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On (Λ, p)-closed sets and the related notions in topological spaces

Chawalit Boonpok[1], Chokchai Viriyapong[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

Abstract. This article deals with the concepts of Λ_p-sets and (Λ, p)-closed sets which are defined by utilizing the notions of preopen sets and preclosed sets. We also introduce and characterize some new low separation axioms. Characterizations of Λ_p-R_0 spaces are given. Moreover, we introduce the concept of weakly (Λ, p)-continuous functions. In particular, several characterizations of weakly (Λ, p)-continuous functions are established.

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Key Words and Phrases: Λ_p-set, (Λ, p)-closed set, Λ_p-R_0 space, weakly (Λ, p)-continuous function

1. Introduction

In 1982, Mashhour et al. [16] introduced the notion of preopen sets which is also known under the name of locally dense sets [7] in the literature. Since then, this notion received wide usage in general topology. Kar and Bhattachryya [12] introduced new separation axioms pre-T_0, pre-T_1 and pre-T_2 by using preopen sets due to Mashhour et al. [16]. Caldas [3] and Jafari [11] introduced independently the notions of p-D-sets and a separation axiom p-D_1 which is strictly between pre-T_0 and pre-T_1. Caldas et al. [4] introduced two new classes of topological spaceS called pre-R_0 and pre-R_1 spaces in terms of concept of preopen sets and investigated some of their fundamental properties. Mashhour et al. [15] introduced and studied the concept of supra topological spaces by dropping a finite intersection condition of topological spaces. El-Shafei et al. [9] defined some concepts on supra topological spaces using supra preopen sets and investigated main properties. Al-shami et al. [2] introduced and investigated new separation axioms, namely supra semi T_i-spaces ($i = 0, 1, 2, 3, 4$). In [1], the present author introduce the version of complete Hausdorffness and complete regularity on supra topological spaces and discussed their

*Corresponding author.
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Email addresses: chawalit.b@msu.ac.th (C. Boonpok), chokchai.v@msu.ac.th (C. Viriyapong)

fundamental properties. Cammaroto and Noiri [6] defined Λ_m-sets and generalized Λ_m-sets in an m-spaces (X, m) which is equivalent to a generalized topological spaces [14] and investigated properties of several low separation axioms of topologies constructed by the families of these sets. Ganster et al. [10] introduced the notions of a pre-Λ-set and a pre-V-set in a topological space and studied the fundamental properties of pre-Λ-sets and pre-V-sets. Caldas et al. [5] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 spaces by using the notions of (Λ, θ)-open sets and (Λ, θ)-closure operators. The concept of weak continuity due to Levine [13] is one of the most important weak forms of continuity in topological spaces. Rose [18] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Noiri [17] introduced the concept of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of such functions. The paper is organized as follows. In section 3, we obtain fundamental properties of Λ_p-sets and investigate low separation axioms of an Alexandorff spaces (X, Λ_p). In section 4, we introduce the concept of (Λ, p)-closed sets and investigate properties of several low separation axioms of topologies constructed by the families of these sets. In section 5, we investigate some characterizations of Λ_p-R_0 spaces. In the last section, we introduce the concept of weakly (Λ, p)-continuous functions and investigate several characterizations of such functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is said to be *preopen* [16] (resp. *preclosed* [16]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq A$). By $PO(X, \tau)$ and $PC(X, \tau)$, we denote the collection of all preopen sets and the collection of all preclosed sets of a topological space (X, τ), respectively. The intersection of all preclosed sets containig A is called the *preclosure* [8] of A and is denoted by $p\mathrm{Cl}(A)$.

Definition 1. *A topological space (X, τ) is said to be:*

(1) pre-T_0 [12] *if, for each pair of distinct points of X, there exists a preopen set containing one of the points but not the other;*

(2) pre-T_1 [12] *if, for each pair of distinct points x and y of X, there exists a pair of preopen sets one containing x but not y and the other containing y but not x;*

(3) pre-R_0 [4] *if every preopen set contains the preclosure of each of its singletons.*

Definition 2. *Let A be a subset of a topological space (X, τ). A subset $\Lambda_p(A)$* [10] *is defined as follows: $\Lambda_p(A) = \cap\{O \in PO(X, \tau) | A \subseteq O\}$.*

Lemma 1. [10] *For subsets A, B and $A_i(i \in I)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_p(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_p(A) \subseteq \Lambda_p(B)$.

(3) $\Lambda_p(\Lambda_p(A)) = \Lambda_p(A)$.

(4) $\Lambda_p(\cap\{A_i \mid i \in I\}) \subseteq \cap\{\Lambda_p(A_i) \mid i \in I\}$.

(5) $\Lambda_p(\cup\{A_i \mid i \in I\}) = \cup\{\Lambda_p(A_i) \mid i \in I\}$.

3. Λ_p-sets and a topological space (X, Λ_p)

In this section, we obtain fundamental properties of Λ_p-sets and investigate low separation axioms of an Alexandorff space (X, Λ_p).

Definition 3. *A subset A of a topological space (X, τ) is called a Λ_p-set (pre-Λ-set* [10]*) if $A = \Lambda_p(A)$. The family of all Λ_p-sets of (X, τ) is denoted by $\Lambda_p(X, \tau)$ (or simply Λ_p).*

Lemma 2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_p(A)$ *is a* Λ_p*-set.*

(2) If A is preopen, then A is a Λ_p-set.

Proof. This follows readily from Lemma 1.

Lemma 3. [10] *For subsets A and $A_i(i \in I)$ of a topological space (X, τ), the following properties hold:*

(1) $\emptyset$ and X are pre-Λ-sets.

(2) Every union of pre-Λ-sets is a pre-Λ_p-set.

(3) Every intersection of pre-Λ-sets is a pre-Λ-set.

Theorem 1. *For a topological space (X, τ), the pair (X, Λ_p) is an Alexandroff space.*

Proof. This is an immediate consequence of Lemma 3.

Theorem 2. *Let (X, τ) be a topological space. Then, $\Lambda_p = \Lambda_{\Lambda_p}$.*

Proof. By Lemma 2, $PO(X, \tau) \subseteq \Lambda_p$. Let A be any subset of X. Then,

$$\Lambda_{\Lambda_p} = \cap\{U \mid A \subseteq U, U \in \Lambda_p\} \subseteq \{U \mid A \subseteq U, U \in PO(X, \tau)\} = \Lambda_p(A).$$

Thus, $\Lambda_{\Lambda_p}(A) \subseteq \Lambda_p(A)$. Now, we suppose that $x \notin \Lambda_{\Lambda_p}(A)$. Then, there exists $U \in \Lambda_p$ such that $A \subseteq U$ and $x \notin U$. Since $x \notin U$, there exists $V \in PO(X, \tau)$ such that $U \subseteq V$ and $x \notin V$. Therefore, $x \notin \Lambda_p(A)$. This shows that $\Lambda_{\Lambda_p}(A) \supseteq \Lambda_p(A)$ and hence $\Lambda_p(A) = \Lambda_{\Lambda_p}(A)$.

Theorem 3. *A topological space* (X, τ) *is pre-*R_0 *if and only if the topological space* (X, Λ_p) *is* R_0*.*

Proof. Let $V \in \Lambda_p$ and let $x \in V$. Then, $x \in \Lambda_p(V) = \cap\{U \mid V \subseteq U, U \in PO(X,\tau)\}$ and $x \in U$ for any $U \in PO(X,\tau)$ containing V. Since (X,τ) is pre-R_0, $p\mathrm{Cl}(\{x\}) \subseteq U$ for every $U \in PO(X,\tau)$ containing V. Thus,

$$p\mathrm{Cl}(\{x\}) \subseteq \cap\{U \mid V \subseteq U, U \in PO(X,\tau)\} = \Lambda_p(V) = V.$$

Since $PO(X,\tau) \subseteq \Lambda_p$, Λ_p-$\mathrm{Cl}(\{x\}) \subseteq p\mathrm{Cl}(\{x\}) \subseteq V$, where Λ_p-$\mathrm{Cl}(\{x\})$ denotes the closure of the singleton $\{x\}$ in the topological space (X, Λ_p). This shows that (X, Λ_p) is R_0.

Conversely, suppose that (X, Λ_p) is R_0. Let $V \in \Lambda_p$ and $x \in V$. Since $PO(X,\tau) \subseteq \Lambda_p$, we have Λ_p-$\mathrm{Cl}(\{x\}) \subseteq V$. Since $X - \Lambda_p$-$\mathrm{Cl}(\{x\}) \in \Lambda_p$,

$$X - \Lambda_p\text{-}\mathrm{Cl}(\{x\}) = \cap\{U \mid X - \Lambda_p\text{-}\mathrm{Cl}(\{x\}) \subseteq U, U \in PO(X,\tau)\}.$$

Then, there exists $U \in PO(X,\tau)$ such that $X - \Lambda_p$-$\mathrm{Cl}(\{x\}) \subseteq U$ and $x \notin U$. Thus, $x \in X - U \subseteq \Lambda_p$-$\mathrm{Cl}(\{x\}) \subseteq V$. Since $X - U$ is preclosed, $p\mathrm{Cl}(\{x\}) \subseteq X - U \subseteq V$. Consequently, we obtain (X,τ) is pre-R_0.

Theorem 4. *A topological space* (X, τ) *is pre-*T_0 *if and only if the topological space* (X, Λ_p) *is* T_0*.*

Proof. This is obvious since $PO(X,\tau) \subseteq \Lambda_p$.

