EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 1, 2022, 261-280
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Infra pre-open sets and their applications to generate new types of operators and maps

Tareq M. Al-shami[1]*, Hakeem A. Othman[2,3]

[1] *Department of Mathematics, Sana'a University, Sana'a, Yemen*
[2] *Department of Mathematics, AL-Qunfudhah University college, Umm Al-Qura University, Saudi Arabia*
[3] *Department of Mathematics, Rada'a College of Education and Science, Albaydha University, Albaydha, Yemen*

Abstract. Herein, we introduce the concepts of infra soft pre-open infra soft pre-closed sets which are respectively generalizations of infra soft open and infra soft closed sets. We characterize them and investigate their behaviours under infra soft homeomorphism maps and finite product of soft spaces. Then, we apply infra soft pre-open infra soft pre-closed sets to define the operators of infra pre-interior, infra pre-closure, infra pre-limit and infra pre-boundary. We discuss their main properties and show the interrelations between them. In the end, we introduce new types of soft maps using infra soft pre-open and infra soft pre-closed sets and explore their essential properties.

2020 Mathematics Subject Classifications: 54A40, 54C08, 54C99

Key Words and Phrases: Infra soft pre-open set, infra soft pre-interior points, infra soft pre-closure points, infra soft pre-continuity

1. Introduction

Soft set is a new mathematical tool to address uncertainity/vagueness; it was introduced in 1999 by Molodtsov [34]. He proved its efficiency by applying successfully in many areas. Aktaş and Çağman [1] showed that rough set and fuzzy set, which are two approaches to handle uncertainty, may be considered soft sets. In the literature, one can note many authors have been applied soft sets to model some phenomena and problems in different disciplines such as decision-making problems [28, 38] and computer science [22].

Maji et al. [33], in 2003, formulated the basic operations and operators between soft sets like the difference between two soft sets, a complement of a soft set, and intersection and union operators. To remove anomaly appeared in their definitions and keep some

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i1.4275

Email addresses: tareqalshami83@gmail.com (T.M. Al-shami), haoali@uqu.edu.sa;hakim_albdoie@yahoo.com (H.A. Othman)

crisp properties in the soft set theory, Ali et al. [19] initiated new operations and operators between soft sets. Attempts were still in this path to produce new operators and relations like those introduced in [15, 36].

In 2011, Çağman et al. [23] and Shabir and Naz [37] applied soft sets to define a soft topology. Whereas, Çağman et al. defined a soft topology over an absolute soft set and different sets of parameters, Shabir and Naz defined a soft topology over a fixed set of universe and a fixed set of parameters. This article follows Shabir and Naz' definition. The main concepts and notions of general topology were studied in soft topology such as basis [18], separation axioms [27], compactness [6, 21], connectedness [32], bioperators [20], covering properties [13, 14, 25], generalized open sets [2–4] and Bipolarity [5].

Soft topologies were generalized to various structures such as infra soft topologies [10] which is the frame of this study. The inducements of continuous investigation of infra soft topologies are that several topological features are still valid via the frame of infra soft topologies, also, easily building the examples that elucidate the interrelationships among the topological notions and concepts. In [9, 11], the authors discussed these advantages for compact and connected spaces. Some studies have en recently conducted in frame of infra soft topologies such as [7, 12, 16, 17]

Extension of soft open sets was a goal of some papers. Some types of these extensions were investigated such as soft semi-open and soft pre-open sets which were presented in [24] and [30], respectively. The target of this work is to scrutinize the behaviours of soft pre-open sets via infra soft topological spaces. As we shall show many properties of soft pre-open sets are still valid for infra soft pre-open sets which offers a flexible frame (in lieu of soft topologies) to study the topological notions and the interrelationships between them.

We layout the remainder of this article as following. In Sect. 2, we survey the related literature and locate the current study in its context. Sect. 3 is first of the three main sections of this study. It introduces the concept of infra soft pre-open sets and establishes its characterization. Sect. 4 is the second main section which defines and discusses the concepts of infra pre-interior, infra pre-closure, infra pre-limit and infra pre-boundary soft points of a soft set. Sect. 5 is the last main section which initiates and explores new types of soft maps namely infra soft pre-continuous, infra soft pre-open, infra soft pre-closed and infra soft pre-homeomorphism maps. Finally, Sect. 6 gives some conclusions and proposes some future works.

2. Preliminaries

In this part, we recall the concepts and findings that help us to understand this article.

2.1. Soft set theory

Definition 1. *[34] Consider Σ as a parameters set and 2^X the power set of X which is the universe. We call (Ω, Σ) a soft set over X if $\Omega : \Sigma \to 2^X$ is a crisp map. A soft set is* *expressed as* $(\Omega, \Sigma) = \{(\eta, \Omega(\eta)) : \eta \in \Sigma$ *and* $\Omega(\eta) \in 2^X\}$.

A class of all soft sets over X *under a set of parameters* Σ *is symbolized by* $C(X_\Sigma)$.

Definition 2. *[19] A complement of a soft set* (Ω, Σ)*, denoted by* (Ω^c, Σ)*, provided that a map* $\Omega^c : \Sigma \to 2^X$ *is given by* $\Omega^c(\eta) = X \setminus \Omega(\eta)$ *for each* $\eta \in \Sigma$.

Definition 3. *[33] Let* (Ω, Σ) *be a soft set on* X *such that* $\Omega(\eta) = \emptyset$ *(resp.,* $\Omega(\eta) = X$*) for each* $\eta \in \Sigma$*. Then we say that* (Ω, Σ) *is a null (resp., an absolute) soft set over* X.

The null and absolute soft sets are respectively symbolized by Φ *and* $\widetilde{X}$.

Definition 4. *[26, 27] We call a soft set* (Ω, Σ) *stable (resp., finite, countable) if all components are equal (resp., finite, countable). Otherwise, we call* (Ω, Σ) *unstable (resp., infinite, uncountable).*

Definition 5. *[35] We call a soft set* (Ω, Σ) *a soft point on* X *if there is* $\eta \in \Sigma$ *such that* $\Omega(\eta) = x \in X$ *and* $\Omega(\eta') = \emptyset$ *for each* $\eta' \neq \eta$*. Henceforth,* δ^x_η *denotes a soft point.*

Definition 6. *[19] The intersection of soft sets* (Ω, Σ) *and* (Ψ, Δ) *on* X*, symbolized by* $(\Omega, \Sigma)\widetilde{\bigcap}(\Psi, \Delta)$*, is a soft set* (Υ, T)*, where* $T = \Sigma \cap \Delta \neq \emptyset$*, and a map* $\Upsilon : T \to 2^X$ *is given by* $\Upsilon(\eta) = \Omega(\eta) \cap \Psi(\eta)$ *for each* $\eta \in T$.

