathrm{P}}(b) < t \qquad ((1.24)) \\ &\Rightarrow a \star b \in L^-(\nu_{\mathrm{P}}, t). \end{aligned}$$

Hence, $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are NUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0, 1], U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are NUPFs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\mathrm{P}}(a \star b) < \mu_{\mathrm{P}}(b)$. Choose $t = \mu_{\mathrm{P}}(a \star b) \in [0, 1]$. Then $\mu_{\mathrm{P}}(b) > t$. Thus $b \in U^{+}(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U^{+}(\mu_{\mathrm{P}}, t)$ is a NUPF of $\mathcal{U}$ and so $a \star b \in U^{+}(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(a \star b) > t = \mu_{\mathrm{P}}(a \star b)$, a contradiction. Hence, $\mu_{\mathrm{P}}(a \star b) \geq \mu_{\mathrm{P}}(b)$ for all $a, b \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\nu_{\mathrm{P}}(a \star b) > \nu_{\mathrm{P}}(b)$. Choose $t = \nu_{\mathrm{P}}(a \star b) \in [0, 1]$. Then $\nu_{\mathrm{P}}(b) < t$. Thus $b \in L^{-}(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L^{-}(\nu_{\mathrm{P}}, t)$ is a NUPF of $\mathcal{U}$ and so $a \star b \in L^{-}(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(a \star b) < t = \nu_{\mathrm{P}}(a \star b)$, a contradiction. Hence, $\nu_{\mathrm{P}}(a \star b) \leq \nu_{\mathrm{P}}(b)$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFNUPF of $\mathcal{U}$.

Theorem 6. P *is a PFUPF of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPFs for every* $t \in [0, 1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFUPF of $\mathcal{U}$. Let $t \in [0, 1]$ be such that $U(\mu_{\mathrm{P}}, t)$, $L(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a) \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a) \geq t && ((1.25)) \\ &\Rightarrow 0 \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \star b, a \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a \star b) \geq t, \mu_{\mathrm{P}}(a) \geq t \\ &\Rightarrow \min\{\mu_{\mathrm{P}}(a \star b), \mu_{\mathrm{P}}(a)\} \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(b) \geq \min\{\mu_{\mathrm{P}}(a \star b), \mu_{\mathrm{P}}(a)\} \geq t && ((1.27)) \\ &\Rightarrow b \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a) \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a) \leq t && ((1.26)) \\ &\Rightarrow 0 \in L(\nu_{\mathrm{P}}, t), \end{aligned}$$

and

$$\begin{aligned} a \star b, a \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a \star b) \leq t, \nu_{\mathrm{P}}(a) \leq t \\ &\Rightarrow \max\{\mu_{\mathrm{P}}(a \star b), \nu_{\mathrm{P}}(a)\} \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(b) \leq \max\{\nu_{\mathrm{P}}(a \star b), \nu_{\mathrm{P}}(a)\} \leq t && ((1.28)) \\ &\Rightarrow b \in L(\nu_{\mathrm{P}}, t). \end{aligned}$$

Hence, $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are UPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0, 1], U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are UPFs of $\mathcal{U}$ if the sets are nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_{\mathrm{P}}(a) \in [0, 1]$. Then $\mu_{\mathrm{P}}(a) \geq t$. Thus $a \in U(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}}, t)$ is a UPF of $\mathcal{U}$ and so $0 \in U(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(0) \geq t = \mu_{\mathrm{P}}(a)$.

Choose $t = \min\{\mu_P(a \star b), \mu_P(a)\} \in [0,1]$. Then $\mu_P(a \star b) \geq t$ and $\mu_P(a) \geq t$. Thus $a \star b, a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is a UPF of $\mathcal{U}$ and so $b \in U(\mu_P, t)$. Thus $\mu_P(b) \geq t = \min\{\mu_P(a \star b), \mu_P(a)\}$.

Choose $t = \nu_P(a) \in [0,1]$. The $\nu_P(a) \leq t$. Thus $a \in L(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_P, t)$ is a UPF of $\mathcal{U}$ and so $0 \in U(\nu_P, t)$. Thus $\nu_P(0) \leq t = \nu_P(a)$.

Choose $t = \max\{\nu_P(a \star b), \nu_P(a)\} \in [0,1]$. Then $\nu_P(a \star b) \leq t$ and $\nu_P(a) \leq t$. Thus $a \star b, a \in L(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\mu_P, t)$ is a UPF of $\mathcal{U}$ and so $b \in L(\mu_P, t)$. Thus $\nu_P(b) \leq t = \max\{\nu_P(a \star b), \nu_P(a)\}$.

Hence, P is a PFUPF of $\mathcal{U}$.

Theorem 7. P *is a PFUPF of* $\mathcal{U}$ *if and only if* $U^+(\mu_P, t)$ *and* $L^-(\nu_P, t)$ *are, if the sets are nonempty, UPFs of* $\mathcal{U}$ *for every* $t \in [0,1]$.

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFUPF of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U^+(\mu_P, t)$, $L^-(\nu_P, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$\begin{aligned} a \in U^+(\mu_P, t) &\Rightarrow \mu_P(a) > t \\ &\Rightarrow \mu_P(0) \geq \mu_P(a) > t \qquad ((1.25)) \\ &\Rightarrow 0 \in U^+(\mu_P, t), \end{aligned}$$

$$\begin{aligned} a \star b, a \in U^+(\mu_P, t) &\Rightarrow \mu_P(a \star b) > t, \mu_P(a) > t \\ &\Rightarrow \min\{\mu_P(a \star b), \mu_P(a)\} > t \\ &\Rightarrow \mu_P(b) \geq \min\{\mu_P(a \star b), \mu_P(a)\} > t \qquad ((1.27)) \\ &\Rightarrow b \in U^+(\mu_P, t), \end{aligned}$$

$$\begin{aligned} a \in L^-(\nu_P, t) &\Rightarrow \nu_P(a) < t \\ &\Rightarrow \nu_P(0) \leq \nu_P(a) < t \qquad ((1.26)) \\ &\Rightarrow 0 \in L^-(\nu_P, t), \end{aligned}$$

and

$$\begin{aligned} a \star b, a \in L^-(\nu_P, t) &\Rightarrow \nu_P(a \star b) < t, \nu_P(a) < t \\ &\Rightarrow \max\{\nu_P(a \star b), \nu_P(a)\} < t \\ &\Rightarrow \nu_P(b) \leq \max\{\nu_P(a \star b), \nu_P(a)\} < t \qquad ((1.28)) \\ &\Rightarrow b \in L^-(\nu_P, t). \end{aligned}$$

