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On ϕ-β-Absorbing Submodules

Thawatchai Khumprapussorn

Department of Mathematics, School of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, THAILAND

Abstract. In this paper, we extend the concept of β-absorbing submodules to ϕ-β-absorbing submodules over a commutative ring with nonzero identity which is a generalization of 2-absorbing submodules. Let $\mathcal{S}(M)$ be the set of all submodules of M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is called a ϕ-β-absorbing submodule, if for each $r, s \in R$ and $m \in M$ with $rsm \in P \backslash \phi(P)$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Some of the properties and characterizations of ϕ-β-absorbing submodules are investigated.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and all modules are unital left R-modules. We recall that a proper submodule P of a left R-module M is called a prime submodule of M if for every $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in (P : M)$. Various generalizations of prime submodules have been studied. For example, see [5], [1] and [6], a proper submodule P of a left R-module M is called a 2-absorbing (resp. weakly 2-absorbing, almost 2-absorbing) submodule if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P$ (resp. $rsm \in P \backslash \{0\}$, $rsm \in P \backslash (P : M)P$), we have $rs \in (P : M)$ or $rm \in P$ or $sm \in P$. According to [4], $n\mathbb{Z}$ is a 2-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0$ or n is a prime number or $n = pq$ where p and q are prime numbers.

Let $\mathcal{S}(M)$ be the set of all submodules of M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. In this paper, we assume that $\phi(P) \subseteq P$. In [3], the authors introduced the concept of ϕ-2-absorbing submodule which is a generalization of 2-absorbing submodules. A proper submodule P of a left R-module M is a ϕ-2-absorbing submodule if whenever $a, b \in R, m \in M$ with $abm \in P$ and $abm \notin \phi(P)$, then $am \in P$ or $bm \in P$ or $ab \in (P : M)$. In addition, the notion of ϕ-2-absorbing submodule is also a generalization of both weakly 2-absorbing submodule and almost 2-absorbing submodule which depends on the definition of ϕ.

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Email address: `thawatchai.kh@kmitl.ac.th` (T.Khumprapussorn)

Let $(G, +)$ be a group and H is a subgroup of G. We denote the symbol $\beta(H)$ by $\{h + h \mid h \in H\}$ and $\alpha(H)$ by $\{h \mid h + h \in H\}$. We see that $\beta(H) \subseteq H \subseteq \alpha(H)$. If I is an ideal of R, then both of $\alpha(I)$ and $\beta(I)$ are ideals of R. Moreover, if N is a submodule of M, then both of $\alpha(N)$ and $\beta(N)$ are submodules of M.

In [2], a proper submodule P of a left R-module M is called β-absorbing if for any element $r, s \in R$ and $m \in M$ such that $rsm \in P$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. The characterization of β-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}$ was also given. On the $\mathbb{Z}$-module $\mathbb{Z}$, $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0$ or $n = 32$ or n is a prime number or $n = pq$ where p and q are prime numbers or $n = 2^3p$ where p is prime number or $n = 2pq$ where p and q are prime numbers. The characterization of β-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}$ explains that β-absorbing submodules need not to be 2-absorbing submodules. Also, in [2], a proper submodule P of M is a weakly β-absorbing submodule of M if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P\backslash\{0\}$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

In this research, we extend the notion of β-absorbing submodules to ϕ-β-absorbing submodules. First, we introduce notions of ϕ-β-absorbing submodules. A proper submodule P of a left R-module M is called a ϕ-β-absorbing submodule of M if for any element $r, s \in R$ and $m \in M$ such that $rsm \in P\backslash\phi(P)$, we have $rs+rs \in (P : M)$ or $r(m+m) \in P$ or $s(m + m) \in P$. In case $\phi_0(N) = \{0\}$ for all submodule N of M, we have weakly β-absorbing submodules and ϕ_0-β-absorbing submodules are equivalent. This case inspired us to investigate some basic properties of ϕ-β-absorbing submodules in section 2, whereas section 3 contains the characterizations of ϕ-β-absorbing submodules.

2. On ϕ-β-absorbing submodules

In this section, we define ϕ-β-absorbing submodules and obtain some related results.