Definition 7. *[33] The union of soft sets* (Ω, Σ) *and* (Ψ, Δ) *on* X*, symbolized by* $(\Omega, \Sigma)\widetilde{\bigcup}(\Psi, \Delta)$*, is a soft set* (Υ, T)*, where* $T = \Sigma \cup \Delta$ *and a map* $T : \Sigma \to 2^X$ *is given as follows:*

$$\Upsilon(\eta) = \begin{cases} \Omega(\eta) & : \eta \in \Sigma \setminus \Delta \\ \Psi(\eta) & : \eta \in \Delta \setminus \Sigma \\ \Omega(\eta) \cup \Psi(\eta) & : \eta \in \Sigma \cap \Delta \end{cases}$$

Definition 8. *[29] A soft set* (Ω, Σ) *is a subset of a soft set* (Ψ, Δ)*, symbolized by* $(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Delta)$*, if* $\Sigma \subseteq \Delta$ *and* $\Omega(\eta) \subseteq \Psi(\eta)$ *for all* $\eta \in \Sigma$*. If* $(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Delta)$ *and* $(\Psi, \Delta)\widetilde{\subseteq}(\Omega, \Sigma)$*, then* (Ω, Σ) *and* (Ψ, Δ) *are called soft equal.*

Definition 9. *[21] The Cartesian product of* (Ω, Σ) *and* (Ψ, Δ)*, symbolized by* $(\Omega \times \Psi, \Sigma \times \Delta)$*, is defined as* $(\Omega \times \Psi)(\eta, \eta') = \Omega(\eta) \times \Psi(\eta')$ *for each* $(\eta, \eta') \in \Sigma \times \Delta$.

Definition 10. *[31] A soft map* f_τ *from* $C(X_\Sigma)$ *to* $C(\mathcal{S}_\Delta)$ *is a pair of crisp maps* f *and* τ*, where* $f : X \to \mathcal{S}$*,* $\tau : \Sigma \to \Delta$*. Let* $(\Omega, \mathcal{M})$ *and* $(\Psi, \mathcal{N})$ *be respectively subsets of* $C(X_\Sigma)$ *and* $C(\mathcal{S}_\Delta)$*. Then the image of* $(\Omega, \mathcal{M})$ *and pre-image of* $(\Psi, \mathcal{N})$ *are given by the following.*

(i) $f_\tau(\Omega, \mathcal{M}) = (f(\Omega), \Delta)$ *is a soft set in* $C(\mathbb{V}_\Delta)$ *such that*

$$f(\Omega)(\omega) = \begin{cases} \widetilde{\bigcup}_{\eta \in \tau^{-1}(\omega) \bigcap \mathcal{M}} f(\Omega(\eta)) & : \tau^{-1}(\omega) \neq \emptyset \\ \emptyset & : \tau^{-1}(\omega) = \emptyset \end{cases}$$

for each $\omega \in \Delta$.

(ii) $f_\tau^{-1}(\Psi, \mathcal{N}) = (f^{-1}(\Psi), \Sigma)$ *is a soft set in* $C(X_\Sigma)$ *such that*

$$f^{-1}(\Psi)(\eta) = \begin{cases} f^{-1}(\Psi(\tau(\eta))) & : \tau(\eta) \in \mathcal{N} \\ \emptyset & : \tau(\eta) \notin \mathcal{N} \end{cases}$$

for each $\eta \in \Sigma$.

Definition 11. *[31] We call a soft map* $f_\tau : C(X_\Sigma) \to C(\mathcal{S}_\Delta)$ *injective (resp., surjective, bijective) if* f *and* τ *are injective (resp., surjective, bijective).*

2.2. Infra soft topological spaces

Definition 12. *[10] A family* ξ *of soft sets over* X *with* Σ *as a parameters set is said to be an infra soft topology on* X *if it is closed under finite intersection and* Φ *is a member of* ξ.
The triple (X, ξ, Σ) *is called an infra soft topological space (briefly, ISTS). We call a member of* ξ *an infra soft open set and called its complement an infra soft closed set. We call* (X, ξ, Σ) *stable if all its infra soft open sets are stable.*

Definition 13. *[10] Let* (Ω, Σ) *be a subset of* (X, ξ, Σ).

(i) *the intersection of all infra soft closed subsets of* (X, ξ, Σ) *which contains a soft set* (Ω, Σ) *is called the infra soft closure points of* (Ω, Σ). *It is denoted by* $Cl(\Omega, \Sigma)$.

(ii) *the union of all infra soft open subsets of* (X, ξ, Σ) *which are contained in a soft set* (Ω, Σ) *is called the infra soft interior points of* (Ω, Σ). *It is denoted by* $Int(\Omega, \Sigma)$.

It was showed in [10] that $Cl(\Omega, \Sigma)$ and $Int(\Omega, \Sigma)$ need not be infra soft closed and infra soft open, respectively. Through this paper, (Ω, Σ) is called ξ-infra soft open (resp., ξ-infra soft closed) if $Int(\Omega, \Sigma) = (\Omega, \Sigma)$ (resp., $Cl(\Omega, \Sigma)) = (\Omega, \Sigma)$.

Proposition 1. *[10] Let* (Ω, Σ) *and* (Ψ, Σ) *subsets of an ISTS* (X, ξ, Σ). *Then*

(i) $Cl[(\Omega, \Sigma)\widetilde{\bigcup}(\Psi, \Sigma)] = Cl(\Omega, \Sigma)\widetilde{\bigcup}Cl(\Psi, \Sigma)$, *and*

(ii) $Int[(\Omega, \Sigma)\widetilde{\bigcap}(\Psi, \Sigma)] = Int(\Omega, \Sigma)\widetilde{\bigcap}Int(\Psi, \Sigma)$.