Hence, $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are UPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are UPFs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a \in \mathcal{U}$ such that $\mu_P(0) < \mu_P(a)$. Choose $t = \mu_P(0) \in [0,1]$. Then $\mu_P(a) > t$. Thus $a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is a UPF of $\mathcal{U}$ and so $0 \in U^{+}(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(0) > t = \mu_{\mathrm{P}}(0)$, a contradiction. Hence, $\mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_{\mathrm{P}}(b) < \min\{\mu_{\mathrm{P}}(a \star b), \mu_{\mathrm{P}}(a)\}$. Choose $t = \mu_{\mathrm{P}}(b) \in [0,1]$. Then $\mu_{\mathrm{P}}(a \star b) > t$ and $\mu_{\mathrm{P}}(a) > t$. Thus $a \star b, a \in U^{+}(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U^{+}(\mu_{\mathrm{P}}, t)$ is a UPF of $\mathcal{U}$ and so $b \in U^{+}(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(b) > t = \mu_{\mathrm{P}}(b)$, a contradiction. Hence, $\mu_{\mathrm{P}}(b) \geq \min\{\mu_{\mathrm{P}}(a \star b), \mu_{\mathrm{P}}(a)\}$ for all $a, b \in \mathcal{U}$.

Suppose there exists $a \in \mathcal{U}$ such that $\nu_{\mathrm{P}}(0) > \nu_{\mathrm{P}}(a)$. Choose $t = \nu_{\mathrm{P}}(0) \in [0,1]$. Then $\nu_{\mathrm{P}}(a) < t$. Thus $a \in L^{-}(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L^{-}(\nu_{\mathrm{P}}, t)$ is a UPF of $\mathcal{U}$ and so $0 \in L^{-}(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(0) < t = \nu_{\mathrm{P}}(0)$, a contradiction. Hence, $\nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\nu_{\mathrm{P}}(b) > \max\{\nu_{\mathrm{P}}(a \star b), \nu_{\mathrm{P}}(a)\}$. Choose $t = \nu_{\mathrm{P}}(b) \in [0,1]$. Then $\nu_{\mathrm{P}}(a \star b) < t$ and $\nu_{\mathrm{P}}(a) < t$. Thus $a \star b, a \in L^{-}(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L^{-}(\nu_{\mathrm{P}}, t)$ is a UPF of $\mathcal{U}$ and so $b \in L^{-}(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(b) < t = \nu_{\mathrm{P}}(b)$, a contradiction. Hence, $\nu_{\mathrm{P}}(b) \leq \max\{\nu_{\mathrm{P}}(a \star b), \nu_{\mathrm{P}}(a)\}$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFUPF of $\mathcal{U}$.

Theorem 8. P *is a PFUPI of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPIs for every* $t \in [0,1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFUPI of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U(\mu_{\mathrm{P}}, t)$, $L(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$\begin{aligned} a \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a) \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a) \geq t \qquad ((1.25)) \\ &\Rightarrow 0 \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \star (b \star c), b \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a \star (b \star c)) \geq t, \mu_{\mathrm{P}}(b) \geq t \\ &\Rightarrow \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\} \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(a \star c) \geq \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\} \geq t \qquad ((1.29)) \\ &\Rightarrow a \star c \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a) \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a) \leq t \qquad ((1.26)) \\ &\Rightarrow 0 \in L(\nu_{\mathrm{P}}, t), \end{aligned}$$

and

$$\begin{aligned} a \star (b \star c), b \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a \star (b \star c)) \leq t, \nu_{\mathrm{P}}(b) \leq t \\ &\Rightarrow \max\{\mu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\} \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(a \star c) \leq \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\} \leq t \qquad ((1.30)) \\ &\Rightarrow a \star c \in L(\nu_{\mathrm{P}}, t). \end{aligned}$$

Hence, $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are UPIs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are UPIs of $\mathcal{U}$ if the sets are nonempty. Let $a, b, c \in \mathcal{U}$.

Choose $t = \mu_{\mathrm{P}}(a) \in [0,1]$. Then $\mu_{\mathrm{P}}(a) \geq t$. Thus $a \in U(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $0 \in U(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(0) \geq t = \mu_{\mathrm{P}}(a)$.

Choose $t = \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\} \in [0,1]$. Then $\mu_{\mathrm{P}}(a \star (b \star c)) \geq t$ and $\mu_{\mathrm{P}}(b) \geq t$. Thus $a \star (b \star c), b \in U(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $a \star c \in U(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(a \star c) \geq t = \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\}$.

Choose $t = \nu_{\mathrm{P}}(a) \in [0,1]$. The $\nu_{\mathrm{P}}(a) \leq t$. Thus $a \in L(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $0 \in U(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(0) \leq t = \nu_{\mathrm{P}}(a)$.

Choose $t = \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\} \in [0,1]$. Then $\nu_{\mathrm{P}}(a \star (b \star c)) \leq t$ and $\nu_{\mathrm{P}}(b) \leq t$. Thus $a \star (b \star c), b \in L(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L(\mu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $a \star c \in L(\mu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(a \star c) \leq t = \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\}$.