Definition 1. *A proper submodule P of an R-module M is said to be a ϕ-β-absorbing submodule of M if whenever $r, s \in R$ and $m \in M$ such that $rsm \in P\backslash\phi(P)$, then $rs+rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.*

Every β-absorbing is a weakly β-absorbing submodule but the converse does not necessarity hold. As mentioned above, β-absorbing submodules and weakly β-absorbing submodules are special cases of ϕ-β-absorbing submodules.

Theorem 1. *If P is a ϕ-β-absorbing submodule of M and $(P : M)^2\beta(P) \nsubseteq \phi(P)$, then P is a β-absorbing submodule of M.*

Proof. Assume that P is a ϕ-β-absorbing submodule of M and $(P : M)^2\beta(P) \nsubseteq \phi(P)$. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P$. If $rsm \notin \phi(P)$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Next, assume that $rsm \in \phi(P)$.

Case 1. $rsP \nsubseteq \phi(P)$.

Then $rsp_0 \notin \phi(P)$ for some $p_0 \in P$. Hence $rs(m + p_0) \in P\backslash\phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $rs + rs \in (P : M)$ or $r(m + p_0 + m + p_0) \in P$ or

$s(m+p_0+m+p_0) \in P$. Since $p_0 \in P$, we have $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Case 2. $rsP \subseteq \phi(P)$.

Subcase 2.1 $s(P:M)m \nsubseteq \phi(P)$.

There exists an element $a_0 \in (P:M)$ such that $sa_0m \notin \phi(P)$. Thus $(r+a_0)sm = rsm+a_0sm \in P \setminus \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $(r+a_0)s+(r+a_0)s \in (P:M)$ or $(r+a_0)(m+m) \in P$ or $s(m+m) \in P$. Then $rs+a_0s+rs+a_0s \in (P:M)$ or $r(m+m)+a_0(m+m) \in P$ or $s(m+m) \in P$. Since $a_0 \in (P:M)$, $a_0M \subseteq P$. This implies that $a_0s+a_0s \in (P:M)$ and $a_0(m+m) \in P$. Therefore $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Subcase 2.2 $s(P:M)m \subseteq \phi(P)$.

Since $(P:M)^2\beta(P) \nsubseteq \phi(P)$, we have that $kt(n+n) \notin \phi(P)$ for some $k,t \in (P:M)$ and $n \in P$. If $rkm \notin \phi(P)$, then $r(k+s)m = rkm+rsm \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $r(k+s)+r(k+s) \in (P:M)$ or $r(m+m) \in P$ or $(k+s)(m+m) \in P$. Since $k \in (P:M)$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$. Similarly, if $rtm \notin \phi(P)$, then $r(t+s)m = rtm+rsm \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $r(t+s)+r(t+s) \in (P:M)$ or $r(m+m) \in P$ or $(t+s)(m+m) \in P$. Since $t \in (P:M)$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$. From now on, we assume that

$$rkm \in \phi(P) \text{ and } rtm \in \phi(P). \tag{1}$$

If $ktm \notin \phi(P)$, then $(k+r)(t+s)m \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $(k+r)(t+s)+(k+r)(t+s) \in (P:M)$ or $(k+r)(m+m) \in P$ or $(t+s)(m+m) \in P$. Since $k,t \in (P:M)$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$. Now, we assume that

$$ktm \in \phi(P). \tag{2}$$

Subsubcase 2.2.1 $kr(n+n) \notin \phi(P)$ or $st(n+n) \notin \phi(P)$.

Suppose that $kr(n+n) \notin \phi(P)$. Then $r(s+k)(n+n+m) \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $r(s+k)+r(s+k) \in (P:M)$ or $r(n+n+m+n+n+m) \in P$ or $(s+k)(n+n+m+n+n+m) \in P$. This implies that $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Next, suppose that $st(n+n) \notin \phi(P)$. Then $s(r+t)(n+n+m) \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $s(r+t)+s(r+t) \in (P:M)$ or $s(n+n+m+n+n+m) \in P$ or $(r+t)(n+n+m+n+n+m) \in P$. This implies that $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Subsubcase 2.2.2 $kr(n+n) \in \phi(P)$ and $st(n+n) \in \phi(P)$.

Then $(s+k)(r+t)(n+n+m) \notin \phi(P)$. Since P is a ϕ-β-absorbing submodule of M, $(s+k)(r+t)+(s+k)(r+t) \in (P:M)$ or $(s+k)(n+n+m+n+n+m) \in P$ or $(r+t)(n+n+m+n+n+m) \in P$. Since $k,t \in (P:M)$, we have $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Therefore P is a β-absorbing submodule of M.