Proposition 2. *[10] Let* (Ω, Σ) *be an infra soft open set. Then*

$$(\Omega, \Sigma)\widetilde{\bigcap}Cl(\Psi, \Sigma)\widetilde{\subseteq}Cl[(\Omega, \Sigma)\widetilde{\bigcup}(\Psi, \Sigma)] \text{ for any subset } (\Psi, \Sigma) \text{ of } (X, \xi, \Sigma).$$

Proposition 3. *[10] Let* (Ω, Σ) *be an infra soft closed set. Then*

$$Int[(\Omega, \Sigma)\widetilde{\bigcup}(\Psi, \Sigma)]\widetilde{\subseteq}(\Omega, \Sigma)\widetilde{\bigcup}Int(\Psi, \Sigma) \text{ for any subset } (\Omega, \Sigma) \text{ of } (X, \xi, \Sigma).$$

Definition 14. *A soft map* $f_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ *is said to be an infra soft homeomorphism if it is bijective, infra soft continuous (i.e,the image of every infra soft open set is infra soft open), and infra soft open (i.e,the image of every infra soft open set is infra soft open).*

We call a property which is kept by any infra soft homeomorphism an infra soft topological property (in short, IST property).

Definition 15. *[10] Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ be a soft map and $\mathcal{M} \neq \emptyset$ be a subset of X. A soft map $E_{\tau_{|\mathcal{M}}} : (\mathcal{M}, \xi_{\mathcal{M}}, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ which given by $E_{\tau_{|\mathcal{M}}}(\delta_\eta^m) = E_\tau(\delta_\eta^m)$ for every $\delta_\eta^m \in \widetilde{\mathcal{M}}$ is called a restriction soft map of E_τ on $\mathcal{M}$.*

Proposition 4. *Let $\{(X_k, \xi_k, \Sigma_k) : k \in K\}$ be a family of ISTSs. Then $\boldsymbol{\xi} = \{\prod_{k\in K}(\eta_k, \Sigma_k) : (\eta_k, \Sigma_k) \in \tau_k\}$ is an infra soft topology on $\boldsymbol{T} = \prod_{k\in K} X_k$ under a set of parameters $\boldsymbol{B} = \prod_{k\in K} \Sigma_k$.*

We call $\boldsymbol{\xi}$ given in proposition above, a product of infra soft topologies, and $(\mathbf{T}, \boldsymbol{\xi}, \mathbf{B})$ a product of infra soft spaces.

3. Main properties of infra soft pre-open sets

In this section, we define infra soft pre-open and infra soft pre-closed sets which are the core concepts of this article. We characterize them and investigate some of their properties. We show that the class of infra soft pre-open sets forms a supra soft topology and discuss under what conditions this class forms a soft topology. We complete this section by proving that this class is kept under infra soft homeomorphism maps and finite product of soft spaces.

Definition 16. *A subset (Ω, Σ) of an ISTS (X, ξ, Σ) is said to be infra soft pre-open if $(\Omega, \Sigma)\widetilde{\subseteq} Int(Cl(\Omega, \Sigma))$. Its complement is said to be an infra soft pre-closed set.*

Proposition 5. *If (Ω, Σ) is an infra soft pre-open subset of an ISTS (X, ξ, Σ), then $Cl(\Omega, \Sigma)$ is infra soft semi-open.*

Proof. Since (Ω, Σ) is an infra soft pre-open set, $(\Omega, \Sigma)\widetilde{\subseteq} Int(Cl(\Omega, \Sigma))$. Therefore, $Cl(\Omega, \Sigma) \widetilde{\subseteq} Cl(Int(Cl(\Omega, \Sigma)))\widetilde{\subseteq} Cl(\Omega, \Sigma)$. Thus, $Cl(\Omega, \Sigma) = Cl(Int(Cl(\Omega, \Sigma)))$. Hence, $Cl(\Omega, \Sigma)$ is infra soft semi-open.

In the next two results, we present some characterizations for infra soft pre-open and infra soft pre-closed sets.

Proposition 6. *A subset (Ω, Σ) of an ISTS (X, ξ, Σ) is infra soft pre-open iff there exists an ξ-infra soft open set (Ψ, Σ) such that $(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Sigma)\widetilde{\subseteq} Cl(\Omega, \Sigma)$.*

Proof. Necessity: Let (Ω, Σ) be an infra soft pre-open set. Then $(\Omega, \Sigma)\widetilde{\subseteq} Int(Cl(\Omega, \Sigma))$ $\widetilde{\subseteq} Cl(\Omega, \Sigma)$. Putting $(\Psi, \Sigma) = Int(Cl(\Omega, \Sigma))$. Then $Int(\Psi, \Sigma) = Int(Int(Cl(\Omega, \Sigma))) = (\Psi, \Sigma)$. Therefore, (Ψ, Σ) is an ξ-infra soft open set.
Sufficiency: Let (Ψ, Σ) is an ξ-infra soft open set such that $(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Sigma)\widetilde{\subseteq} Cl(\Omega, \Sigma)$. Then $Int(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Sigma)\widetilde{\subseteq} Int(Cl(\Omega, \Sigma))$. Therefore, $(\Omega, \Sigma)\widetilde{\subseteq} Int(Cl(\Omega, \Sigma))$ which means that (Ω, Σ) is an infra soft pre-open set.

Proposition 7. *A subset* (Ω, Σ) *of an ISTS* (X, ξ, Σ) *is infra soft pre-closed iff there exists an* ξ*-infra soft closed set* (Ψ, Σ) *such that* $Int(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Sigma)\widetilde{\subseteq}(\Omega, \Sigma)$.

Proof. Similar to the proof of Proposition 6.

Proposition 8. *The class of infra soft pre-open sets is closed under arbitrary unions.*

Proof. Consider $\{(\Omega_j, \Sigma) : j \in J\}$ as a family of infra soft pre-open sets. Suppose that $J \neq \emptyset$. Then $(\Omega_j, \Sigma)\widetilde{\subseteq}Int(Cl(\Omega_j, \Sigma))$ for each $j \in J$. Consequentially, $\widetilde{\bigcup}_{j\in J}(\Omega_j, \Sigma)\widetilde{\subseteq}\widetilde{\bigcup}_{j\in J}Int(Cl(\Omega_j, \Sigma))\ \widetilde{\subseteq}Int(Cl(\widetilde{\bigcup}_{j\in J}(\Omega_j, \Sigma)))$. Hence, $\widetilde{\bigcup}_{j\in J}(\Omega_j, \Sigma)$ is infra soft pre-open.

Corollary 1. *The class of infra soft pre-closed sets is closed under arbitrary intersections.*

Corollary 2. *The class of infra soft pre-open subsets of an ISTS* (X, ξ, Σ) *forms a supra soft topology over* X.