Hence, P is a PFUPI of $\mathcal{U}$.

Theorem 9. P *is a PFUPI of* $\mathcal{U}$ *if and only if* $U^+(\mu_{\mathrm{P}}, t)$ *and* $L^-(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, UPIs of* $\mathcal{U}$ *for every* $t \in [0,1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFUPI of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U^+(\mu_{\mathrm{P}}, t)$, $L^-(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$\begin{aligned} a \in U^+(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a) > t \\ &\Rightarrow \mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a) > t \qquad ((1.25)) \\ &\Rightarrow 0 \in U^+(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \star (b \star c), b \in U^+(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a \star (b \star c)) > t, \mu_{\mathrm{P}}(b) > t \\ &\Rightarrow \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\} > t \\ &\Rightarrow \mu_{\mathrm{P}}(a \star c) \geq \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\} > t \qquad ((1.29)) \\ &\Rightarrow a \star c \in U^+(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \in L^-(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a) < t \\ &\Rightarrow \nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a) < t \qquad ((1.26)) \\ &\Rightarrow 0 \in L^-(\nu_{\mathrm{P}}, t), \end{aligned}$$

and

$$\begin{aligned} a \star (b \star c), b \in L^-(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a \star (b \star c)) < t, \nu_{\mathrm{P}}(b) < t \\ &\Rightarrow \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\} < t \\ &\Rightarrow \nu_{\mathrm{P}}(a \star c) \leq \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\} < t \qquad ((1.30)) \\ &\Rightarrow a \star c \in L^-(\nu_{\mathrm{P}}, t). \end{aligned}$$

Hence, $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are UPIs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are UPIs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a \in \mathcal{U}$ such that $\mu_{\mathrm{P}}(0) < \mu_{\mathrm{P}}(a)$. Choose $t = \mu_{\mathrm{P}}(0) \in [0,1]$. Then $\mu_{\mathrm{P}}(a) > t$. Thus $a \in U^+(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $0 \in U^+(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(0) > t = \mu_{\mathrm{P}}(0)$, a contradiction. Hence, $\mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_{\mathrm{P}}(a \star c) < \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\}$. Choose $t = \mu_{\mathrm{P}}(a \star c) \in [0,1]$. Then $\mu_{\mathrm{P}}(a \star (b \star c)) > t$ and $\mu_{\mathrm{P}}(b) > t$. Thus $a \star (b \star c), b \in U^+(\mu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $a \star c \in U^+(\mu_{\mathrm{P}}, t)$. Thus $\mu_{\mathrm{P}}(a \star c) > t = \mu_{\mathrm{P}}(a \star c)$, a contradiction. Hence, $\mu_{\mathrm{P}}(a \star c) \geq \min\{\mu_{\mathrm{P}}(a \star (b \star c)), \mu_{\mathrm{P}}(b)\}$ for all $a, b, c \in \mathcal{U}$.

Suppose there exists $a \in \mathcal{U}$ such that $\nu_{\mathrm{P}}(0) > \nu_{\mathrm{P}}(a)$. Choose $t = \nu_{\mathrm{P}}(0) \in [0,1]$. Then $\nu_{\mathrm{P}}(a) < t$. Thus $a \in L^-(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $0 \in L^-(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(0) < t = \nu_{\mathrm{P}}(0)$, a contradiction. Hence, $\nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b, c \in \mathcal{U}$ such that $\nu_{\mathrm{P}}(a \star c) > \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\}$. Choose $t = \nu_{\mathrm{P}}(a) \in [0,1]$. Then $\nu_{\mathrm{P}}(a \star (b \star c)) < t$ and $\nu_{\mathrm{P}}(b) < t$. Thus $a \star (b \star c), b \in L^-(\nu_{\mathrm{P}}, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_{\mathrm{P}}, t)$ is a UPI of $\mathcal{U}$ and so $a \star c \in L^-(\nu_{\mathrm{P}}, t)$. Thus $\nu_{\mathrm{P}}(a \star c) < t = \nu_{\mathrm{P}}(a \star c)$, a contradiction. Hence, $\nu_{\mathrm{P}}(a \star c) \leq \max\{\nu_{\mathrm{P}}(a \star (b \star c)), \nu_{\mathrm{P}}(b)\}$ for all $a, b, c \in \mathcal{U}$.

Therefore, P is a PFUPI of $\mathcal{U}$.

Theorem 10. P *is a PFSUPI of* $\mathcal{U}$ *if and only if* $U(\mu_{\mathrm{P}}, t)$ *and* $L(\nu_{\mathrm{P}}, t)$ *are, if the sets are nonempty, SUPIs for every* $t \in [0,1]$.

Proof. Assume $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a PFSUPI of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U(\mu_{\mathrm{P}}, t)$, $L(\nu_{\mathrm{P}}, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$\begin{aligned} a \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}(a) \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(a) \geq t && ((1.25)) \\ &\Rightarrow 0 \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} (c \star b) \star (c \star a), b \in U(\mu_{\mathrm{P}}, t) &\Rightarrow \mu_{\mathrm{P}}((c \star b) \star (c \star a)) \geq t, \mu_{\mathrm{P}}(b) \geq t \\ &\Rightarrow \min\{\mu_{\mathrm{P}}((c \star b) \star (c \star a)), \mu_{\mathrm{P}}(b)\} \geq t \\ &\Rightarrow \mu_{\mathrm{P}}(a) \geq \min\{\mu_{\mathrm{P}}((c \star b) \star (c \star a)), \mu_{\mathrm{P}}(b)\} \geq t && ((1.31)) \\ &\Rightarrow a \in U(\mu_{\mathrm{P}}, t), \end{aligned}$$