Corollary 1. *[1] If P is a weakly β-absorbing submodule of M and $(P:M)^2\beta(P) \neq \{0\}$, then P is a β-absorbing submodule of M.*

Next, we use the function $\phi_i : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ by the following meaning, for any submodule N of an R-module M and natural number n with $n \geq 2$,

$$\begin{aligned}
\phi_{\emptyset}(N) &= \emptyset \\
\phi_0(N) &= \{0\} \\
\phi_1(N) &= (N:M)\beta(N) \\
\phi_n(N) &= (N:M)^n\beta(N) \\
\phi_{\omega}(N) &= \bigcap_{i=1}^{\infty}(N:M)^i\beta(N)
\end{aligned}$$

Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\varphi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be functions. We write $\phi \leq \varphi$ if $\phi(N) \subseteq \varphi(N)$ for all $N \in \mathcal{S}(M)$. Then $\phi_{\emptyset} \leq \phi_0 \leq \phi_{\omega} \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Proposition 1. *Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\varphi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be functions such that $\phi \leq \varphi$. If P is a ϕ-β-absorbing submodule of M, then P is a φ-β-absorbing submodule of M.*

Proof. This proof is straightforward.

Let M_1 be a R_1-module and M_2 be a R_2-module. Then $M = M_1 \times M_2$ is an $R_1 \times R_2$-module by $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$. Next, let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function and P be a submodule of M_1. Then

$$(P \times M_2)\backslash\phi(P \times M_2) \subseteq (P \times M_2)\backslash(\{0\} \times M_2) = (P\backslash\{0\}) \times M_2.$$

We have the following results.

Proposition 2. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. If P is a ϕ_0-β-absorbing submodule of M_1 with $\{0\} \times M_2 \subseteq \phi(P \times M_2)$, then $P \times M_2$ is a ϕ-β-absorbing submodule of M.*

Proof. Assume that P is a ϕ_0-β-absorbing submodule of M_1 with $\{0\} \times M_2 \subseteq \phi(P \times M_2)$. Let $(r_1, r_2), (s_1, s_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ be such that $(r_1, r_2)(s_1, s_2)(m_1, m_2) \in (P \times M_2)\backslash\phi(P \times M_2)$. Then $r_1s_1m_1 \in P\backslash\{0\}$. Since P is a ϕ_0-β-absorbing submodule of M_1, $r_1s_1 + r_1s_1 \in (P:M)$ or $r_1(m_1 + m_1) \in P$ or $s_1(m_1 + m_1) \in P$. This implies that $(r_1, r_2)(s_1, s_2) + (r_1, r_2)(s_1, s_2) \in (P \times M_2 : M_1 \times M_2)$ or $(r_1, r_2)[(m_1, m_2) + (m_1, m_2)] \in P \times M_2$ or $(s_1, s_2)[(m_1, m_2) + (m_1, m_2)] \in P \times M_2$. Therefore $P \times M_2$ is a ϕ-β-absorbing submodule of M.

Proposition 3. *Let M_i be an R_i-module and $\phi_{M_i} : \mathcal{S}(M_i) \to \mathcal{S}(M_i) \cup \{\emptyset\}$ be a function where $i = 1, 2$. For this result, we define $\phi : \mathcal{S}(M_1 \times M_2) \to \mathcal{S}(M_1 \times M_2) \cup \{\emptyset\}$ by $\phi = \phi_{M_1} \times \phi_{M_2}$. If $P_1 \times P_2$ is a ϕ-β-absorbing submodule of $M_1 \times M_2$, then P_i is a ϕ_{M_i}-β-absorbing submodule of M_i.*