To illustrate that the class of infra soft pre-open sets does not form an infra soft topology, we present the following example.

Example 1. *Let* $X = \{x_1, x_2, x_3, x_4\}$ *and* $\Sigma = \{\eta_1, \eta_2\}$. *Then* $\xi = \{\Phi, \widetilde{X}, (\Omega_1, \Sigma), (\Omega_2, \Sigma)\}$ *is an infra soft topology on* X *with* Σ *as a set of parameters, where*

$(\Omega_1, \Sigma) = \{(\eta_1, \{x_1\}), (\eta_2, \{x_1\})\}$ *and*

$(\Omega_2, \Sigma) = \{(\eta_1, \{x_2\}), (\eta_2, \{x_2\})\}$.

Let $(\Omega_5, \Sigma) = \{(\eta_1, X), (\eta_2, \{x_1, x_3\})\}$ *and* $(\Omega_6, \Sigma) = \{(\eta_1, \{x_1, x_3\}), (\eta_2, X)\}$. *Then* (Ω_5, Σ) *and* (Ω_6, Σ) *are infra soft pre-open sets because* $Int(Cl(\Omega_5, \Sigma)) = \widetilde{X}$ *and* $Int(Cl(\Omega_6, \Sigma)) = \widetilde{X}$. *But* $(\Omega_5, \Sigma)\widetilde{\bigcap}(\Omega_6, \Sigma)$ *is not infra soft pre-open because* $Int(Cl[(\Omega_5, \Sigma)\widetilde{\bigcap}(\Omega_6, \Sigma)]) = \{(\eta_1, \{x_1\}), (\eta_2, \{x_1\})\}\widetilde{\not\supseteq}[(\Omega_5, \Sigma)\widetilde{\bigcap}(\Omega_6, \Sigma)]$.

Proposition 9. *The intersection of infra soft open and infra soft pre-open sets is an infra soft pre-open set.*

Proof. Let (Ω_1, Σ) be an infra soft open set and (Ω_2, Σ) be an infra soft pre-open set. Then $(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)\widetilde{\subseteq}(\Omega_1, \Sigma)\widetilde{\bigcap}Int(Cl(\Omega_2, \Sigma)) = Int[(\Omega_1, \Sigma)\widetilde{\bigcap}Cl(\Omega_2, \Sigma)]$; by Proposition 2 we obtain $Int[(\Omega_1, \Sigma)\widetilde{\bigcap}Cl(\Omega_2, \Sigma)]\widetilde{\subseteq}Int(Cl[(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)]$. Hence, $(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)$ is an infra soft pre-open set, as required.

Corollary 3. *The union of infra soft closed and infra soft pre-closed sets is an infra soft pre-closed set.*

Definition 17. *An ISTS* (X, ξ, Σ) *is said to be infra soft hyperconnected if the intersection of any two non-null* ξ*-infra soft open sets is non-null. Otherwise,* (X, ξ, Σ) *is said to be infra soft dishyperconnected.*

Proposition 10. *The intersection of two infra soft pre-open subsets of an infra soft hyperconnected space is an infra soft pre-open set.*

Proof. Let (Ω_1, Σ) and (Ω_2, Σ) be infra soft pre-open sets. If one of them is the null soft set, then we obtain the desired result. Suppose that (Ω_1, Σ) and (Ω_2, Σ) are non-null. According to Proposition 6 there are two ξ-infra soft open sets $(\Psi_1, \Sigma) \neq \Phi$ and $(\Psi_2, \Sigma) \neq \Phi$ such that $(\Omega_1, \Sigma)\widetilde{\subseteq}(\Psi_1, \Sigma)\widetilde{\subseteq}Cl(\Omega_1, \Sigma)$ and $(\Omega_2, \Sigma)\widetilde{\subseteq}(\Psi_2, \Sigma)\widetilde{\subseteq}Cl(\Omega_2, \Sigma)$. By hypothesis of infra soft hyperconnectedness, $(\Psi_1, \Sigma)\widetilde{\bigcap}(\Psi_2, \Sigma)$ is a non-null ξ-infra soft open set. Now, $(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)\widetilde{\subseteq}(\Psi_1, \Sigma)\widetilde{\bigcap}(\Psi_2, \Sigma)\widetilde{\subseteq}Cl[(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)]$. Hence, $(\Omega_1, \Sigma)\widetilde{\bigcap}(\Omega_2, \Sigma)$ is an infra soft pre-open set.

Lemma 1. *Let $E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_2, \xi_2, \Sigma_2)$ be an infra soft homeomorphism map. Then for any subset (Ω, Σ_1) we have the next two results.*

(i) $E_\tau(Int(\Omega, \Sigma_1)) = Int(E_\tau(\Omega, \Sigma_1))$.

(ii) $E_\tau(Cl(\Omega, \Sigma_1)) = Cl(E_\tau(\Omega, \Sigma_1))$.

Proof. To prove (i), let $\delta^s_{\eta'} \in E_\tau(Int(\Omega, \Sigma_1))$. Then there is $\delta^x_\eta \in Int(\Omega, \Sigma_1)$ such that $E_\tau(\delta^x_\eta) = \delta^s_{\eta'}$. This means there exists an infra soft open set (Ψ, Σ_1) such that $\delta^x_\eta \in (\Psi, \Sigma_1)\widetilde{\subseteq}(\Omega, \Sigma_1)$. Therefore, $\delta^s_{\eta'} = E_\tau(\delta^x_\eta) \in E_\tau(\Psi, \Sigma_1)\widetilde{\subseteq}E_\tau(\Omega, \Sigma_1)$. This implies that $\delta^s_{\eta'} \in Int(E_\tau(\Omega, \Sigma_1))$. Thus, $E_\tau(Int(\Omega, \Sigma_1))\widetilde{\subseteq}Int(E_\tau(\Omega, \Sigma_1))$. Conversely, let $\delta^s_{\eta'} \in Int(E_\tau(\Omega, \Sigma_1))$. Then there exists an infra soft open set (Ψ, Σ_2) such that $\delta^s_{\eta'} \in (\Psi, \Sigma_2)\widetilde{\subseteq}E_\tau(\Omega, \Sigma_1)$. Therefore, $E^{-1}_\tau(\delta^s_{\eta'}) \in E^{-1}_\tau(\Psi, \Sigma_2)\widetilde{\subseteq}(\Omega, \Sigma_1)$. Automatically, we obtain $E^{-1}_\tau(\delta^s_{\eta'}) \in Int(\Omega, \Sigma_1)$. So that, $\delta^s_{\eta'} \in E_\tau(Int(\Omega, \Sigma_1))$. Thus, $Int(E_\tau(\Omega, \Sigma_1))\widetilde{\subseteq}E_\tau(Int(\Omega, \Sigma_1))$. Hence, the proof is complete.