$$\begin{aligned} a \in L(\nu_{\mathrm{P}}, t) &\Rightarrow \nu_{\mathrm{P}}(a) \leq t \\ &\Rightarrow \nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(a) \leq t && ((1.26)) \\ &\Rightarrow 0 \in L(\nu_{\mathrm{P}}, t), \end{aligned}$$

and

$$\begin{aligned}(c\star b)\star(c\star a), b\in L(\nu_{\mathrm{P}},t) &\Rightarrow \nu_{\mathrm{P}}((c\star b)\star(c\star a))\leq t, \nu_{\mathrm{P}}(b)\leq t\\ &\Rightarrow \max\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)), \nu_{\mathrm{P}}(b)\}\leq t\\ &\Rightarrow \nu_{\mathrm{P}}(a)\leq \max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)), \nu_{\mathrm{P}}(b)\}\leq t \qquad ((1.32))\\ &\Rightarrow a\in L(\nu_{\mathrm{P}},t).\end{aligned}$$

Hence, $U(\mu_{\mathrm{P}},t)$ and $L(\nu_{\mathrm{P}},t)$ are SUPIs of $\mathcal{U}$.

Conversely, assume for all $t\in[0,1], U(\mu_{\mathrm{P}},t)$ and $L(\nu_{\mathrm{P}},t)$ are SUPIs of $\mathcal{U}$ if the sets are nonempty. Let $a,b,c\in\mathcal{U}$.

Choose $t=\mu_{\mathrm{P}}(a)\in[0,1]$. Then $\mu_{\mathrm{P}}(a)\geq t$. Thus $a\in U(\mu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $0\in U(\mu_{\mathrm{P}},t)$. Thus $\mu_{\mathrm{P}}(0)\geq t=\mu_{\mathrm{P}}(a)$.

Choose $t=\min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}\in[0,1]$. Then $\mu_{\mathrm{P}}((c\star b)\star(c\star a))\geq t$ and $\mu_{\mathrm{P}}(b)\geq t$. Thus $(c\star b)\star(c\star a), b\in U(\mu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $U(\mu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $a\in U(\mu_{\mathrm{P}},t)$. Thus $\mu_{\mathrm{P}}(a)\geq t=\min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}$.

Choose $t=\nu_{\mathrm{P}}(a)\in[0,1]$. The $\nu_{\mathrm{P}}(a)\leq t$. Thus $a\in L(\nu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $L(\nu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $0\in U(\nu_{\mathrm{P}},t)$. Thus $\nu_{\mathrm{P}}(0)\leq t=\nu_{\mathrm{P}}(a)$.

Choose $t=\max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}\in[0,1]$. Then $\nu_{\mathrm{P}}((c\star b)\star(c\star a))\leq t$ and $\nu_{\mathrm{P}}(b)\leq t$. Thus $(c\star b)\star(c\star a), b\in L(\mu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $L(\mu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $a\in L(\mu_{\mathrm{P}},t)$. Thus $\nu_{\mathrm{P}}(a)\geq t=\max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}$.

Hence, P is a PFSUPI of $\mathcal{U}$.

Theorem 11. P *is a PFSUPI of* $\mathcal{U}$ *if and only if* $U^+(\mu_{\mathrm{P}},t)$ *and* $L^-(\nu_{\mathrm{P}},t)$ *are, if the sets are nonempty, SUPIs of* $\mathcal{U}$ *for every* $t\in[0,1]$.

Proof. Assume $\mathrm{P}=(\mu_{\mathrm{P}},\nu_{\mathrm{P}})$ is a PFSUPI of $\mathcal{U}$. Let $t\in[0,1]$ be such that $U^+(\mu_{\mathrm{P}},t)$, $L^-(\nu_{\mathrm{P}},t)\neq\emptyset$. Let $a,b,c\in\mathcal{U}$. Then

$$\begin{aligned}a\in U^+(\mu_{\mathrm{P}},t) &\Rightarrow \mu_{\mathrm{P}}(a)>t\\ &\Rightarrow \mu_{\mathrm{P}}(0)\geq\mu_{\mathrm{P}}(a)>t \qquad ((1.25))\\ &\Rightarrow 0\in U^+(\mu_{\mathrm{P}},t),\end{aligned}$$

$$\begin{aligned}(c\star b)\star(c\star a), b\in U^+(\mu_{\mathrm{P}},t) &\Rightarrow \mu_{\mathrm{P}}((c\star b)\star(c\star a))>t, \mu_{\mathrm{P}}(b)>t\\ &\Rightarrow \min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}>t\\ &\Rightarrow \mu_{\mathrm{P}}(a)\geq\min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}>t \qquad ((1.31))\\ &\Rightarrow a\in U^+(\mu_{\mathrm{P}},t),\end{aligned}$$

$$\begin{aligned}a\in L^-(\nu_{\mathrm{P}},t) &\Rightarrow \nu_{\mathrm{P}}(a)<t\\ &\Rightarrow \nu_{\mathrm{P}}(0)\leq\nu_{\mathrm{P}}(a)<t \qquad ((1.26))\\ &\Rightarrow 0\in L^-(\nu_{\mathrm{P}},t),\end{aligned}$$

and

$$\begin{aligned}(c\star b)\star(c\star a), b\in L^-(\nu_{\mathrm{P}},t) &\Rightarrow \nu_{\mathrm{P}}((c\star b)\star(c\star a))<t, \nu_{\mathrm{P}}(b)<t\\ &\Rightarrow \max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}<t\\ &\Rightarrow \nu_{\mathrm{P}}(a)\leq \max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}<t \qquad ((1.32))\\ &\Rightarrow a\in L^-(\nu_{\mathrm{P}},t).\end{aligned}$$

Hence, $U^+(\mu_{\mathrm{P}},t)$ and $L^-(\nu_{\mathrm{P}},t)$ are SUPIs of $\mathcal{U}$.