Conversely, let x and y be any pair of distinct points of X. Since (X, Λ_p) is T_0, there exists $V \in \Lambda_p$ such that either $x \in V$ and $y \notin V$ or $x \notin V$ and $y \in V$. In case $x \in V$ and $y \notin V$, there exists $U \in PO(X,\tau)$ such that $V \subseteq U$ and $y \notin U$. However, since $x \in V, x \in U$. In case $x \notin V$ and $y \in V$, similarly there exists $U \in PO(X,\tau)$ such that $x \notin U$ and $y \in U$. Thus, (X,τ) is pre-T_0.

Lemma 4. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X,τ) *is pre-*T_1*;*

(2) For each $x \in X$*, the singleton* $\{x\}$ *is preclosed in* (X,τ)*.*

(3) For each $x \in X$*, the singleton* $\{x\}$ *is a* Λ_p*-set.*

Proof. (1) ⇒ (2): Let y be any point of X and let $x \in X - \{y\}$. Then, there exists a preopen set V_x such that $x \in V_x$ and $y \notin V_x$. Thus, $X - \{y\} = \cup_{x \in X - \{y\}} V_x$ and hence the singleton $\{y\}$ is preclosed in X.

(2) ⇒ (3): Let x be any point of X and let $y \in X - \{x\}$. Then,

$$x \in (X - \{y\}) \in PO(X,\tau)$$

and $\Lambda_p(\{x\}) \subseteq X - \{y\}$. Therefore, $y \notin \Lambda_p(\{x\})$ and hence $\Lambda_p(\{x\}) \subseteq \{x\}$. Thus, $\Lambda_p(\{x\}) = \{x\}$. This shows that $\{x\}$ is a Λ_p-set.

(3) $\Rightarrow$ (1): Suppose that the singleton $\{x\}$ is a Λ_p-set for each $x \in X$. Let x and y be any distinct points. Then, $y \notin \Lambda_p(\{x\})$ and there exists a preopen set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, $x \notin \Lambda_p(\{y\})$ and there exists a preopen set U_y such that $y \in U_y$ and $x \notin U_y$. This shows that (X, τ) is pre-T_1.

Theorem 5. *A topological space (X, τ) is pre-T_1 if and only if the topological space (X, Λ_p) is discrete.*

Proof. Suppose that (X, τ) is pre-T_1. Let $x \in X$. By Lemma 4, $\{x\}$ is a Λ_p-set and hence $\{x\}$ is open in (X, Λ_p). Thus, every subset of X is open in (X, Λ_p). This shows that (X, Λ_p) is discrete.

Conversely, suppose that a topological space (X, Λ_p) is discrete. For any point $x \in X$, $\{x\}$ is open in (X, Λ_p) and hence $\{x\}$ is a Λ_p-set, by Lemma 4, we have (X, τ) is pre-T_1.

Corollary 1. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is pre-T_1;

(2) (X, τ) is pre-R_0 and pre-T_0;

(3) (X, Λ_p) is R_0 and T_0;

(4) (X, Λ_p) is T_1;

(5) (X, Λ_p) is discrete.

Proof. (1) $\Rightarrow$ (2): By Lemma 4, every pre-T_1 space is pre-R_0 and pre-T_0.

(2) $\Rightarrow$ (1): Since (X, τ) is pre-T_0, for any distinct point x, y of X, there exists a preopen set U of X such that $x \in U$ and $y \notin U$. Hence, $p\mathrm{Cl}(\{x\}) \subseteq U$ since (X, τ) is pre-R_0. Thus, $x \notin X - p\mathrm{Cl}(\{x\})$ and hence $y \in X - U \subseteq X - p\mathrm{Cl}(\{x\}) \in PO(X, \tau)$. This shows that (X, τ) is pre-T_1.

(2) $\Leftrightarrow$ (3): This is an immediate consequence of Theorem 3 and Theorem 4.

(3) $\Leftrightarrow$ (4): This proof is obvious.

(5) $\Leftrightarrow$ (1): This is an immediate consequence of Theorem 5.

4. (Λ, p)-closed sets

In this section, we introduce the notion of (Λ, p)-closed sets in topological spaces. Moreover, some properties of (Λ, p)-closed sets are discussed.

Definition 4. *A subset A of a topological space (X, τ) is called (Λ, p)-closed if $A = T \cap C$, where T is a Λ_p-set and C is a preclosed set. The collection of all (Λ, p)-closed sets in a topological space (X, τ) is denoted by $\Lambda_p C(X, \tau)$.*

Theorem 6. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A *is* (Λ, p)*-closed;*

(2) $A = T \cap pCl(A)$*, where* T *is a* Λ_p*-set;*

(3) $A = \Lambda_p(A) \cap pCl(A)$*.*

Proof. (1) ⇒ (2): Let $A = T \cap C$,where T is a Λ_p-set and C is a preclosed set. Since $A \subseteq C$, we have $p\text{Cl}(A) \subseteq C$ and hence $A = T \cap C \supseteq T \cap p\text{Cl}(A) \supseteq A$. Consequently, we obtain $A = T \cap p\text{Cl}(A)$.

(2) ⇒ (3): Let $A = T \cap p\text{Cl}(A)$, where T is a Λ_p-set. Since $A \subseteq T$, $\Lambda_p(A) \subseteq \Lambda_p(T) = T$ and hence $A \subseteq \Lambda_p(A) \cap p\text{Cl}(A) \subseteq T \cap p\text{Cl}(A) = A$. Thus, $A = \Lambda_p(A) \cap p\text{Cl}(A)$.

(3) ⇒ (1): Since $\Lambda_p(A)$ is a Λ_p-set, $p\text{Cl}(A)$ is preclosed and $A = \Lambda_p(A) \cap p\text{Cl}(A)$. This shows that A is (Λ, p)-closed.

Definition 5. *A subset* A *of a topological space* (X, τ) *is said to be* (Λ, p)*-open if the complement of* A *is* (Λ, p)*-closed. The collection of all* (Λ, p)*-open sets in a topological space* (X, τ) *is denoted by* $\Lambda_p O(X, \tau)$*.*

Theorem 7. *For a subset* $A_\gamma (\gamma \in \Gamma)$ *of a topological space* (X, τ)*, the following properties hold:*

(1) If A_γ *is* (Λ, p)*-closed for each* $\gamma \in \Gamma$*, then* $\cap\{A_\gamma \mid \gamma \in \Gamma\}$ *is* (Λ, p)*-closed.*

(2) If A_γ *is* (Λ, p)*-open for each* $\gamma \in \Gamma$*, then* $\cup\{A_\gamma \mid \gamma \in \Gamma\}$ *is* (Λ, p)*-open.*

Proof. (1) Suppose that A_γ is (Λ, p)-closed for each $\gamma \in \Gamma$. Then, for each γ, there exist a Λ_p-set T_γ and a preclosed set C_γ such that $A_\gamma = T_\gamma \cap C_\gamma$. Thus,

$$\cap_{\gamma \in \Gamma} A_\gamma = \cap_{\gamma \in \Gamma} (T_\gamma \cap C_\gamma) = (\cap_{\gamma \in \Gamma} T_\gamma) \cap (\cap_{\gamma \in \Gamma} C_\gamma).$$

Since $\cap_{\gamma \in \Gamma} C_\gamma$ is a preclosed set and by Lemma 3, we have $\cap_{\gamma \in \Gamma} T_\gamma$ is a Λ_p-set. This shows that $\cap_{\gamma \in \Gamma} A_\gamma$ is (Λ, p)-closed.

(2) Let A_γ be (Λ, p)-open for each $\gamma \in \Gamma$. Then, $X - A_\gamma$ is (Λ, p)-closed, by (1), we have $X - \cup_{\gamma \in \Gamma} A_\gamma = \cap_{\gamma \in \Gamma} (X - A_\gamma)$ is (Λ, p)-closed and hence $\cup_{\gamma \in \Gamma} A_\gamma$ is (Λ, p)-open.

Theorem 8. *Let* (X, τ) *be a pre-*R_0 *space. For each* $x \in X$*,* $\{x\}$ *is* (Λ, p)*-closed if and only if* $\{x\}$ *is preclosed.*

Proof. Suppose that $\{x\}$ is a (Λ, p)-closed set. By Theorem 6,

$$\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\}).$$

For any preopen set U containing x, $p\text{Cl}(\{x\}) \subseteq U$ and hence $p\text{Cl}(\{x\}) \subseteq \Lambda_p(\{x\})$. Thus, $\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\}) \supseteq p\text{Cl}(\{x\})$. This shows that $\{x\}$ is preclosed.

Conversely, suppose that $\{x\}$ is a preclosed set. Since $\{x\} \subseteq \Lambda_p(\{x\})$, we have $\Lambda_p(\{x\}) \cap p\text{Cl}(\{x\}) = \Lambda_p(\{x\}) \cap \{x\} = \{x\}$, by Theorem 6, $\{x\}$ is (Λ, p)-closed.

Theorem 9. *A topological space (X, τ) is pre-T_0 if and only if for each $x \in X$, the singleton $\{x\}$ is (Λ, p)-closed.*

Proof. Suppose that (X, τ) is pre-T_0. For each $x \in X$, it is obvious that

$$\{x\} \subseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\}).$$

If $y \neq x$, (i) there exists a preopen set V_x such that $y \notin V_x$ and $x \in V_x$ or (ii) there exists a preopen set V_y such that $x \notin V_y$ and $y \in V_y$. In case of (i), $y \notin \Lambda_p(\{x\})$ and $y \notin \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$. Thus, $\{x\} \supseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$. In case (ii), $y \notin p\text{Cl}(\{x\})$ and $y \notin \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$. This shows that $\{x\} \supseteq \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$. Consequently, we obtain $\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$.