Following similar arguments, one can prove (ii).

Proposition 11. *The infra soft homeomorphism image of an infra soft pre-open set is an infra soft pre-open set.*

Proof. Consider $E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_2, \xi_2, \Sigma_2)$ as an infra soft continuous map and let (Ω, Σ_1) be an infra soft pre-open subset of (X_1, ξ_1, Σ_1). Then $E_\tau(\Omega, \Sigma_1)\widetilde{\subseteq}E_\tau(Int(Cl(\Omega, \Sigma_1)))$. It follows from the above lemma that $E_\tau(\Omega, \Sigma_1)\widetilde{\subseteq}Int(Cl(E_\tau(\Omega, \Sigma_1)))$. Hence, $E_\tau(\Omega, \Sigma_1)$ is an infra soft pre-open subset of (X_2, ξ_2, Σ_2), as required.

Lemma 2. *Consider (Ω_1, Σ_1) and (Ω_2, Σ_2) as subsets of (X_1, ξ_1, Σ_1) and (X_2, ξ_2, Σ_2), respectively. Then*

(i) $Cl[(\Omega_1, \Sigma_1) \times (\Omega_2, \Sigma_2)] = Cl(\Omega_1, \Sigma_1) \times Cl(\Omega_2, \Sigma_2)$.

(ii) $Int[(\Omega_1, \Sigma_1) \times (\Omega_2, \Sigma_2)] = Int(\Omega_1, \Sigma_1) \times Int(\Omega_2, \Sigma_2)$.

Proof. (i): Let $\delta^{(t,s)}_{(\eta,\vartheta)} \notin Cl[(\Omega_1, \Sigma_1) \times (\Omega_2, \Sigma_2)]$. Then there is an infra soft open subset $(\Psi_1, \Sigma_1) \times (\Psi_2, \Sigma_2)$ of $\widetilde{X_1} \times \widetilde{X_2}$ containing $\delta^{(t,s)}_{(\eta,\vartheta)}$ such that $[(\Omega_1, \Sigma_1) \times (\Omega_2, \Sigma_2)]\widetilde{\bigcap}[(\Psi_1, \Sigma_1) \times (\Psi_2, \Sigma_2)] = \Phi_{\Sigma_1 \times \Sigma_2}$. This implies that $(\Omega_1, \Sigma_1)\widetilde{\bigcap}(\Psi_1, \Sigma_1) = \Phi_{\Sigma_1}$ or $(\Omega_2, \Sigma_2)\widetilde{\bigcap}(\Psi_2, \Sigma_2) =$

Φ_{Σ_2}. Therefore, $\delta_\eta^t \notin Cl(\Omega_1,\Sigma_1)$ or $\delta_\vartheta^s \notin Cl(\Omega_2,\Sigma_2)$. Thus, $\delta_{(\eta,\vartheta)}^{(t,s)} \notin [Cl(\Omega_1,\Sigma_1) \times Cl(\Omega_2,\Sigma_2)]$. Hence, $Cl(\Omega_1,\Sigma_1) \times Cl(\Omega_2,\Sigma_2)\widetilde{\subseteq} Cl[(\Omega_1,\Sigma_1) \times (\Omega_2,\Sigma_2)]$. Conversely, let $\delta_{(\eta,\vartheta)}^{(t,s)} \notin Cl(\Omega_1,\Sigma_1) \times Cl(\Omega_2,\Sigma_2)$. Then $\delta_\eta^x \notin Cl(\Omega_1,\Sigma_1)$ or $\delta_\vartheta^s \notin Cl(\Omega_2,\Sigma_2)$. Suppose, without loss of generality, that $\delta_\eta^x \notin Cl(\Omega_1,\Sigma_1)$. Then there is an infra soft open subset (Ψ_1,Σ_1) of (X_1,ξ_1,Σ_1) containing δ_η^x such that $(\Omega_1,\Sigma_1)\widetilde{\bigcap}(\Psi_1,\Sigma_1) = \Phi_{\Sigma_1}$. Obviously, $(\Psi_1,\Sigma_1) \times \widetilde{X_2}$ is an infra soft open subset of $\widetilde{X_1} \times \widetilde{X_2}$ containing $\delta_{(\eta,\vartheta)}^{(t,s)}$ such that $[(\Psi_1,\Sigma_1)\times\widetilde{X_2}]\widetilde{\bigcap}[(\Omega_1,\Sigma_1)\times(\Omega_2,\Sigma_2)] = \Phi_{\Sigma_1\times\Sigma_2}$. Therefore, $\delta_{(\eta,\vartheta)}^{(t,s)} \notin Cl[(\Omega_1,\Sigma_1)\times(\Omega_2,\Sigma_2)]$. Thus, $Cl[(\Omega_1,\Sigma_1) \times (\Omega_2,\Sigma_2)]\widetilde{\subseteq} Cl(\Omega_1,\Sigma_1) \times Cl(\Omega_2,\Sigma_2)$. Hence, the proof is complete.

Following similar arguments, one can prove (ii).

Proposition 12. *The product of infra soft pre-open sets is an infra soft pre-open set.*

Proof. Let (Ω_1,Σ_1) and (Ω_2,Σ_2) be infra soft pre-open subsets of (X_1,ξ_1,Σ_1) and (X_2,ξ_2,Σ_2), respectively. Then $(\Omega_1,\Sigma_1) \times (\Omega_2,\Sigma_2)\widetilde{\subseteq} Int(Cl(\Omega_1,\Sigma_1)) \times Int(Cl(\Omega_2,\Sigma_2))$. According to the above lemma, we obtain $(\Omega_1,\Sigma_1)\times(\Omega_2,\Sigma_2)\widetilde{\subseteq} Int(Cl[(\Omega_1,\Sigma_1)\times(\Omega_2,\Sigma_2)])$ which means that $(\Omega_1,\Sigma_1) \times (\Omega_2,\Sigma_2)$ is an infra soft pre-open subset of $\widetilde{X_1} \times \widetilde{X_2}$.