Conversely, assume for all $t\in[0,1], U^+(\mu_{\mathrm{P}},t)$ and $L^-(\nu_{\mathrm{P}},t)$ are SUPIs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a\in\mathcal{U}$ such that $\mu_{\mathrm{P}}(0)<\mu_{\mathrm{P}}(a)$. Choose $t=\mu_{\mathrm{P}}(0)\in[0,1]$. Then $\mu_{\mathrm{P}}(a)>t$. Thus $a\in U^+(\mu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $U^+(\mu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $0\in U^+(\mu_{\mathrm{P}},t)$. Thus $\mu_{\mathrm{P}}(0)>t=\mu_{\mathrm{P}}(0)$, a contradiction. Hence, $\mu_{\mathrm{P}}(0)\geq\mu_{\mathrm{P}}(a)$ for all $a\in\mathcal{U}$.

Suppose there exist $a,b,c\in\mathcal{U}$ such that $\mu_{\mathrm{P}}(a)<\min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}$. Choose $t=\mu_{\mathrm{P}}(a)\in[0,1]$. Then $\mu_{\mathrm{P}}((c\star b)\star(c\star a))>t$ and $\mu_{\mathrm{P}}(b)>t$. Thus $(c\star b)\star(c\star a),b\in U^+(\mu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $U^+(\mu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $a\in U^+(\mu_{\mathrm{P}},t)$. Thus $\mu_{\mathrm{P}}(a)>t=\mu_{\mathrm{P}}(a)$, a contradiction. Hence, $\mu_{\mathrm{P}}(a)\geq\min\{\mu_{\mathrm{P}}((c\star b)\star(c\star a)),\mu_{\mathrm{P}}(b)\}$ for all $a,b,c\in\mathcal{U}$.

Suppose there exists $a\in\mathcal{U}$ such that $\nu_{\mathrm{P}}(0)>\nu_{\mathrm{P}}(a)$. Choose $t=\nu_{\mathrm{P}}(0)\in[0,1]$. Then $\nu_{\mathrm{P}}(a)<t$. Thus $a\in L^-(\nu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $L^-(\nu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $0\in L^-(\nu_{\mathrm{P}},t)$. Thus $\nu_{\mathrm{P}}(0)<t=\nu_{\mathrm{P}}(0)$, a contradiction. Hence, $\nu_{\mathrm{P}}(0)\leq\nu_{\mathrm{P}}(a)$ for all $a\in\mathcal{U}$.

Suppose there exist $a,b,c\in\mathcal{U}$ such that $\nu_{\mathrm{P}}(a)>\max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}$. Choose $t=\nu_{\mathrm{P}}(a)\in[0,1]$. Then $\nu_{\mathrm{P}}((c\star b)\star(c\star a))<t$ and $\nu_{\mathrm{P}}(b)<t$. Thus $(c\star b)\star(c\star a),b\in L^-(\nu_{\mathrm{P}},t)\neq\emptyset$. As a hypothesis, we get $L^-(\nu_{\mathrm{P}},t)$ is a SUPI of $\mathcal{U}$ and so $a\in L^-(\nu_{\mathrm{P}},t)$. Thus $\nu_{\mathrm{P}}(a)<t=\nu_{\mathrm{P}}(a)$, a contradiction. Hence, $\nu_{\mathrm{P}}(a)\leq\max\{\nu_{\mathrm{P}}((c\star b)\star(c\star a)),\nu_{\mathrm{P}}(b)\}$ for all $a,b,c\in\mathcal{U}$.

Therefore, P is a PFSUPI of $\mathcal{U}$.

Theorem 12. P *is a PFSUPI of* $\mathcal{U}$ *if and only if* $E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))$ *and* $E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))$ *are SUPIs of* $\mathcal{U}$.

Proof. Assume $\mathrm{P}=(\mu_{\mathrm{P}},\nu_{\mathrm{P}})$ is a PFSUPI of $\mathcal{U}$. Since P is constant, we have

$$(\forall a\in\mathcal{U})\left(\begin{array}{c}\mu_{\mathrm{P}}(a)=\mu_{\mathrm{P}}(0)\\ \nu_{\mathrm{P}}(a)=\nu_{\mathrm{P}}(0)\end{array}\right).$$

Thus $a\in E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))$ and $a\in E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))$ and so $E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))=\mathcal{U}$ and $E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))=\mathcal{U}$. Hence, $E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))$ and $E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))$ are SUPIs of $\mathcal{U}$.

Conversely, assume $E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))$ and $E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))$ are SUPIs of $\mathcal{U}$. Then $E(\mu_{\mathrm{P}},\mu_{\mathrm{P}}(0))=\mathcal{U}$ and $E(\nu_{\mathrm{P}},\nu_{\mathrm{P}}(0))=\mathcal{U}$. We consider

$$(\forall a\in\mathcal{U})\left(\begin{array}{c}\mu_{\mathrm{P}}(a)=\mu_{\mathrm{P}}(0)\\ \nu_{\mathrm{P}}(a)=\nu_{\mathrm{P}}(0)\end{array}\right).$$

Thus P is constant, that is, P is a PFSUPI of $\mathcal{U}$.

The following lemma shows the relationships between t-level subsets of approximations and approximations of t-level subsets.