Conversely, suppose that (X, τ) is not pre-T_0. There exist two distinct points x, y such that (i) $y \in V_x$ for every preopen set V_x containing x and (ii) $x \in V_y$ for every preopen set V_y containing y. From (i) and (ii), we obtain $y \in \Lambda_p(\{x\})$ and $y \in p\text{Cl}(\{x\})$, respectively. Therefore, we have $y \in \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$. By Theorem 6, $\{x\} = \Lambda_p(\{x\}) \cap p\text{Cl}(\{x\})$ since $\{x\}$ is (Λ, p)-closed. This is contrary to $x \neq y$.

Definition 6. *Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, p)-cluster point of A if $A \cap U \neq \emptyset$ for every (Λ, p)-open set U of X containing x. The set of all (Λ, p)-cluster points of A is called the (Λ, p)-closure of A and is denoted by $A^{(\Lambda,p)}$.*

Lemma 5. *For subsets A, B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq A^{(\Lambda,p)}$ and $[A^{(\Lambda,p)}]^{(\Lambda,p)} = A^{(\Lambda,p)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,p)} \subseteq B^{(\Lambda,p)}$.

(3) $A^{(\Lambda,p)} = \cap\{F | A \subseteq F$ and F is (Λ, p)-closed$\}$.

(4) $A^{(\Lambda,p)}$ is (Λ, p)-closed.

(5) A is (Λ, p)-closed if and only if $A = A^{(\Lambda,p)}$.

Remark 1. *Every Λ_p-set is (Λ, p)-closed.*

The converse of Remark 1 is not true in general as shown by the following example.

Example 1. *Let $X = \{-2, -1\}$ with a topology $\tau = \{\emptyset, \{-2\}, X\}$. Then, $\{-1\}$ is a (Λ, p)-closed set, but $\{-1\}$ is not a Λ_p-set.*

Lemma 6. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is preclosed, then A is (Λ, p)-closed.

(2) A is (Λ, p)-closed if and only if $A = \Lambda_p(A) \cap A^{(\Lambda,p)}$.

Proof. (1) It is sufficient to observe that $A = X \cap A$, where the whole set X is a Λ_p-set.

(2) Let A be a (Λ, p)-closed set. Then, there exist a Λ_p-set T and a preclosed set C such that $A = T \cap C$. Since $A \subseteq T$, we have $A \subseteq \Lambda_p(A) \subseteq \Lambda_p(T) = T$. Since C is preclosed, by (1), C is (Λ, p)-closed. Since $A \subseteq C$, $A \subseteq A^{(\Lambda,p)} \subseteq C^{(\Lambda,p)} = C$ and hence

$$A \subseteq \Lambda_p(A) \cap A^{(\Lambda,p)} \subseteq T \cap C = A.$$

Thus, $A = \Lambda_p(A) \cap A^{(\Lambda,p)}$.

Conversely, let $A = \Lambda_p(A) \cap A^{(\Lambda,p)}$. Since $\Lambda_p(A)$ is a Λ_p-set, by Remark 1, $\Lambda_p(A)$ is (Λ, p)-closed. Since $A^{(\Lambda,p)}$ is (Λ, p)-closed, by Theroem 7(1), $\Lambda_p(A) \cap A^{(\Lambda,p)}$ is (Λ, p)-closed and hence A is (Λ, p)-closed.

The following example shows that the converse of Lemma 6(1) is not true in general.

Example 2. *Let $X = \{-2, -1, 0, 1, 2\}$ with a topology $\tau = \{\emptyset, \{-2\}, \{2\}, \{-2, 2\}, X\}$. Then, $\{-2, 2\}$ is (Λ, p)-closed, but $\{-2, 2\}$ is not preclosed.*

Definition 7. *Let A be a subset of a topological space (X, τ). A subset $\Lambda_{(\Lambda,p)}(A)$ is defined as follows: $\Lambda_{(\Lambda,p)}(A) = \cap\{U \in \Lambda_pO(X, \tau) \mid A \subseteq U\}$.*

Lemma 7. *For subsets A, B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,p)}(A)$.

(2) If $A \subseteq B$, then $\Lambda_{(\Lambda,p)}(A) \subseteq \Lambda_{(\Lambda,p)}(B)$.

(3) $\Lambda_{(\Lambda,p)}[\Lambda_{(\Lambda,p)}(A)] = \Lambda_{(\Lambda,p)}(A)$;

(4) If A is (Λ, p)-open, then $\Lambda_{(\Lambda,p)}(A) = A$.

Lemma 8. *Let (X, τ) be a topological space and let $x, y \in X$. Then, $y \in \Lambda_{(\Lambda,p)}(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,p)}$.*

Proof. Let $y \notin \Lambda_{(\Lambda,p)}(\{x\})$. Then, there exists a (Λ, p)-open set V containing x such that $y \notin V$. Hence, $x \notin \{y\}^{(\Lambda,p)}$. The converse is similarly shown.

A subset N_x of a topological space (X, τ) is said to be (Λ, p)-*neighbourhood* of a point $x \in X$ if there exists a (Λ, p)-open set U such that $x \in U \subseteq N_x$.

Lemma 9. *A subset A of a topological space (X, τ) is (Λ, p)-open if and only if A is (Λ, p)-neighbourhood of each $x \in A$.*

Definition 8. *Let A be a subset of a topological space (X, τ). A subset $\langle x \rangle_p$ is defined as follows: $\langle x \rangle_p = \Lambda_{(\Lambda,p)}(\{x\}) \cap \{x\}^{(\Lambda,p)}$.*

Theorem 10. *For a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{(\Lambda,p)}(A) = \{x \in X \mid A \cap \{x\}^{(\Lambda,p)} \neq \emptyset\}$ for each subset A of X.

(2) For each $x \in X$, $\Lambda_{(\Lambda,p)}(\langle x \rangle_p) = \Lambda_{(\Lambda,p)}(\{x\})$.

(3) For each $x \in X$, $(\langle x\rangle_p)^{(\Lambda,p)} = \{x\}^{(\Lambda,p)}$.

(4) If U *is* (Λ,p)*-open and* $x \in U$, *then* $\langle x\rangle_p \subseteq U$.

(5) If F *is* (Λ,p)*-closed and* $x \in F$, *then* $\langle x\rangle_p \subseteq F$.

Proof. (1) Suppose that $A \cap \{x\}^{(\Lambda,p)} = \emptyset$. Then, we have $x \notin X - \{x\}^{(\Lambda,p)}$ which is a (Λ,p)-open set containing A. Therefore, $x \notin \Lambda_{(\Lambda,p)}(A)$. Consequently, we have $\Lambda_{(\Lambda,p)}(A) \subseteq \{x \in X \mid A \cap \{x\}^{(\Lambda,p)} \neq \emptyset\}$. Next, let $x \in X$ such that $A \cap \{x\}^{(\Lambda,p)} \neq \emptyset$ and suppose that $x \notin \Lambda_{(\Lambda,p)}(A)$. Then, there exists a (Λ,p)-open set U containing A and $x \notin U$. Let $y \in A \cap \{x\}^{(\Lambda,p)}$. Hence, U is a (Λ,p)-neighbourhood of y which does not contain x. By this contradiction $x \in \Lambda_{(\Lambda,p)}(A)$.

(2) Let $x \in X$. Then, we have $\{x\} \subseteq \{x\}^{(\Lambda,p)} \cap \Lambda_{(\Lambda,p)}(\{x\}) = \langle x\rangle_p$. By Lemma 7, $\Lambda_{(\Lambda,p)}(\{x\}) \subseteq \Lambda_{(\Lambda,p)}(\langle x\rangle_p)$. Next, we show the opposite implication. Suppose that $y \notin \Lambda_{(\Lambda,p)}(\{x\})$. Then, there exists a (Λ,p)-open set V such that $x \in V$ and $y \notin V$. Since $\langle x\rangle_p \subseteq \Lambda_{(\Lambda,p)}(\{x\}) \subseteq \Lambda_{(\Lambda,p)}(V) = V$, we have $\Lambda_{(\Lambda,p)}(\langle x\rangle_p) \subseteq V$. Since $y \notin V$, we have $y \notin \Lambda_{(\Lambda,p)}(\langle x\rangle_p)$. Thus, $\Lambda_{(\Lambda,p)}(\langle x\rangle_p) \subseteq \Lambda_{(\Lambda,p)}(\{x\})$ and hence $\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\langle x\rangle_p)$.

(3) By the definition of $\langle x\rangle_p$, we have $\{x\} \subseteq \langle x\rangle_p$ and $\{x\}^{(\Lambda,p)} \subseteq (\langle x\rangle_p)^{(\Lambda,p)}$ by Lemma 5. On the other hand, we have $\langle x\rangle_p \subseteq \{x\}^{(\Lambda,p)}$ and $(\langle x\rangle_p)^{(\Lambda,p)} \subseteq (\{x\}^{(\Lambda,p)})^{(\Lambda,p)} = \{x\}^{(\Lambda,p)}$. Thus, $(\langle x\rangle_p)^{(\Lambda,p)} = \{x\}^{(\Lambda,p)}$.