4. Infra pre-interior, infra pre-closure, infra pre-limit and infra pre-boundary soft points of a soft set

The goal of this part is to introduce the concepts of infra soft pre-interior and infra soft pre-closure, infra soft pre-limit and infra soft pre-boundary soft points of a soft set. We explore their essential properties and explain the interrelationships between them with the aid of illustrative examples.

Definition 18. *Let (Ω,Σ) be a subset of (X,ξ,Σ). Then:*

(i) *the infra soft pre-interior of (Ω,Σ), denoted by $pInt(\Omega,\Sigma)$, is the union of all infra soft pre-open sets that are contained in (Ω,Σ).*

(ii) *the infra soft pre-closure of (Ω,Σ), denoted by $pCl(\Omega,\Sigma)$, is the intersection of all infra soft pre-closed sets containing (Ω,Σ).*

Proposition 13. *We have the following properties.*

(i) (Ω,Σ) *is an infra soft pre-open subset of* (X,ξ,Σ) *iff* $pInt(\Omega,\Sigma) = (\Omega,\Sigma)$.

(ii) (Ω,Σ) *is an infra soft pre-closed subset of* (X,ξ,Σ) *iff* $pCl(\Omega,\Sigma) = (\Omega,\Sigma)$.

Proof. It comes from Proposition 8 and Corollary 1.

Note that the the above two properties are not valid for infra soft open and infra soft closed sets.

Proposition 14. *Let* (Ω, Σ) *be a subset of* (X, ξ, Σ).

(i) $\delta^x_\eta \in pInt(\Omega, \Sigma)$ *iff there is an infra soft pre-open set* (Ψ, Σ) *such that* $\delta^x_\eta \in (\Psi, \Sigma)\widetilde{\subseteq}(\Omega, \Sigma)$.

(ii) $\delta^x_\eta \in pCl(\Omega, \Sigma)$ *iff the intersection of any infra soft pre-open set* (Ψ, Σ) *containing* δ^x_η *and* (Ω, Σ) *is non-null.*

Proof. The proof of (i) is obvious, so we prove (ii).

Let $\delta^x_\eta \in pCl(\Omega, \Sigma)$. Then every infra soft pre-closed set contains (Ω, Σ) contains δ^x_η as well. Suppose that there exists an infra soft pre-open set (Ψ, Σ) containing δ^x_η such that $(\Omega, \Sigma)\widetilde{\bigcap}(\Psi, \Sigma) = \Phi$. Therefore, $(\Omega, \Sigma)\widetilde{\subseteq}(\Psi^c, \Sigma)$ which means that $\delta^x_\eta \notin pCl(\Omega, \Sigma)$. This is a contradiction. Conversely, suppose that there exists an infra soft pre-open set (Ψ, Σ) containing δ^x_η such that $(\Omega, \Sigma)\widetilde{\bigcap}(\Psi, \Sigma) = \Phi$. Therefore, $pCl(\Omega, \Sigma)\widetilde{\subseteq}(\Psi^c, \Sigma)$ which means that $\delta^x_\eta \notin pCl(\Omega, \Sigma)$. Hence, we obtain the desired result.

Proposition 15. *Let* (Ω, Σ) *be a subset of* (X, ξ, Σ). *Then:*

(i) $(pInt(\Omega, \Sigma))^c = pCl(\Omega^c, \Sigma)$.

(ii) $(pCl(\Omega, \Sigma))^c = pInt(\Omega^c, \Sigma)$.

Proof. (i): $(pInt(\Omega, \Sigma))^c = \{\widetilde{\bigcup\limits_{j\in J}}(\Psi_j, \Sigma) : (\Psi_j, \Sigma)$ is an infra soft pre-open set contained in $(\Omega, \Sigma)\}^c = \widetilde{\bigcap\limits_{j\in J}}\{(\Psi^c_j, \Sigma) : (\Psi^c_j, \Sigma)$ is an infra soft pre-closed set containing $(\Omega^c, \Sigma)\} = pCl(\Omega^c, \Sigma)$.

The proof of (ii) is similar to (i).

Proposition 16. *Let* (Ψ, Σ) *be an infra soft open set and* (Λ, Σ) *be an infra soft closed set in* (X, ξ, Σ). *Then:*

(i) $(\Psi, \Sigma)\widetilde{\bigcap}pCl(\Omega, \Sigma)\widetilde{\subseteq}pCl((\Psi, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma))$.

(ii) $pInt((\Lambda, \Sigma)\widetilde{\bigcup}(\Omega, \Sigma))\widetilde{\subseteq}(\Lambda, \Sigma)\widetilde{\bigcup}pInt(\Omega, \Sigma)$.

Proof. (i): Let $\delta^x_\eta \in (\Psi, \Sigma)\widetilde{\bigcap}pCl(\Omega, \Sigma)$. Then $\delta^x_\eta \in (\Psi, \Sigma)$ and $\delta^x_\eta \in pCl(\Omega, \Sigma)$. This implies that $(\Gamma, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma) \neq \Phi$ for every infra soft pre-open set (Γ, Σ) containing δ^x_η. It follows from Proposition 9 that $(\Psi, \Sigma)\widetilde{\bigcap}(\Gamma, \Sigma)$ is an infra soft pre-open set containing δ^x_η. Therefore, $[(\Psi, \Sigma)\widetilde{\bigcap}(\Gamma, \Sigma)]\widetilde{\bigcap}(\Omega, \Sigma) \neq \Phi$. Now, $(\Gamma, \Sigma)\widetilde{\bigcap}[(\Psi, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma)] \neq \Phi$ which means that $\delta^x_\eta \in pCl((\Psi, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma))$. Hence, $(\Psi, \Sigma)\widetilde{\bigcap}pCl(\Omega, \Sigma)\widetilde{\subseteq}pCl((\Psi, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma))$.

One can prove (ii) following similar arguments.

Theorem 1. *Let* (Ω, Σ) *and* (Ψ, Σ) *be subsets of* (X, ξ, Σ). *Then we have the following properties.*

(i) $pInt(\widetilde{X}) = \widetilde{X}$.

(ii) $pInt(\Omega,\Sigma)\widetilde{\subseteq}(\Omega,\Sigma)$.

(iii) *If* $(\Psi,\Sigma)\widetilde{\subseteq}(\Omega,\Sigma)$, *then* $pInt(\Psi,\Sigma)\widetilde{\subseteq}pInt(\Omega,\Sigma)$.