Lemma 1. *Let ρ be a CR on $\mathcal{U}$ and $t \in [0,1]$. Then the following statements hold:*

(1) $U(\overline{\mu}_{\mathrm{P}}, t) = \rho^-(U(\mu_{\mathrm{P}}, t))$,

(2) $U^+(\overline{\mu}_{\mathrm{P}}, t) = \rho^-(U^+(\mu_{\mathrm{P}}, t))$,

(3) $L(\overline{\nu}_{\mathrm{P}}, t) = \rho^+(L(\nu_{\mathrm{P}}, t))$,

(4) $L^-(\overline{\nu}_{\mathrm{P}}, t) = \rho^+(L^-(\nu_{\mathrm{P}}, t))$,

(5) $U(\underline{\mu}_{\mathrm{P}}, t) = \rho^+(U(\mu_{\mathrm{P}}, t))$,

(6) $U^+(\underline{\mu}_{\mathrm{P}}, t) = \rho^+(U^+(\mu_{\mathrm{P}}, t))$,

(7) $L(\underline{\nu}_{\mathrm{P}}, t) = \rho^-(L(\nu_{\mathrm{P}}, t))$, *and*

(8) $L^-(\underline{\nu}_{\mathrm{P}}, t) = \rho^-(L^-(\nu_{\mathrm{P}}, t))$.

Proof. (1) Let $a \in \mathcal{U}$. Then

$$
\begin{aligned}
a \in U(\overline{\mu}_{\mathrm{P}}, t) &\Leftrightarrow \overline{\mu}_{\mathrm{P}}(a) \geq t && \text{(Definition 12)}\\
&\Leftrightarrow \sup_{u \in (a)_\rho} \{\mu_{\mathrm{P}}(u)\} \geq t && \text{(Definition 9)}\\
&\Leftrightarrow \exists a \in (a)_\rho, \mu_{\mathrm{P}}(u) \geq t\\
&\Leftrightarrow \exists a \in (a)_\rho \cap U(\mu_{\mathrm{P}}, t) \neq \emptyset && \text{(Definition 12)}\\
&\Leftrightarrow a \in \rho^-(U(\mu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}
$$

(2) Let $a \in \mathcal{U}$. Then

$$
\begin{aligned}
a \in U^+(\overline{\mu}_{\mathrm{P}}, t) &\Leftrightarrow \overline{\mu}_{\mathrm{P}}(a) > t && \text{(Definition 12)}\\
&\Leftrightarrow \sup_{u \in (a)_\rho} \{\mu_{\mathrm{P}}(u)\} > t && \text{(Definition 9)}\\
&\Leftrightarrow \exists a \in (a)_\rho, \mu_{\mathrm{P}}(u) > t\\
&\Leftrightarrow \exists a \in (a)_\rho \cap U^+(\mu_{\mathrm{P}}, t) \neq \emptyset && \text{(Definition 12)}\\
&\Leftrightarrow a \in \rho^-(U^+(\mu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}
$$

(3) Let $a \in \mathcal{U}$. Then

$$
\begin{aligned}
a \in L(\overline{\nu}_{\mathrm{P}}, t) &\Leftrightarrow \overline{\nu}_{\mathrm{P}}(a) \leq t && \text{(Definition 12)}\\
&\Leftrightarrow \inf_{u \in (a)_\rho} \{\nu_{\mathrm{P}}(u)\} \leq t && \text{(Definition 9)}\\
&\Leftrightarrow \forall a \in (a)_\rho, \nu_{\mathrm{P}}(u) \leq t
\end{aligned}
$$

$$\begin{aligned}
&\Leftrightarrow \forall a \in (a)_\rho, a \in L(\nu_{\mathrm{P}}, t) && \text{(Definition 12)}\\
&\Leftrightarrow (a)_\rho \subseteq L(\nu_{\mathrm{P}}, t) &&\\
&\Leftrightarrow a \in \rho^+(L(\nu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}$$

(4) Let $a \in \mathcal{U}$. Then

$$\begin{aligned}
a \in L^-(\overline{\nu}_{\mathrm{P}}, t) &\Leftrightarrow \overline{\nu}_{\mathrm{P}}(a) < t && \text{(Definition 12)}\\
&\Leftrightarrow \inf_{u \in (a)_\rho} \{\nu_{\mathrm{P}}(u)\} < t && \text{(Definition 9)}\\
&\Leftrightarrow \forall a \in (a)_\rho, \nu_{\mathrm{P}}(u) < t &&\\
&\Leftrightarrow \forall a \in (a)_\rho, a \in L^-(\nu_{\mathrm{P}}, t) && \text{(Definition 12)}\\
&\Leftrightarrow (a)_\rho \subseteq L^-(\nu_{\mathrm{P}}, t) &&\\
&\Leftrightarrow a \in \rho^+(L^-(\nu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}$$

(5) Let $a \in \mathcal{U}$. Then

$$\begin{aligned}
a \in U(\underline{\mu}_{\mathrm{P}}, t) &\Leftrightarrow \underline{\mu}_{\mathrm{P}}(a) \geq t && \text{(Definition 12)}\\
&\Leftrightarrow \inf_{u \in (a)_\rho} \{\mu_{\mathrm{P}}(u)\} \geq t && \text{(Definition 9)}\\
&\Leftrightarrow \forall a \in (a)_\rho, \mu_{\mathrm{P}}(u) \geq t &&\\
&\Leftrightarrow \forall a \in (a)_\rho, a \in U(\mu_{\mathrm{P}}, t) && \text{(Definition 12)}\\
&\Leftrightarrow (a)_\rho \subseteq U(\mu_{\mathrm{P}}, t) &&\\
&\Leftrightarrow a \in \rho^+(U(\mu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}$$

(6) Let $a \in \mathcal{U}$. Then

$$\begin{aligned}
a \in U^+(\underline{\mu}_{\mathrm{P}}, t) &\Leftrightarrow \underline{\mu}_{\mathrm{P}}(a) > t && \text{(Definition 12)}\\
&\Leftrightarrow \inf_{u \in (a)_\rho} \{\mu_{\mathrm{P}}(u)\} > t && \text{(Definition 9)}\\
&\Leftrightarrow \forall a \in (a)_\rho, \mu_{\mathrm{P}}(u) > t &&\\
&\Leftrightarrow \forall a \in (a)_\rho, a \in U^+(\mu_{\mathrm{P}}, t) && \text{(Definition 12)}\\
&\Leftrightarrow (a)_\rho \subseteq U^+(\mu_{\mathrm{P}}, t) &&\\
&\Leftrightarrow a \in \rho^+(U^+(\mu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}$$