(4) Let U be a (Λ,p)-open set and let $x \in U$. By Lemma 7, $\Lambda_{(\Lambda,p)}(\{x\}) \subseteq U$ and hence $\langle x\rangle_p \subseteq U$.

(5) Let F be a (Λ,p)-closed set and let $x \in F$. By Lemma 5, we have

$$\langle x\rangle_p = \{x\}^{(\Lambda,p)} \cap \Lambda_{(\Lambda,p)}(\{x\}) \subseteq \{x\}^{(\Lambda,p)} \subseteq F^{(\Lambda,p)} = F.$$

Lemma 10. *For any points* x *and* y *in a topological space* (X,τ), *the following properties are equivalent:*

(1) $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$;

(2) $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$. There exists a point $z \in X$ such that $z \in \Lambda_{(\Lambda,p)}(\{x\})$ and $z \notin \Lambda_{(\Lambda,p)}(\{y\})$ or $z \in \Lambda_{(\Lambda,p)}(\{y\})$ and $z \notin \Lambda_{(\Lambda,p)}(\{x\})$. We prove only the first case being the second analogous. From $z \in \Lambda_{(\Lambda,p)}(\{x\})$ it follows that $\{x\} \cap \{z\}^{(\Lambda,p)} \neq \emptyset$ which implies $x \in \{z\}^{(\Lambda,p)}$. By $z \notin \Lambda_{(\Lambda,p)}(\{y\})$, $\{y\} \cap \{z\}^{(\Lambda,p)} = \emptyset$. Since $x \in \{z\}^{(\Lambda,p)}$, $\{x\}^{(\Lambda,p)} \subseteq \{z\}^{(\Lambda,p)}$ and $\{y\} \cap \{x\}^{(\Lambda,p)} = \emptyset$. Therefore, it follows that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Thus, $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$ implies that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Then, there exists a point $z \in X$ such that $z \in \{x\}^{(\Lambda,p)}$ and $z \notin \{y\}^{(\Lambda,p)}$ or $z \in \{y\}^{(\Lambda,p)}$ and $z \notin \{x\}^{(\Lambda,p)}$. We prove only the first case being the second analogous. It follows that there exists a (Λ,p)-open set containing z and therefore x but not y, namely, $y \notin \Lambda_{(\Lambda,p)}(\{x\})$ and hence $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$.

Lemma 11. *For any points x and y in a topological space (X,τ), the following properties hold:*

(1) $y \in \Lambda_{(\Lambda,p)}(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,p)}$;

(2) $\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\{y\})$ if and only if $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$.

Proof. (1) Let $x \notin \{y\}^{(\Lambda,p)}$. Then, there exists a (Λ,p)-open set U such that $x \in U$ and $y \notin U$. Therefore, $y \notin \Lambda_{(\Lambda,p)}(\{x\})$. The converse is similarly shown.

(2) Suppose that $\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\{y\})$ for any points x and y in X. Since

$$x \in \Lambda_{(\Lambda,p)}(\{x\}),$$

$x \in \Lambda_{(\Lambda,p)}(\{y\})$ and by (1), $y \in \{x\}^{(\Lambda,p)}$. By Lemma 5, $\{y\}^{(\Lambda,p)} \subseteq \{x\}^{(\Lambda,p)}$. Similarly, we have $\{x\}^{(\Lambda,p)} \subseteq \{y\}^{(\Lambda,p)}$ and hence $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$.

Conversely, suppose that $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$. Since $x \in \{x\}^{(\Lambda,p)}$, $x \in \{y\}^{(\Lambda,p)}$ and by (1), $y \in \Lambda_{(\Lambda,p)}(\{x\})$. By Lemma 7, $\Lambda_{(\Lambda,p)}(\{y\}) \subseteq \Lambda_{(\Lambda,p)}(\Lambda_{(\Lambda,p)}(\{x\})) = \Lambda_{(\Lambda,p)}(\{x\})$. Similarly, we have $\Lambda_{(\Lambda,p)}(\{x\}) \subseteq \Lambda_{(\Lambda,p)}(\{y\})$ and hence $\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\{y\})$.

5. Characterizations of Λ_p-R_0 spaces

In this section, we introduce the concept of Λ_p-R_0 spaces. Moreover, some characterizations of Λ_p-R_0 spaces are investigated.

Definition 9. *A topological space (X,τ) is called a Λ_p-R_0 space if, for each (Λ,p)-open set U and each $x \in U$, $\{x\}^{(\Lambda,p)} \subseteq U$.*

Theorem 11. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_p-R_0;

(2) for each (Λ,p)-closed set F and each $x \in X - F$, there exists a (Λ,p)-open set U such that $F \subseteq U$ and $x \notin U$;

(3) for each (Λ,p)-closed set F and each $x \in X - F$, $F \cap \{x\}^{(\Lambda,p)} = \emptyset$;

(4) for each $x, y \in X$, $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$ or $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let F be a (Λ,p)-closed set and let $x \in X - F$. Then, we have $\{x\}^{(\Lambda,p)} \subseteq X - F$. Let $U = X - \{x\}^{(\Lambda,p)}$, then U is a (Λ,p)-open set such that $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let F be a (Λ,p)-closed set and let $x \in X - F$. There exists a (Λ,p)-open set U such that $F \subseteq U$ and $x \notin U$. Thus, $U \cap \{x\}^{(\Lambda,p)} = \emptyset$ and hence $F \cap \{x\}^{(\Lambda,p)} = \emptyset$.

(3) $\Rightarrow$ (4): Let x, y be distinct points of X. Suppose that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. By (3), $x \in \{y\}^{(\Lambda,p)}$ and $y \in \{x\}^{(\Lambda,p)}$. Thus, $\{x\}^{(\Lambda,p)} \subseteq \{y\}^{(\Lambda,p)} \subseteq \{x\}^{(\Lambda,p)}$ and hence $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$.

(4) $\Rightarrow$ (1): Let U be a (Λ, p)-open set and let $x \in U$. For each $y \notin U$, we have $U \cap \{y\}^{(\Lambda,p)} = \emptyset$ and hence $x \notin \{y\}^{(\Lambda,p)}$. Therefore, $\{y\}^{(\Lambda,p)} \neq \{x\}^{(\Lambda,p)}$. By (4),

$$\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset.$$

Since $X - U$ is (Λ, p)-closed, $y \in \{y\}^{(\Lambda,p)} \subseteq X - U$ and $\cup_{y \in X-U}\{y\}^{(\Lambda,p)} = X - U$. Thus, $\{x\}^{(\Lambda,p)} \cap (X - U) = \{x\}^{(\Lambda,p)} \cap [\cup_{y\in X-U}\{y\}^{(\Lambda,p)}] = \cup_{y\in X-U}[\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)}] = \emptyset$ and hence $\{x\}^{(\Lambda,p)} \subseteq U$. This shows that (X, τ) is Λ_p-R_0.

Corollary 2. *A topological space (X, τ) is Λ_p-R_0 if and only if, for each $x, y \in X$, $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$ implies $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$.*

Proof. This is obvious by Theorem 11.

Conversely, let U be a (Λ, p)-open set and let $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,p)} = \emptyset$. Thus, $x \notin \{y\}^{(\Lambda,p)}$ and $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. By the hypothesis, $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$ and hence $y \notin \{x\}^{(\Lambda,p)}$. Therefore, $\{x\}^{(\Lambda,p)} \subseteq U$. Thus, (X, τ) is Λ_p-R_0.

Theorem 12. *A topological space (X, τ) is Λ_p-R_0 if and only if, for each $x, y \in X$, $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$ implies $\Lambda_{(\Lambda,p)}(\{x\}) \cap \Lambda_{(\Lambda,p)}(\{y\}) = \emptyset$.*

Proof. Suppose that $\Lambda_{(\Lambda,p)}(\{x\}) \cap \Lambda_{(\Lambda,p)}(\{y\}) \neq \emptyset$. Let $z \in \Lambda_{(\Lambda,p)}(\{x\}) \cap \Lambda_{(\Lambda,p)}(\{y\})$. Then, $z \in \Lambda_{(\Lambda,p)}(\{x\})$ and by Lemma 11, $x \in \{z\}^{(\Lambda,p)}$. Thus, $x \in \{z\}^{(\Lambda,p)} \cap \{x\}^{(\Lambda,p)}$ and by Corollary 2, $\{z\}^{(\Lambda,p)} = \{x\}^{(\Lambda,p)}$. Similarly, we have $\{z\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$ and by Lemma 11, $\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\{y\})$.

Conversely, we shows the sufficiency by using Corollary 2. Suppose that

$$\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}.$$

By Lemma 11, $\Lambda_{(\Lambda,p)}(\{x\}) \neq \Lambda_{(\Lambda,p)}(\{y\})$ and hence $\Lambda_{(\Lambda,p)}(\{x\}) \cap \Lambda_{(\Lambda,p)}(\{y\}) = \emptyset$. Therefore, $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$. In fact, assume $z \in \{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)}$. Then, $z \in \{x\}^{(\Lambda,p)}$ implies $x \in \Lambda_{(\Lambda,p)}(\{z\})$ and hence $x \in \Lambda_{(\Lambda,p)}(\{z\}) \cap \Lambda_{(\Lambda,p)}(\{x\})$. By the hypothesis, $\Lambda_{(\Lambda,p)}(\{z\}) = \Lambda_{(\Lambda,p)}(\{x\})$ and by Lemma 11, $\{z\}^{(\Lambda,p)} = \{x\}^{(\Lambda,p)}$. Similarly, we have $\{z\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$ and hence $\{x\}^{(\Lambda,p)} = \{y\}^{(\Lambda,p)}$. This contradicts that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Thus, $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$. This shows that (X, τ) is Λ_p-R_0.