Proof. Assume that $P_1 \times P_2$ is a ϕ-β-absorbing submodule of $M_1 \times M_2$. To show that P_1 is a ϕ_{M_1}-β-absorbing submodule of M_1, let $r, s \in R_1$ and $m \in M_1$ be such that $rsm \in P_1 \backslash \phi_{M_1}(P_1)$. Since $\phi(P_1 \times P_2) = \phi_{M_1}(P_1) \times \phi_{M_2}(P_2)$, we have $(r, 1)(s, 1)(m, 0) = (rsm, 0) \in (P_1 \times P_2) \backslash \phi(P_1 \times P_2)$. Since $P_1 \times P_2$ is a ϕ-β-absorbing submodule of $M_1 \times M_2$, $(r, 1)(s, 1) + (r, 1)(s, 1) \in (P_1 \times M_2 : M_1 \times M_2)$ or $(r, 1)[(m, 0) + (m, 0)] \in P_1 \times M_2$ or $(s, 1)[(m, 0) + (m, 0)] \in P_1 \times M_2$. This implies that $rs + rs \in (P_1 : M_1)$ or $r(m + m) \in P_1$ or $s(m + m) \in P_1$. Hence P_1 is a ϕ_{M_1}-β-absorbing submodule of M_1. Similarly, we can show that P_2 is a ϕ_{M_2}-β-absorbing submodule of M_2.

3. Characterizations of ϕ-β-absorbing submodules

The purpose of this section is to investigate characterizations of ϕ-β-absorbing submodules. For a submodule N of a left R-module M and $r \in R$, we define the symbol N_r by $\{m \in M \mid rm \in N\}$.

Theorem 2. *Let P be a submodule of M. Then the following statements are equivalent :*

(i) P is a ϕ-β-absorbing submodule of M.

(ii) For all $r, s \in R$, if $rs + rs \notin (P : M)$, then $P_{rs} \subseteq \alpha(P_r) \cup \alpha(P_s) \cup \phi(P)_{rs}$.

Proof. $(i) \to (ii)$ Assume that P is a ϕ-β-absorbing submodule of M. Let $r, s \in R$ be such that $rs + rs \notin (P : M)$ and $m \in P_{rs}$. Then $rsm \in P$. If $rsm \in \phi(P)$, then $m \in \phi(P)_{rs}$. Assume that $rsm \in P \backslash \phi(P)$. Since P is a ϕ-β-absorbing submodule of M and $rs + rs \notin (P : M)$, $r(m + m) \in P$ or $s(m + m) \in P$. Thus $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. This shows that $P_{rs} \subseteq \alpha(P_r) \cup \alpha(P_s) \cup \phi(P)_{rs}$.

$(ii) \to (i)$ Assume that (ii) holds. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P \backslash \phi(P)$ and $rs + rs \notin (P : M)$. Then $m \in P_{rs}$. By our assumptions, $m \in \alpha(P_r) \cup \alpha(P_s)$. Hence $r(m + m) \in P$ or $s(m + m) \in P$. Therefore P is a ϕ-β-absorbing submodule of M.

For a submodule N of a left R-module M and $m \in M$, we define the symbol $(N : m)$ by $\{r \in R \mid rm \in N\}$.

Theorem 3. *Let P be a submodule of M. Then the following statements are equivalent :*

(i) P is a ϕ-β-absorbing submodule of M.

(ii) For all $s \in R$ and $m \in M$, if $sm \notin \alpha(P)$, then
$(P : sm) \subseteq \alpha((P : sM)) \cup \alpha((P : m)) \cup (\phi(P) : sm)$.

Proof. $(i) \to (ii)$ Assume that P is a ϕ-β-absorbing submodule of M. Let $s \in R$ and $m \in M$ be such that $sm \notin \alpha(P)$. Let $r \in (P : sm)$. Then $rsm \in P$. If $rsm \in \phi(P)$, then $r \in (\phi(P) : sm)$. Assume that $rsm \in P \backslash \phi(P)$. Since P is a ϕ-β-absorbing submodule of M and $sm \notin \alpha(P)$, $rs + rs \in (P : M)$ or $r(m + m) \in P$. Thus $r \in \alpha((P : sM))$ or $r \in \alpha((P : m))$. This proves that $(P : sm) \subseteq \alpha((P : sM)) \cup \alpha((P : m)) \cup (\phi(P) : sm)$.

$(ii) \to (i)$ Assume that (ii) holds. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P \backslash \phi(P)$ and $sm \notin \alpha(P)$. Then $r \in (P : sm)$ and $r \notin (\phi(P) : sm)$. This implies that $r \in \alpha((P : sM)) \cup \alpha((P : m))$. Hence $rs + rs \in (P : M)$ or $r(m + m) \in P$. This shows that P is a ϕ-β-absorbing submodule of M.