(iv) $pInt(pInt(\Omega,\Sigma)) = pInt(\Omega,\Sigma)$.

(v) $pInt(\Psi,\Sigma)\widetilde{\bigcap}pInt(\Omega,\Sigma)\widetilde{\subseteq}pInt((\Psi,\Sigma)\widetilde{\bigcap}(\Omega,\Sigma))$.

Proof. (i): Since $\widetilde{X}$ is infra soft pre-open, $pInt(\widetilde{X}) = \widetilde{X}$.
(ii) and (iii) are obvious.
(iv): It is clear that $pInt(pInt(\Omega,\Sigma))$ is the largest infra soft pre-open set contained in $pInt(\Omega,\Sigma)$; however, $pInt(\Omega,\Sigma)$ is an infra soft pre-open set; hence, $pInt(pInt(\Omega,\Sigma)) = pInt(\Omega,\Sigma)$.
(v): It comes from (iii).

Theorem 2. *Let* (Ω,Σ) *and* (Ψ,Σ) *be subsets of* (X,ξ,Σ). *Then we have the following properties.*

(i) $pCl(\Phi) = \Phi$.

(ii) $(\Omega,\Sigma)\widetilde{\subseteq}pCl(\Omega,\Sigma)$.

(iii) *If* $(\Psi,\Sigma)\widetilde{\subseteq}(\Omega,\Sigma)$, *then* $pCl(\Psi,\Sigma)\widetilde{\subseteq}pCl(\Omega,\Sigma)$.

(iv) $pCl(pCl(\Omega,\Sigma))\widetilde{\subseteq}pCl(\Omega,\Sigma)$.

(v) $pCl((\Psi,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)) = pCl(\Psi,\Sigma)\widetilde{\bigcup}pCl(\Omega,\Sigma)$.

Proof. It can be proved following similar arguments given in the proof of Theorem 1.

The next example shows that the inclusion relations given in the above two theorems are proper.

Example 2. *Let* $X = \{x_1, x_2\}$ *and* $\Sigma = \{\eta_1, \eta_2\}$. *Then* $\xi = \{\Phi, \widetilde{X}, (\Omega_j,\Sigma) : j = 1, 2, 3\}$ *is an infra soft topology on* X *over* X *with* Σ *as a set of parameters, where*
$(\Omega_1,\Sigma) = \{(\eta_1, \{x_1\}), (\eta_2, \emptyset)\}$;
$(\Omega_2,\Sigma) = \{(\eta_1, \emptyset), (\eta_2, \{x_1\})\}$ *and*
$(\Omega_3,\Sigma) = \{(\eta_1, X), (\eta_2, \{x_2\})\}$.
Let $(\Psi_1,\Sigma) = \{(\eta_1, \{x_2\}), (\eta_2, \{x_1\})\}$. *Then* $pInt(\Psi_1,\Sigma) = \{(\eta_1, \emptyset), (\eta_2, \{x_1\})\}\widetilde{\subset}(\Psi_1,\Sigma)$ *and* $pCl(\Psi_1,\Sigma) = \{(\eta_1, \{x_2\}), (\eta_2, X)\}\widetilde{\supset}(\Psi_1,\Sigma)$. *Also, consider* $(\Psi_2,\Sigma) = \{(\eta_1, \{x_2\}), (\eta_2, \emptyset)\}$. *Then* $pCl((\Psi_1,\Sigma)\widetilde{\bigcup}(\Psi_2,\Sigma)) = \{(\eta_1, \{x_2\}), (\eta_2, X)\}\widetilde{\supseteq}pCl(\Psi_1,\Sigma)\widetilde{\bigcap}pCl(\Psi_2,\Sigma) = \{(\eta_1, \{x_2\}), (\eta_2, \{x_2\})\}$.

Definition 19. *A soft point* δ^x_η *is said to be an infra soft pre-limit point of a subset* (Ω,Σ) *of* (X,ξ,Σ) *provided that* $[(\Psi,\Sigma)\backslash\delta^x_\eta]\widetilde{\bigcap}(\Omega,\Sigma) \neq \Phi$ *for every infra soft pre-open set* (Ψ,Σ) *containing* δ^x_η.

The soft set of all infra soft pre-limit points of (Ω,Σ) *is said to be an infra pre-derived soft set. It is denoted by* $(\Omega,\Sigma)^{ps\prime}$.

Proposition 17. *Consider* (Ψ,Σ) *and* (Ω,Σ) *as subsets of* (X,ξ,Σ)*. Then*

(i) $\Phi^{ps\prime}=\Phi$ *and* $\widetilde{X}^{ps\prime}\widetilde{\subseteq}\widetilde{X}$.

(ii) If $(\Psi,\Sigma)\widetilde{\subseteq}(\Omega,\Sigma)$*, then* $(\Psi,\Sigma)^{ps\prime}\widetilde{\subseteq}(\Omega,\Sigma)^{ps\prime}$.

(iii) If $\delta^x_\eta\in(\Omega,\Sigma)^{ps\prime}$*, then* $\delta^x_\eta\in((\Omega,\Sigma)\setminus\delta^x_\eta)^{ps\prime}$.

(iv) $(\Psi,\Sigma)^{ps\prime}\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime}\widetilde{\subseteq}((\Psi,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma))^{ps\prime}$.

Proof. Straightforward.

Theorem 3. *Let* (Ω,Σ) *be a subset of* (X,ξ,Σ)*. Then*

(i) If (Ω,Σ) *is an infra soft pre-closed set, then* $(\Omega,\Sigma)^{ps\prime}\subseteq(\Omega,\Sigma)$.

(ii) $((\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime})^{ps\prime}\widetilde{\subseteq}(\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime}$.

(iii) $pCl(\Omega,\Sigma)=(\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime}$.

Proof.

(i) Consider (Ω,Σ) as an infra soft pre-closed set such that $\delta^x_\eta\notin(\Omega,\Sigma)$. Then $\delta^x_\eta\in(\Omega^c,\Sigma)$. Now, (Ω^c,Σ) is an infra soft pre-open set such that $(\Omega^c,\Sigma)\widetilde{\bigcap}(\Omega,\Sigma)=\Phi$ which means that $\delta^x_\eta\notin(\Omega,\Sigma)^{ps\prime}$. Thus, $(\Omega,\Sigma)^{ps\prime}\widetilde{\subseteq}(\Omega,\Sigma)$.