Theorem 13. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_p-R_0;

(2) $x \in \{y\}^{(\Lambda,p)}$ if and only if $y \in \{x\}^{(\Lambda,p)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ) is Λ_p-R_0. Let $x \in \{y\}^{(\Lambda,p)}$. By Lemma 11, $y \in \Lambda_{(\Lambda,p)}(\{x\})$ and hence $\Lambda_{(\Lambda,p)}(\{x\}) \cap \Lambda_{(\Lambda,p)}(\{y\}) \neq \emptyset$. By Theorem 12, we have

$$\Lambda_{(\Lambda,p)}(\{x\}) = \Lambda_{(\Lambda,p)}(\{y\})$$

and hence $x \in \Lambda_{(\Lambda,p)}(\{y\})$. Thus, $y \in \{x\}^{(\Lambda,p)}$ by Lemma 11. The converse is similarly shown.

(2) ⇒ (1): Let U be a (Λ, p)-open set and let $x \in U$. If $y \notin U$, then $\{y\}^{(\Lambda,p)} \cap U = \emptyset$. Thus, $x \notin \{y\}^{(\Lambda,p)}$ and hence $y \notin \{x\}^{(\Lambda,p)}$. This implies that $\{x\}^{(\Lambda,p)} \subseteq U$. Therefore, (X, τ) is Λ_p-R_0.

Theorem 14. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_p-R_0;

(2) for each nonempty subset A of X and each (Λ, p)-open set U such that $A \cap U \neq \emptyset$, there exists a (Λ, p)-closed set F such that $A \cap F \neq \emptyset$ and $F \subseteq U$;

(3) $F = \Lambda_{(\Lambda,p)}(F)$ for each (Λ, p)-closed set F;

(4) $\{x\}^{(\Lambda,p)} = \Lambda_{(\Lambda,p)}(\{x\})$ for each $x \in X$.

Proof. (1) ⇒ (2): Let A be a nonempty subset of X and let $U \in \Lambda_pO(X, \tau)$ such that $A \cap U \neq \emptyset$. Then, there exists $x \in A \cap U$ and hence $\{x\}^{(\Lambda,p)} \subseteq U$. Put $F = \{x\}^{(\Lambda,p)}$, then F is (Λ, p)-closed, $A \cap F \neq \emptyset$ and $F \subseteq U$.

(2) ⇒ (3): Let F be a (Λ, p)-closed set. By Lemma 7, we have $F \subseteq \Lambda_{(\Lambda,p)}(F)$. Next, we show $F \supseteq \Lambda_{(\Lambda,p)}(F)$. Let $x \notin F$. Then, $x \in (X - F) \in \Lambda_pO(X, \tau)$ and by (2), there exists a (Λ, p)-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, $F \subseteq U \in \Lambda_pO(X, \tau)$ and $x \notin U$. Thus, $x \notin \Lambda_{(\Lambda,p)}(F)$. This shows that $F \supseteq \Lambda_{(\Lambda,p)}(F)$.

(3) ⇒ (4): Let $x \in X$ and let $y \notin \Lambda_{(\Lambda,p)}(\{x\})$. There exists a (Λ, p)-open set U such that $x \in U$ and $y \notin U$. Therefore, $\{y\}^{(\Lambda,p)} \cap U = \emptyset$. By (3), we have

$$\Lambda_{(\Lambda,p)}(\{y\}^{(\Lambda,p)}) \cap U = \emptyset.$$

Since $x \notin \Lambda_{(\Lambda,p)}(\{y\}^{(\Lambda,p)})$, there exists a (Λ, p)-open set G such that $\{y\}^{(\Lambda,p)} \subseteq G$ and $x \notin G$. Hence, $\{x\}^{(\Lambda,p)} \cap G = \emptyset$. Since $y \in G$, we have $y \notin \{x\}^{(\Lambda,p)}$ and hence $\{x\}^{(\Lambda,p)} \subseteq \Lambda_{(\Lambda,p)}(\{x\})$. Moreover, $\{x\}^{(\Lambda,p)} \subseteq \Lambda_{(\Lambda,p)}(\{x\}) \subseteq \Lambda_{(\Lambda,p)}(\{x\}^{(\Lambda,p)}) = \{x\}^{(\Lambda,p)}$. Consequently, we obtain $\{x\}^{(\Lambda,p)} = \Lambda_{(\Lambda,p)}(\{x\})$.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (1): Let $U \in \Lambda_pO(X, \tau)$ and let $x \in U$. If $y \notin U$, then $\{y\}^{(\Lambda,p)} \cap U = \emptyset$ and $x \notin \{y\}^{(\Lambda,p)}$. By Lemma 11, $y \notin \Lambda_{(\Lambda,p)}(\{x\})$ and by (5), $y \notin \{x\}^{(\Lambda,p)}$. Thus, $\{x\}^{(\Lambda,p)} \subseteq U$ and hence (X, τ) is Λ_p-R_0.

Corollary 3. *A topological space (X, τ) is Λ_p-R_0 if and only if $\{x\}^{(\Lambda,p)} \subseteq \Lambda_{(\Lambda,p)}(\{x\})$ for each $x \in X$.*

Proof. This is obvious by Theorem 14.

Conversely, let $x \in \{y\}^{(\Lambda,p)}$. By Lemma 11, we have $y \in \Lambda_{(\Lambda,p)}(\{x\})$ and hence $y \in \{x\}^{(\Lambda,p)}$. Similarly, if $y \in \{x\}^{(\Lambda,p)}$, then $x \in \{y\}^{(\Lambda,p)}$. It follows from Theorem 13 that (X, τ) is Λ_p-R_0.

Theorem 15. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* Λ_p-R_0*;*

(2) $\langle x \rangle_p = \{x\}^{(\Lambda,p)}$ *for each* $x \in X$*;*

(3) $\langle x \rangle_p$ *is* (Λ, p)*-closed for each* $x \in X$*.*

Proof. (1) ⇒ (2): By Theorem 14, $\{x\}^{(\Lambda,p)} = \Lambda_{(\Lambda,p)}(\{x\})$ for each $x \in X$ and hence $\{x\}^{(\Lambda,p)} = \{x\}^{(\Lambda,p)} \cap \Lambda_{(\Lambda,p)}(\{x\}) = \langle x \rangle_p$.

(2) ⇒ (1): Since $\{x\}^{(\Lambda,p)} = \langle x \rangle_p$ for each $x \in X$, we have $\{x\}^{(\Lambda,p)} \subseteq \Lambda_{(\Lambda,p)}(\{x\})$. By Corollary 3, (X, τ) is Λ_p-R_0.

(2) ⇔ (3): This is a consequence of Lemma 7.

6. Characterizations of weakly (Λ, p)-continuous functions

In this section, we introduce the notion of weakly (Λ, p)-continuous functions and obtain several characterizations of weakly (Λ, p)-continuous functions.

Definition 10. *Let A be a subset of a topological space (X, τ). The union of all (Λ, p)-open sets contained in A is called the (Λ, p)-interior of A and is denoted by $A_{(\Lambda,p)}$.*

Lemma 12. *Let A and B be subsets of a topological space (X, τ). For the (Λ, p)-interior, the following properties hold:*

(1) $A_{(\Lambda,p)} \subseteq A$ *and* $[A_{(\Lambda,p)}]_{(\Lambda,p)} = A_{(\Lambda,p)}$.

(2) If $A \subseteq B$*, then* $A_{(\Lambda,p)} \subseteq B_{(\Lambda,p)}$.

(3) $A_{(\Lambda,p)} = \cup\{G \mid G \subseteq A \text{ and } G \text{ is } (\Lambda, p)\text{-open}\}$.

(4) $A_{(\Lambda,p)}$ *is* (Λ, p)*-open.*

(5) A is (Λ, p)-open if and only if $A_{(\Lambda,p)} = A$.

(6) $[X - A]^{(\Lambda,p)} = X - A_{(\Lambda,p)}$.

Definition 11. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly (Λ, p)-continuous at a point $x \in X$ if, for each (Λ, p)-open set V containing $f(x)$, there exists a (Λ, p)-open set U containing x such that $f(U) \subseteq V^{(\Lambda,p)}$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, p)-continuous if f has this property at each point $x \in X$.*

Theorem 16. *A function $f : (X, \tau) \to (Y, \sigma)$ is weakly (Λ, p)-continuous at $x \in X$ if and only if for each (Λ, p)-open set V containing $f(x)$, $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$.*

Proof. Let V be a (Λ, p)-open set containing $f(x)$. Then, there exists a (Λ, p)-open set U containing x such that $f(U) \subseteq V^{(\Lambda,p)}$ and hence $x \in U \subseteq f^{-1}(V^{(\Lambda,p)})$. Thus, $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$.

Conversely, let V be a (Λ, p)-open set containing $f(x)$. By the hypothesis, we have $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$. There exists a (Λ, p)-open set U such that $x \in U \subseteq f^{-1}(V^{(\Lambda,p)})$; hence $f(U) \subseteq V^{(\Lambda,p)}$. This shows that f is weakly (Λ, p)-continuous at $x \in X$.