(7) Let $a \in \mathcal{U}$. Then

$$\begin{aligned}
a \in L(\underline{\nu}_{\mathrm{P}}, t) &\Leftrightarrow \underline{\nu}_{\mathrm{P}}(a) \leq t && \text{(Definition 12)}\\
&\Leftrightarrow \sup_{u \in (a)_\rho} \{\nu_{\mathrm{P}}(u)\} \leq t && \text{(Definition 9)}\\
&\Leftrightarrow \exists a \in (a)_\rho, \nu_{\mathrm{P}}(u) \leq t &&\\
&\Leftrightarrow \exists a \in (a)_\rho \cap L(\nu_{\mathrm{P}}, t) \neq \emptyset && \text{(Definition 12)}\\
&\Leftrightarrow a \in \rho^-(L(\nu_{\mathrm{P}}, t)). && \text{(Definition 7)}
\end{aligned}$$

(8) Let $a \in \mathcal{U}$. Then

$$\begin{aligned} a \in L^{-}(\underline{\nu}_{\mathrm{P}}, t) &\Leftrightarrow \underline{\nu}_{\mathrm{P}}(a) < t && \text{(Definition 12)} \\ &\Leftrightarrow \sup_{u \in (a)_\rho} \{\nu_{\mathrm{P}}(u)\} < t && \text{(Definition 9)} \\ &\Leftrightarrow \exists a \in (a)_\rho, \nu_{\mathrm{P}}(u) < t \\ &\Leftrightarrow \exists a \in (a)_\rho \cap L^{-}(\nu_{\mathrm{P}}, t) \neq \emptyset && \text{(Definition 12)} \\ &\Leftrightarrow a \in \rho^{-}(L^{-}(\nu_{\mathrm{P}}, t)). && \text{(Definition 7)} \end{aligned}$$

The following theorems show the relationships between RPFSs and their t-level subsets.

Theorem 13. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPS of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPS and a LoRUPS of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 2 and Lemmas 1 (1) and (3).

Theorem 14. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPS of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPS and a LoRUPS of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3 and Lemmas 1 (2) and (4).

Theorem 15. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFNUPF of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRNUPF and a LoRNUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 4 and Lemmas 1 (1) and (3).

Theorem 16. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFNUPF of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRNUPF and a LoRNUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 4 and Lemmas 1 (1) and (3).

Theorem 17. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPF of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPF and a LoRUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 6 and Lemmas 1 (1) and (3).

Theorem 18. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPF of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPF and a LoRUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 7 and Lemmas 1 (2) and (4).

Theorem 19. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPI of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPI and a LoRUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 8 and Lemmas 1 (1) and (3).

Theorem 20. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFUPI of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPI and a LoRUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 9 and Lemmas 1 (2) and (4).

Theorem 21. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFSUPI of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRSUPI and a LoRSUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 10 and Lemmas 1 (1) and (3).

Theorem 22. *Let ρ be a CR on $\mathcal{U}$. Then* P *is an UpRPFSUPI of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRSUPI and a LoRSUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 11 and Lemmas 1 (2) and (4).

Theorem 23. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPS of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPS and a LoRUPS of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 2 and Lemmas 1 (5) and (7).

Theorem 24. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPS of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPS and a LoRUPS of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3 and Lemmas 1 (6) and (8).

Theorem 25. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFNUPF of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRNUPF and a LoRNUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 4 and Lemmas 1 (5) and (7).

Theorem 26. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFNUPF of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRNUPF and a LoRNUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 5 and Lemmas 1 (6) and (8).

Theorem 27. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPF of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPF and a LoRUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 6 and Lemmas 1 (5) and (7).

Theorem 28. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPF of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPF and a LoRUPF of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 7 and Lemmas 1 (6) and (8).

Theorem 29. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPI of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPI and a LoRUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 8 and Lemmas 1 (5) and (7).

Theorem 30. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFUPI of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRUPI and a LoRUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 9 and Lemmas 1 (6) and (8).

Theorem 31. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFSUPI of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRSUPI and a LoRSUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 10 and Lemmas 1 (5) and (7).

Theorem 32. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a LoRPFSUPI of $\mathcal{U}$ if and only if $U^+(\mu_{\mathrm{P}}, t)$ and $L^-(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, an UpRSUPI and a LoRSUPI of $\mathcal{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 11 and Lemmas 1 (6) and (8).

Theorem 33. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI) of $\mathcal{U}$ if and only if $U(\mu_{\mathrm{P}}, t)$ and $L(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, RUPSs (resp., RNUPFs, RUPFs, RUPIs, and RSUPIs) of $\mathcal{U}$ for every $t \in [0, 1]$.*

Proof. It is straightforward by Theorems 13 (resp., Theorems 15, 17, 19, 21) and 23 (resp., Theorems 25, 27, 29, 31).

Theorem 34. *Let ρ be a CR on $\mathcal{U}$. Then* P *is a RPFUPS (resp., RPFNUPF, RPFUPF, RPFUPI, and RPFSUPI) of $\mathcal{U}$ if and only if $U^{+}(\mu_{\mathrm{P}}, t)$ and $L^{-}(\nu_{\mathrm{P}}, t)$ are, if the sets are nonempty, RUPSs (resp., RNUPFs, RUPFs, RUPIs, and RSUPIs) of $\mathcal{U}$ for every $t \in [0, 1]$.*

Proof. It is straightforward by Theorems 14 (resp., Theorems 16, 18, 20, 22) and 24 (resp., Theorems 26, 28, 30, 32).