By the definition of ϕ_1, a proper submodule P of M is said to be ϕ_1-β-absorbing provided for each $r, s \in R$ and $m \in M$, if $rsm \in P \backslash (P : M)\beta(P)$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. For an element m of an R-module M, we recall the symbol that $(\{0\} : m) = \{r \in R \mid rm = \{0\}\}$ and denote $(\{0\} : m)$ by $(0 : m)$ for short.

Finally, we show some assumptions which β-absorbing submodules and ϕ_1-β-absorbing submodules are equivalent.

Theorem 4. *Let m be a nonzero element of an R-module M such that $(0 : m) = \{0\}$ and $Rm \neq M$. Then Rm is a β-absorbing submodule of M if and only if Rm is a ϕ_1-β-absorbing submodule of M.*

Proof. $(\to)$ This is obvious.

$(\leftarrow)$ Assume that Rm is not a β-absorbing submodule of M. Then there are $r, s \in R$ and $x \in M$ such that $rsx \in Rm$ and $rs+rs \notin (Rm : M)$ and $r(x+x) \notin Rm$ and $s(x+x) \notin Rm$. If $rsx \notin (Rm : M)\beta(Rm)$, then we are done. Assume that $rsx \in (Rm : M)\beta(Rm)$. Since $r(x + x) \notin Rm$, $r(x + x + m + m) \notin Rm$. Since $rsx \in Rm$ and $rsm \in Rm$, $rs(x + m) \in Rm$. If $rs(x + m) \notin (Rm : M)\beta(Rm)$, then we are done. Suppose that $rs(x + m) \in (Rm : M)\beta(Rm)$. Since $rsx \in (Rm : M)\beta(Rm)$, $rsm \in (Rm : M)\beta(Rm)$. Note that $\beta(Rm) = \beta(R)m$. Hence $rsm \in (Rm : M)\beta(R)m$. Thus $rsm = tm$ for some $t \in (Rm : M)\beta(R)$. So $rs = t \in (Rm : M)\beta(R) \subseteq (Rm : M)$. Consequently, $rs + rs \in (Rm : M)$ which is a contradiction.

For each $r \in R$, we would like to remind that $\{0\}_r = \{m \in M \mid rm = 0\}$.

Theorem 5. *Let M be an R-module and $r \in R$ be such that $rM \neq M$ and $\{0\}_r \subseteq \beta(rM)$. Then rM is a β-absorbing submodule of M if and only if rM is a ϕ_1-β-absorbing submodule of M.*

Proof. $(\to)$ This is obvious.

$(\leftarrow)$ Assume that rM is a ϕ_1-β-absorbing submodule of M. Let $a, b \in R$ and $m \in M$ such that $abm \in rM$. There are 2 cases to be considered :

(i) $abm \notin (rM : M)\beta(rM)$,

(ii) $abm \in (rM : M)\beta(rM)$.

First, we consider Case (i). Since rM is a ϕ_1-β-absorbing submodule of M, $ab + ab \in (rM : M)$ or $a(m + m) \in rM$ or $b(m + m) \in rM$. Next, Case (ii) is considered. Since $abm \in rM$ and $rbm \in rM$, $(ab + rb)m \in rM$. If $(ab + rb)m \notin (rM : M)\beta(rM)$, then $(a + r)b + (a + r)b \in (rM : M)$ or $(a + r)(m + m) \in rM$ or $b(m + m) \in rM$. Since $rb + rb \in (rM : M)$ and $r(m + m) \in rM$, $ab + ab \in (rM : M)$ or $a(m + m) \in rM$ or $b(m + m) \in rM$. Assume that $(ab + rb)m \in (rM : M)\beta(rM)$. Since $\beta(rM) = r\beta(M)$, $(ab + rb)m$ and abm are elements of $r(rM : M)\beta(M)$. This implies that $rbm \in r(rM : M)\beta(M)$. Thus $rbm = ry$ for some $y \in (rM : M)\beta(M)$. So $bm - y \in \{0\}_r \subseteq \beta(rM)$. Therefore $bm = (bm-y)+y \in (rM : M)\beta(rM)+\{0\}_r \subseteq \beta(rM)$. We have $b(m+m) \in rM$. Therefore rM is a β-absorbing submodule of M.

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