Theorem 17. *A function $f : (X, \tau) \to (Y, \sigma)$ is weakly (Λ, p)-continuous if and only if $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$ for every (Λ, p)-open set V of Y.*

Proof. Let V be any (Λ, p)-open set of Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is weakly (Λ, p)-continuous at x, by Theorem 16, $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$ and hence $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$.

Conversely, let $x \in X$ and let V be any (Λ, p)-open set of Y containing $f(x)$. Then, we have $x \in f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$ and hence $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$. Thus, f is weakly (Λ, p)-continuous by Theorem 16.

Theorem 18. *A function $f : (X, \tau) \to (Y, \sigma)$ is weakly (Λ, p)-continuous if and only if $[f^{-1}(V)]^{(\Lambda,p)} \subseteq f^{-1}(V^{(\Lambda,p)})$ for every (Λ, p)-open set V of Y.*

Proof. Let V be any (Λ, p)-open subset of Y and let $x \in f^{-1}(V)$. There exists a (Λ, p)-open set U containing x such that $f(U) \subseteq V^{(\Lambda,p)}$. Since $x \in U \subseteq f^{-1}(V^{(\Lambda,p)})$, we have $x \in [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$ and hence $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$.

Conversely, let $x \in X$ and let V be any (Λ, p)-open set containing $f(x)$. Since

$$V \cap [Y - V^{(\Lambda,p)}] = \emptyset,$$

$f(x) \notin [Y - V^{(\Lambda,p)}]^{(\Lambda,p)}$ and hence $x \notin f^{-1}([Y - V^{(\Lambda,p)}]^{(\Lambda,p)})$. By the hypothesis,

$$x \notin [f^{-1}(Y - V^{(\Lambda,p)})]^{(\Lambda,p)} = [X - f^{-1}(V^{(\Lambda,p)})]^{(\Lambda,p)}$$

and there exists a (Λ, p)-open set U containing x such that $U \cap [X - f^{-1}(V^{(\Lambda,p)})] = \emptyset$. Thus, $f(U) \subseteq V^{(\Lambda,p)}$. This shows that f is is weakly (Λ, p)-continuous.

Theorem 19. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is weakly (Λ, p)-continuous;

(2) $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$ for every (Λ, p)-open subset U of Y;

(3) $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(F)$ for every (Λ, p)-closed subset F of Y;

(4) $[f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(A^{(\Lambda,p)})$ for every subset A of Y;

(5) $f^{-1}(A_{(\Lambda,p)}) \subseteq [f^{-1}([A_{(\Lambda,p)}]^{(\Lambda,p)})]_{(\Lambda,p)}$ for every subset A of Y;

(6) $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* (Λ,p)*-open subset* U *of* Y*.*

Proof. (1) ⇒ (2): It follows from Theorem 17.

(2) ⇒ (3): Let F be any (Λ,p)-closed subset of Y. Then, $Y-F$ is (Λ,p)-open, by (2), $f^{-1}(Y-F) \subseteq [f^{-1}([Y-F]^{(\Lambda,p)})]_{(\Lambda,p)} = [f^{-1}(Y-F_{(\Lambda,p)})]_{(\Lambda,p)} = X-[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)}$. Thus, $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(F)$.

(3) ⇒ (4): Let A be any subset of Y. Since $A^{(\Lambda,p)}$ is (Λ,p)-closed, by (3),

$$[f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(A^{(\Lambda,p)}).$$

(4) ⇒ (5): Let A be any subset of Y. By (4), we have

$$\begin{aligned} f^{-1}(A_{(\Lambda,p)}) &= X - f^{-1}([Y-A]^{(\Lambda,p)}) \\ &\subseteq X - [f^{-1}([[Y-A]^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \\ &= [f^{-1}([A_{(\Lambda,p)}]^{(\Lambda,p)})]_{(\Lambda,p)}. \end{aligned}$$

Thus, we get the result.

(5) ⇒ (6): Let U be any (Λ,p)-open subset of Y. Suppose that $x \notin f^{-1}(U^{(\Lambda,p)})$. Then, $f(x) \notin U^{(\Lambda,p)}$ and so there exists a (Λ,p)-open set V containing x such that $U \cap V = \emptyset$. Thus, $U \cap V^{(\Lambda,p)} = \emptyset$. By (5), $x \in f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$. There exists a (Λ,p)-open set W containing x such that $x \in W \subseteq f^{-1}(V^{(\Lambda,p)})$. Since $U \cap V^{(\Lambda,p)} = \emptyset$ and $f(W) \subseteq V^{(\Lambda,p)}$, we have $W \cap f^{-1}(U) = \emptyset$ and hence $x \notin [f^{-1}(U)]^{(\Lambda,p)}$. This shows that $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(6) ⇒ (1): This is obvious from Theorem 18.

Definition 12. *A subset* A *of a topological space* (X,τ) *is said to be:*

(i) $s(\Lambda,p)$*-open if* $A \subseteq [A_{(\Lambda,p)}]^{(\Lambda,p)}$*;*

(ii) $p(\Lambda,p)$*-open if* $A \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$*;*

(iii) $\beta(\Lambda,p)$*-open if* $A \subseteq [[A^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}$*;*

(iv) $r(\Lambda,p)$*-open if* $A = [A^{(\Lambda,p)}]_{(\Lambda,p)}$*.*

The complement of a $s(\Lambda,p)$-open (resp. $p(\Lambda,p)$-open, $\beta(\Lambda,p)$-open, $r(\Lambda,p)$-open) set is called $s(\Lambda,p)$-closed (resp. $p(\Lambda,p)$-closed, $\beta(\Lambda,p)$-closed, $r(\Lambda,p)$-closed).

Theorem 20. *For a function* $f:(X,\tau)\to(Y,\sigma)$*, the following properties are equivalent:*

(1) f *is weakly* (Λ,p)*-continuous;*

(2) $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(F)$ *for every* $r(\Lambda,p)$*-closed subset* F *of* Y*;*

(3) $[f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* $\beta(\Lambda,p)$*-open subset* U *of* Y*;*

(4) $[f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* $s(\Lambda,p)$*-open subset* U *of* Y*.*

Proof. (1) ⇒ (2): Let F be any $r(\Lambda,p)$-closed subset of Y. Then, $F_{(\Lambda,p)}$ is (Λ,p)-open, by Theorem 19, $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([F_{(\Lambda,p)}]^{(\Lambda,p)})$. Since F is $r(\Lambda,p)$-closed, we have $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([F_{(\Lambda,p)}]^{(\Lambda,p)}) = f^{-1}(F)$.

(2) ⇒ (3): Let U be any $\beta(\Lambda,p)$-open set. Then, $U^{(\Lambda,p)} \subseteq [[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \subseteq U^{(\Lambda,p)}$ and hence $U^{(\Lambda,p)}$ is $r(\Lambda,p)$-closed. By (2), $[f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let U be any (Λ,p)-open subset of Y. Then, we have U is $s(\Lambda,p)$-open and by (4), $[f^{-1}(U)]^{(\Lambda,p)} \subseteq [f^{-1}([U_{(\Lambda,p)}]^{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$. Thus, f is weakly (Λ,p)-continuous by Theorem 19.

Theorem 21. *For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) f *is weakly* (Λ,p)*-continuous;*

(2) $[f^{-1}([U_{(\Lambda,p)}]^{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* $p(\Lambda,p)$*-open subset* U *of* Y*;*

(3) $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* $p(\Lambda,p)$*-open subset* U *of* Y*;*

(4) $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$ *for every* $p(\Lambda,p)$*-open subset* U *of* Y*.*

Proof. (1) ⇒ (2): Let U be any $p(\Lambda,p)$-open subset of Y. Then, we have

$$U^{(\Lambda,p)} = [[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}$$

and hence $U^{(\Lambda,p)}$ is $r(\Lambda,p)$-closed. By Theorem 20, $[f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(2) ⇒ (3): Let U be any $p(\Lambda,p)$-open subset of Y. Then, $U \subseteq [U^{(\Lambda,p)}]_{(\Lambda,p)}$ and by (2), we have $[f^{-1}(U)]^{(\Lambda,p)} \subseteq [f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(3) ⇒ (4): Let U be any $p(\Lambda,p)$-open subset of Y. By (3), we have

$$\begin{aligned} f^{-1}(U) &\subseteq f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)}) \\ &= X - f^{-1}([Y - U^{(\Lambda,p)}]^{(\Lambda,p)}) \\ &= X - [f^{-1}(Y - U^{(\Lambda,p)})]^{(\Lambda,p)} \\ &= [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}. \end{aligned}$$

(4) ⇒ (1): Since every (Λ,p)-open set is $p(\Lambda,p)$-open, by (4) and Theorem 19, it follows that f is weakly (Λ,p)-continuous.