4. Conclusions and Future Works

In this paper, we have introduced the concept of RSs to PFSs in UP-algebras. Then we have introduced fifteen types of RPFSs in UP-algebras, namely UpRPFUPSs, UpRPFNUPFs, UpRPFUPFs, UpRPFUPIs, UpRPFSUPIs, LoRPFUPSs, LoRPFNUPFs, LoRPFUPFs, LoRPFUPIs, LoRPFSUPIs, RPFUPSs, RPFNUPFs, RPFUPFs, RPFUPIs, and RPFSUPIs and so proved their generalizations. In addition, we investigated t-level subsets of RPFSs in UP-algebras in order to discuss the relationships between RPFSs and RSs in UP-algebras.

The following are some essential subjects for our future research of UP-algebras:

(1) to get more results in RPFSs,

(2) to define more types of RPFSs, and

(3) to study the soft set theory of PFSs.

Acknowledgements

This research and innovation activity is funded by National Research Council of Thailand (NRCT).

References

[1] S. S. Ahn and C. Kim. Rough set theory applied to fuzzy filters in BE-algebras. *Commun. Korean Math. Soc.*, 31(3):451–460, 2016.

[2] S. S. Ahn and J. M. Ko. Rough fuzzy ideals in BCK/BCI-algebras. *J. Comput. Anal. Appl.*, 25(1):75–84, 2018.

[3] M. A. Ansari, A. Haidar, and A. N. A. Koam. On a graph associated to UP-algebras. *Math. Comput. Appl.*, 23(4):61, 2018.

[4] M. A. Ansari, A. N. A. Koam, and A. Haider. Rough set theory applied to UP-algebras. *Ital. J. Pure Appl. Math.*, 42:388–402, 2019.

[5] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.*, 20:87–96, 1986.

[6] L. Chen and F. Wang. On rough ideals and rough fuzzy ideals of BCI-algebras. *Fifth international conference on fuzzy systems and knowledge discovery*, 5:281–284, 2008.

[7] R. Chinram and T. Panityakul. Rough Pythagorean fuzzy ideals in ternary semigroups. *J. Math. Computer Sci.*, 20:303–312, 2020.

[8] N. Dokkhamdang, A. Kesorn, and A. Iampan. Generalized fuzzy sets in UP-algebras. *Ann. Fuzzy Math. Inform.*, 16(2):171–190, 2018.

[9] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan. Fuzzy translations of a fuzzy set in UP-algebras. *J. Indones. Math. Soc.*, 23(2):1–19, 2017.

[10] A. Hussain, T. Mahmood, and M. I. Ali. Rough Pythagorean fuzzy ideals in semigroups. *Comput. Appl. Math.*, 38(15), 1986.

[11] A. Iampan. A new branch of the logical algebra: UP-algebras. *J. Algebra Relat. Top.*, 5(1):35–54, 2017.

[12] A. Iampan. Introducing fully UP-semigroups. *Discuss. Math., Gen. Algebra Appl.*, 38(2):297–306, 2018.

[13] A. Iampan. Multipliers and near UP-filters of UP-algebras. *J. Discrete Math. Sci. Cryptography*, 24(3):667–680, 2021.

[14] A. Iampan, M. Songsaeng, and G. Muhiuddin. Fuzzy duplex UP-algebras. *Eur. J. Pure Appl. Math.*, 13(3):459–471, 2020.

[15] Y. B. Jun, S.-Z. Song, and E. H. Roh. Generalized rough sets applied to BCK/BCI-algebras. *Discuss. Math., Gen. Algebra Appl.*, 41(2):343–360, 2021.

[16] T. Klinseesook, S. Bukok, and A. Iampan. Rough set theory applied to UP-algebras. *J. Inf. Optim. Sci.*, 41(3):705–722, 2020.

[17] R. Moradiana, . S. K. Shoarb, and A. Radfarc. Rough sets induced by fuzzy ideals in BCK-algebras. *J. Intell. Fuzzy Syst.*, 30:2397–2404, 2016.

[18] Z. Pawlak. Rough sets. *Int. J. Inform. Comp. Sci.*, 11:341–356, 1982.

[19] C. Prabpayak and U. Leerawat. On ideals and congruences in KU-algebras. *Sci. Magna*, 5(1):54–57, 2009.

[20] A. Satirad, R. Chinram, and A. Iampan. Pythagorean fuzzy sets in UP-algebras and approximations. *AIMS Math.*, 6(6):6002–6032, 2021.

[21] A. Satirad, R. Chinram, P. Julatha, and A. Iampan. Some Pythagorean fuzzy UP-filters of UP-algebras with approximations. Manuscript submitted for publication, November 2021.

[22] A. Satirad, P. Mosrijai, and A. Iampan. Formulas for finding UP-algebras. *Int. J. Math. Comput. Sci.*, 14(2):403–409, 2019.

[23] A. Satirad, P. Mosrijai, and A. Iampan. Generalized power UP-algebras. *Int. J. Math. Comput. Sci.*, 14(1):17–25, 2019.

[24] T. Senapati, Y. B. Jun, and K. P. Shum. Cubic set structure applied in UP-algebras. *Discrete Math. Algorithms Appl.*, 10(4):1850049, 2018.

[25] T. Senapati, G. Muhiuddin, and K. P. Shum. Representation of UP-algebras in interval-valued intuitionistic fuzzy environment. *Ital. J. Pure Appl. Math.*, 38:497–517, 2017.

[26] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan. Fuzzy sets in UP-algebras. *Ann. Fuzzy Math. Inform.*, 12(6):739–756, 2016.

[27] R. R. Yager. Pythagorean fuzzy subsets. *In:Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmomton, Canada*, pages 57–61, 2013.

[28] R. R. Yager and A. M. Abbasov. Pythagorean member grades, complex numbers, and decision making. *Int. J. Intell. Syst.*, 28:436–452, 2013.

[29] L. A. Zadeh. Fuzzy sets. *Inf. Cont.*, 8:338–353, 1965.