Theorem 22. *For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*

(1) f *is weakly* (Λ,p)*-continuous;*

(2) $[f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(A^{(\Lambda,p)})$ *for every subset* A *of* Y*;*

(3) $[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(F)$ *for every* $r(\Lambda,p)$*-closed subset* F *of* Y*;*

(4) $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* (Λ,p)*-open subset* U *of* Y*;*

(5) $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$ *for every* (Λ,p)*-open subset* U *of* Y*;*

(6) $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$ *for every* $p(\Lambda,p)$*-open subset* U *of* Y*;*

(7) $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$ *for every* $p(\Lambda,p)$*-open subset* U *of* Y*.*

Proof. (1) ⇒ (2): Let A be any subset of Y and let $x \in X - f^{-1}(A^{(\Lambda,p)})$. Then, $f(x) \in Y - A^{(\Lambda,p)}$ and there exists a (Λ,p)-open set U containing $f(x)$ such that $U \cap A = \emptyset$ and hence $U^{(\Lambda,p)} \cap [A^{(\Lambda,p)}]_{(\Lambda,p)} = \emptyset$. Since f is weakly (Λ,p)-continuous, there exists a (Λ,p)-open set W containing x such that $f(W) \subseteq U^{(\Lambda,p)}$. Then $W \cap f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)}) = \emptyset$ and hence $x \in X - [f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)}$. This shows that

$$[f^{-1}([A^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(A^{(\Lambda,p)}).$$

(2) ⇒ (3): Let F be any $r(\Lambda,p)$-closed subset of Y. By (2), we have

$$[f^{-1}(F_{(\Lambda,p)})]^{(\Lambda,p)} = [f^{-1}([[F_{(\Lambda,p)}]^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([F_{(\Lambda,p)}]^{(\Lambda,p)}) = f^{-1}(F).$$

(3) ⇒ (4): Let U be any (Λ,p)-open subset of Y. Since $U^{(\Lambda,p)}$ is $r(\Lambda,p)$-closed and by (3), $[f^{-1}(U)]^{(\Lambda,p)} \subseteq [f^{-1}([U^{(\Lambda,p)}]_{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(4) ⇒ (5): Let U be any (Λ,p)-open subset of Y. Since $Y - U^{(\Lambda,p)}$ is (Λ,p)-open, by (4), $X - [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)} = [f^{-1}(Y - U^{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([Y - U^{(\Lambda,p)}]^{(\Lambda,p)}) \subseteq X - f^{-1}(U)$ and hence $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$.

(5) ⇒ (1): Let $x \in X$ and let U be any (Λ,p)-open subset of Y containing $f(x)$. By (5), $x \in f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$. Put $W = [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$. Thus, $f(W) \subseteq U^{(\Lambda,p)}$ and hence f is weakly (Λ,p)-continuous at x. This shows that f is weakly (Λ,p)-continuous.

(1) ⇒ (6): Let U be any $p(\Lambda,p)$-open subset of Y and let $x \in X - f^{-1}(U^{(\Lambda,p)})$. There exists a (Λ,p)-open set V containing $f(x)$ such that $V \cap U = \emptyset$ and hence $[V \cap U]^{(\Lambda,p)} = \emptyset$. Since U is (Λ,p)-open, we have $U \cap V^{(\Lambda,p)} \subseteq [U \cap V]^{(\Lambda,p)} = \emptyset$. Since f is weakly (Λ,p)-continuous and V is a (Λ,p)-open set containing $f(x)$, there exists a (Λ,p)-open set W containing x such that $f(W) \subseteq V^{(\Lambda,p)}$. Then, $f(W) \cap U = \emptyset$ and hence $W \cap f^{-1}(U) = \emptyset$. Thus, $x \in X - [f^{-1}(U)]^{(\Lambda,p)}$. This shows that $[f^{-1}(U)]^{(\Lambda,p)} \subseteq f^{-1}(U^{(\Lambda,p)})$.

(6) ⇒ (7): Let U be any $p(\Lambda,p)$-open subset of Y. Since $Y - U^{(\Lambda,p)}$ is (Λ,p)-open and by (6), we have

$$X - [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)} = [f^{-1}(Y - U^{(\Lambda,p)})]^{(\Lambda,p)} \subseteq f^{-1}([Y - U^{(\Lambda,p)}]^{(\Lambda,p)}) \subseteq X - f^{-1}(U)$$

and hence $f^{-1}(U) \subseteq [f^{-1}(U^{(\Lambda,p)})]_{(\Lambda,p)}$.

(7) ⇒ (1): Let $x \in X$ and let V be any (Λ,p)-open subset of Y containing $f(x)$. Then, V is $p(\Lambda,p)$-open, by (7), $x \in f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,p)})]_{(\Lambda,p)}$. Put $U = [f^{-1}(V^{(\Lambda,p)})]^{(\Lambda,p)}$. Then, $f(U) \subseteq V^{(\Lambda,p)}$ and hence f is weakly (Λ,p)-continuous at x. Thus, f is weakly (Λ,p)-continuous.

Definition 13. *A topological space (X, τ) is said to be Λ_p-T_2 if, for any disjoint pair of points x and y in X, there exist (Λ, p)-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

Definition 14. *A topological space (X, τ) is said to be Λ_p-Urysohn if, for each distinct points $x, y \in X$, there exist (Λ, p)-open sets U and V containing x and y, respectively, such that $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$.*

Theorem 23. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly (Λ, p)-continuous injection and (Y, σ) is Λ_p-Urysohn, then (X, τ) is Λ_p-T_2.*

Proof. Let x, y be distinct points of X. Then, $f(x) \neq f(y)$. Since (Y, σ) is Λ_p-Urysohn, there exist (Λ, p)-open sets U and V containing $f(x)$ and $f(y)$, respectively, such that $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$. Since f is weakly (Λ, p)-continuous, there exist (Λ, p)-open sets G and W containing x and y, respectively, such that $f(G) \subseteq U^{(\Lambda,p)}$ and $f(W) \subseteq V^{(\Lambda,p)}$. This shows that $G \cap W = \emptyset$. Thus, (X, τ) is Λ_p-T_2.

Theorem 24. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly $(\Lambda.p)$-continuous and (Y, σ) is Λ_p-T_2, then f has (Λ, p)-closed point inverses.*

Proof. Let $y \in Y$. We show that $f^{-1}(y) = \{x \in X \mid f(x) = y\}$ is (Λ, p)-closed, or equivalently $G = \{x \in X \mid f(x) \neq y\}$ is (Λ, p)-open. Let $x \in G$. Since $f(x) \neq y$ and (Y, σ) is Λ_p-T_2, there exist disjoint (Λ, p)-open sets U and V such that $f(x) \in U$ and $y \in V$. Since $U \cap V = \emptyset$, $U^{(\Lambda,p)} \cap V = \emptyset$ and hence $y \notin U^{(\Lambda,p)}$. Since f is weakly (Λ, p)-continuous, there exists a (Λ, p)-open set W containing x such that $f(W) \subseteq U^{(\Lambda,p)}$. Now, suppose that W is not contained in G. Then, there exists a point $z \in W$ such that $f(z) = y$. Since $f(W) \subseteq U^{(\Lambda,p)}$, $y = f(z) \in U^{(\Lambda,p)}$. This is a contradiction. Therefore, $W \subseteq G$ and by Lemma 9, G is (Λ, p)-open.

Theorem 25. *Let (X, τ) be a topological space. If for each pair of distinct points x_1 and x_2 in X, there exists a function $f : (X, \tau) \to (Y, \sigma)$ such that*

(1) (Y, σ) is Λ_p-Urysohn,

(2) $f(x_1) \neq f(x_2)$ and

(3) f is weakly (Λ, p)-continuous at x_1 and x_2, then (X, τ) is Λ_p-T_2.

Proof. Let x_1, x_2 be any distinct points of X. By the hypothesis, there exists a function $f : (X, \tau) \to (Y, \sigma)$ which satisfies the conditions (1), (2) and (3). Let $y_i = f(x_i)$ for $i = 1, 2$. Then, $y_1 \neq y_2$. Since (Y, σ) is Λ_p-Urysohn, there exist (Λ, p)-open sets V_i in (Y, σ) containing y_i such that $V_1^{(\Lambda,p)} \cap V_2^{(\Lambda,p)} = \emptyset$. Since f is weakly (Λ, p)-continuous at x_1 and x_2, for $i = 1, 2$, there exist (Λ, p)-open sets U_i in (X, τ) containing x_i such that $f(U_i) \subseteq V_i^{(\Lambda,p)}$. Hence, we get $U_1 \cap U_2 = \emptyset$. This shows that (X, τ) is Λ_p-T_2.

Corollary 4. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly (Λ,p)-continuous injection and (Y,σ) is Λ_p-Urysohn, then (X,τ) is Λ_p-T_2.*

Definition 15. *Let A be a subset of a topological space (X,τ). The $\theta(\Lambda,p)$-closure of A, $A^{\theta(\Lambda,p)}$, is defined as follows:*

$$A^{\theta(\Lambda,p)}=\{x\in X \mid A\cap\ U^{(\Lambda,p)}\neq\emptyset \text{ for each } (\Lambda,p)\text{-open set } U \text{ containing } x\}.$$

A subset A of a topological space (X,τ) is called $\theta(\Lambda,p)$-*closed* if $A=A^{\theta(\Lambda,p)}$. The complement of a $\theta(\Lambda,p)$-closed set is said to be $\theta(\Lambda,p)$-*open*.

Lemma 13. *Let A be a subset of a topological space (X,τ). Then, $x\in A^{(\Lambda,p)}$ if and only if $U\cap A\neq\emptyset$ for every (Λ,p)-open set U containing x.*

Lemma 14. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) If A is (Λ,p)-open in (X,τ), then $A^{(\Lambda,p)}=A^{\theta(\Lambda,p)}$.

(2) $A^{\theta(\Lambda,p)}$ is (Λ,p)-closed for every subset A of X.

Proof. (1) In general, we have $A^{(\Lambda,p)}\subseteq A^{\theta(\Lambda,p)}$. Suppose that $x\notin A^{(\Lambda,p)}$. By Lemma 13, there exists a (Λ,p)-open set U containing x such that $U\cap A=\emptyset$; hence $A\cap U^{(\Lambda,p)}=\emptyset$ since A is (Λ,p)-open. Thus, $x\notin A^{\theta(\Lambda,p)}$. Consequently, we obtain $A^{(\Lambda,p)}=A^{\theta(\Lambda,p)}$.

(2) Let $x\in X-A^{\theta(\Lambda,p)}$. Then, we have $x\notin A^{\theta(\Lambda,p)}$. There exists a (Λ,p)-open set U_x containing x such that $A\cap U_x^{(\Lambda,p)}=\emptyset$ and hence $U_x\cap A^{\theta(\Lambda,p)}=\emptyset$. Therefore, $x\in U_x\subseteq X-A^{\theta(\Lambda,p)}$. Thus, $X-A^{\theta(\Lambda,p)}=\cup_{x\in X-A^{\theta(\Lambda,p)}}U_x$ and hence $X-A^{\theta(\Lambda,p)}$ is (Λ,p)-open. This shows that $A^{\theta(\Lambda,p)}$ is (Λ,p)-closed.

Theorem 26. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) f is weakly (Λ,p)-continuous;

(2) $f(A^{(\Lambda,p)})\subseteq[f(A)]^{\theta(\Lambda,p)}$ for every subset A of X;

(3) $[f^{-1}(B)]^{(\Lambda,p)}\subseteq f^{-1}(B^{\theta(\Lambda,p)})$ for every subset B of Y;

(4) $[f^{-1}(V)]^{(\Lambda,p)}\subseteq f^{-1}(V^{(\Lambda,p)})$ for every (Λ,p)-open subset V of Y.

Proof. (1) ⇒ (2): Let A be any subset of X. Let $x\in A^{(\Lambda,p)}$ and V be any (Λ,p)-open set containing $f(x)$. Since f is weakly (Λ,p)-continuous, there exists a (Λ,p)-open set U containing x such that $f(U)\subseteq V^{(\Lambda,p)}$. Since $x\in A^{(\Lambda,p)}$, we have $U\cap A\neq\emptyset$. It follows that $\emptyset\neq f(U)\cap f(A)\subseteq V^{(\Lambda,p)}\cap f(A)$ and hence $V^{(\Lambda,p)}\cap f(A)\neq\emptyset$. Thus, $f(x)\in[f(A)]^{\theta(\Lambda,p)}$. Consequently, we obtain $f(A^{(\Lambda,p)})\subseteq[f(A)]^{\theta(\Lambda,p)}$.

(2) ⇒ (3): Let B be any subset of Y. By (2), we have

$$f([f^{-1}(B)]^{(\Lambda,p)})\subseteq[f(f^{-1}(B))]^{\theta(\Lambda,p)}\subseteq B^{\theta(\Lambda,p)}$$

and hence $[f^{-1}(B)]^{(\Lambda,p)}\subseteq f^{-1}(B^{\theta(\Lambda,p)})$.

(3) $\Rightarrow$ (4): Let V be any (Λ, p)-open subset of Y. By Lemma 14, $V^{(\Lambda,p)} = V^{\theta(\Lambda,p)}$. Thus, the proof is obvious.

(4) $\Rightarrow$ (1): Let V be any (Λ, p)-open set containing $f(x)$. Since $V \cap [Y - V^{(\Lambda,p)}] = \emptyset$, we have $f(x) \notin [Y - V^{(\Lambda,p)}]^{(\Lambda,p)}$ and hence $x \notin f^{-1}([Y - V^{(\Lambda,p)}]^{(\Lambda,p)})$. Since $Y - V^{(\Lambda,p)}$ is (Λ, p)-open, by (4), $x \notin [f^{-1}([Y - V^{(\Lambda,p)}])]^{(\Lambda,p)}$ and there exists a (Λ, p)-open set U containing x such that $U \cap f^{-1}(Y - V^{(\Lambda,p)}) = \emptyset$; hence $f(U) \cap [Y - V^{(\Lambda,p)}] = \emptyset$. This shows that $f(U) \subseteq V^{(\Lambda,p)}$. Thus, f is weakly (Λ, p)-continuous.

Definition 16. *A topological space (X, τ) is said to be Λ_p-regular if, for each (Λ, p)-closed set F and each $x \notin F$, there exist disjoint (Λ, p)-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 15. *A topological space (X, τ) is Λ_p-regular if and only if for each $x \in X$ and each (Λ, p)-open set U containing x, there exists a (Λ, p)-open set V such that $x \in V \subseteq V^{(\Lambda,p)} \subseteq U$.*

Proof. Let $x \in X$ and let U be a (Λ, p)-open set containing x. Then, $x \notin X - U$ and $X - U$ is (Λ, p)-closed. There exist disjoint (Λ, p)-open sets V and W such that $x \in V$ and $X - U \subseteq W$. Thus, $V \subseteq X - W \subseteq U$. Since $X - W$ is (Λ, p)-closed, we have $V^{(\Lambda,p)} \subseteq X - W \subseteq U$ and hence $x \in V \subseteq V^{(\Lambda,p)} \subseteq U$.

Conversely, let F be a (Λ, p)-closed set and let $x \notin F$. Then, $x \in X - F$. Since $X - F$ is (Λ, p)-open, there exists a (Λ, p)-open set V such that $x \in V \subseteq V^{(\Lambda,p)} \subseteq X - F$ and hence $F \subseteq X - V^{(\Lambda,p)}$. This shows that (X, τ) is Λ_p-regular.

Lemma 16. *Let (X, τ) be a Λ_p-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda,p)} = A^{\theta(\Lambda,p)}$ for every subset A of X.

(2) Every (Λ, p)-open set is $\theta(\Lambda, p)$-open.

Proof. (1) In general, we have $A^{(\Lambda,p)} \subseteq A^{\theta(\Lambda,p)}$ for every subset A of X. Next, we show that $A^{\theta(\Lambda,p)} \subseteq A^{(\Lambda,p)}$. Let $x \in A^{\theta(\Lambda,p)}$ and U be any (Λ, p)-open set containing x. By Lemma 15, there exists a (Λ, p)-open set V such that $x \in V \subseteq V^{(\Lambda,p)} \subseteq U$. Since $x \in A^{\theta(\Lambda,p)}$, it follows that $A \cap V^{(\Lambda,p)} \neq \emptyset$ and hence $U \cap A \neq \emptyset$. Thus, $x \in A^{(\Lambda,p)}$. Consequently, we obtain $A^{\theta(\Lambda,p)} \subseteq A^{(\Lambda,p)}$.

(2) Let V be a (Λ, p)-open set. By (1), we have $X - V = [X - V]^{(\Lambda,p)} = [X - V]^{\theta(\Lambda,p)}$ and hence $X - V$ is $\theta(\Lambda, p)$-closed. Thus, V is $\theta(\Lambda, p)$-open.

Theorem 27. *Let (Y, σ) be a Λ_p-regular space. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) $f^{-1}(B^{\theta(\Lambda,p)})$ is $\theta(\Lambda, p)$-closed in X for every subset B of Y;

(2) f is weakly (Λ, p)-continuous;

(3) $f^{-1}(F)$ is (Λ, p)-closed in X for every $\theta(\Lambda, p)$-closed subset F of Y;

(4) $f^{-1}(V)$ is (Λ, p)-open in X for every $\theta(\Lambda, p)$-open subset V of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Then,

$$[f^{-1}(B)]^{(\Lambda,p)} \subseteq [f^{-1}(B^{\theta(\Lambda,p)})]^{(\Lambda,p)} = f^{-1}(B^{\theta(\Lambda,p)}),$$

by Theorem 26, f is is weakly (Λ, p)-continuous.

(2) ⇒ (3): Let F be any $\theta(\Lambda, p)$-closed subset of Y. By Theorem 26, we have

$$[f^{-1}(F)]^{(\Lambda,p)} \subseteq f^{-1}(F^{\theta(\Lambda,p)}) = f^{-1}(F)$$

and hence $f^{-1}(F)$ is (Λ, p)-closed in X.

(3) ⇒ (4): Let V be any $\theta(\Lambda, p)$-open subset of Y. Then, $Y - V$ is $\theta(\Lambda, p)$-closed, by (3), $X - f^{-1}(V) = F^{-1}(Y - V)$ is (Λ, p)-closed in X. Thus, $f^{-1}(V)$ is (Λ, p)-open.

(4) ⇒ (1): Let B be any subset of Y. By Lemma 14, $B^{\theta(\Lambda,p)}$ is (Λ, p)-closed in Y and by Lemma 16, $Y - B^{\theta(\Lambda,p)}$ is $\theta(\Lambda, p)$-open in Y. Thus, by (4), we have

$$X - f^{-1}(B^{\theta(\Lambda,p)}) = f^{-1}(Y - B^{\theta(\Lambda,p)})$$

is (Λ, p)-open in X and hence $f^{-1}(B^{\theta(\Lambda,p)})$ is $\theta(\Lambda, p)$-closed.

7. Conclusion

Closedness and openness are fundamental with respect to the investigation of general topological spaces. Various types of generalizations of closed sets and open sets in topological spaces have been researched by many mathematicians. This article is devoted to introducing and discussing the concepts of (Λ, p)-closed sets and (Λ, p)-open sets. Moreover, some characterizations of Λ_p-R_0 spaces are explored. Additionally, several characterizations of weakly (Λ, p)-continuous functions are established. The ideas and results of this article may motivate further research.

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