(ii) Consider $\delta^x_\eta\notin(\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime}$. Then $\delta^x_\eta\notin(\Omega,\Sigma)$ and $\delta^x_\eta\notin(\Omega,\Sigma)^{ps\prime}$. Therefore, there exists an infra soft pre-open set (Ψ,Σ) such that

$$(\Psi,\Sigma)\widetilde{\bigcap}(\Omega,\Sigma)=\Phi \tag{1}$$

This implies that

$$(\Psi,\Sigma)\widetilde{\bigcap}(\Omega,\Sigma)^{ps\prime}=\Phi \tag{2}$$

It follows from (1) and (2) that $(\Psi,\Sigma)\widetilde{\bigcap}((\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime})=\Phi$. Thus, $\delta^x_\eta\notin((\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime})^{ps\prime}$. Hence, $((\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime})^{ps\prime}\widetilde{\subseteq}((\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime})$, as required.

(iii) It is clear that $(\Omega,\Sigma)\widetilde{\bigcup}(\Omega,\Sigma)^{ps\prime}\widetilde{\subseteq}pCl(\Omega,\Sigma)$. Conversely, let $\delta^x_\eta\in pCl(\Omega,\Sigma)$. Then for every infra soft pre-open set containing δ^x_η we have $(\Omega,\Sigma)\widetilde{\bigcap}(\Psi,\Sigma)\neq\Phi$. Without loss of generality, let $\delta^x_\eta\notin(\Omega,\Sigma)$. Then $[(\Omega,\Sigma)\setminus\delta^x_\eta]\widetilde{\bigcap}(\Psi,\Sigma)\neq\Phi$. Consequentially, $\delta^x_\eta\in(\Omega,\Sigma)^{ps\prime}$. Hence, the proof is complete.

Definition 20. *The infra soft pre-boundary points of a subset* (Ω,Σ) *of* (X,ξ,Σ)*, denoted by* $pB(\Omega,\Sigma)$*, are all the soft points which belong to the complement of* $pInt(\Omega,\Sigma)\widetilde{\bigcup}pInt(\Omega^c,\Sigma)$.

Proposition 18. *Let* (Ω, Σ) *be a subset of* (X, ξ, Σ)*. Then:*

(i) $pB(\Omega, \Sigma) = pCl(\Omega, \Sigma)\widetilde{\bigcap}pCl((\Omega^c, \Sigma))$.

(ii) $pB(\Omega, \Sigma) = pCl(\Omega, \Sigma) \setminus pInt(\Omega, \Sigma)$.

Proof.

(i)
$$\begin{aligned} pB(\Omega, \Sigma) &= \{\delta_\eta^x \in \widetilde{X} : \delta_\eta^x \notin pInt(\Omega, \Sigma) \text{ and } \delta_\eta^x \notin pInt((\Omega^c, \Sigma))\} \\ &= \{\delta_\eta^x \in \widetilde{X} : \delta_\eta^x \notin (pCl(\Omega^c, \Sigma))^c \text{ and } \delta_\eta^x \notin (pCl(\Omega, \Sigma))^c\} \\ &= \{\delta_\eta^x \in \widetilde{X} : \delta_\eta^x \in pCl(\Omega^c, \Sigma) \text{ and } \delta_\eta^x \in pCl(\Omega, \Sigma)\} \\ &= pCl(\Omega, \Sigma)\widetilde{\bigcap}pCl(\Omega^c, \Sigma) \end{aligned}$$

(ii)
$$\begin{aligned} pB(\Omega, \Sigma) &= pCl(\Omega, \Sigma)\widetilde{\bigcap}pCl(\Omega^c, \Sigma) \\ &= pCl(\Omega, \Sigma)\widetilde{\bigcap}(pInt(\Omega, \Sigma))^c \\ &= pCl(\Omega, \Sigma) \setminus pInt(\Omega, \Sigma) \end{aligned}$$

Corollary 4. *Let* (Ω, Σ) *be a subset of* (X, ξ, Σ)*. Then*

(i) $pB(\Omega, \Sigma) = pB(\Omega^c, \Sigma)$

(ii) $pCl(\Omega, \Sigma) = pInt(\Omega, \Sigma)\widetilde{\bigcup}pB(\Omega, \Sigma)$

Proposition 19. *Let* (Ω, Σ) *be a subset of* (X, ξ, Σ)*. Then*

(i) (Ω, Σ) *is infra soft pre-open iff* $pB(\Omega, \Sigma)\widetilde{\bigcap}(\Omega, \Sigma) = \Phi$.

(ii) (Ω, Σ) *is infra soft pre-closed iff* $pB(\Omega, \Sigma)\widetilde{\subseteq}(\Omega, \Sigma)$.

Proof.

(i) $pB(\Omega, \Sigma) \bigcap (\Omega, \Sigma) = pB(\Omega, \Sigma) \bigcap pInt(\Omega, \Sigma) = \Phi$. Conversely, let $\delta_\eta^x \in (\Omega, \Sigma)$. Then $\delta_\eta^x \in pInt(\Omega, \Sigma)$ or $\delta_\eta^x \in pB(\Omega, \Sigma)$. Since $pB(\Omega, \Sigma) \bigcap (\Omega, \Sigma) = \Phi$, $\delta_\eta^x \in pInt(\Omega, \Sigma)$. Thus, $(\Omega, \Sigma) \subseteq pInt(\Omega, \Sigma)$ which means that $(\Omega, \Sigma) = pInt(\Omega, \Sigma)$. Hence, (Ω, Σ) is infra soft pre-open.

(ii) (Ω, Σ) is infra soft pre-closed $\Leftrightarrow (\Omega^c, \Sigma)$ is infra soft pre-open $\Leftrightarrow pB(\Omega^c, \Sigma) \bigcap (\Omega^c, \Sigma) = \Phi \Leftrightarrow pB(\Omega, \Sigma) \bigcap (\Omega^c, \Sigma) = \Phi \Leftrightarrow pB(\Omega, \Sigma) \subseteq (\Omega, \Sigma)$.

Corollary 5. *A subset* (Ω, Σ) *of* (X, ξ, Σ) *is infra soft pre-open and infra soft pre-closed iff* $pB(\Omega, \Sigma) = \Phi$.

5. Infra soft pre-homeomorphism maps

We devote this section to introducing new types of soft maps called infra soft pre-continuous, infra soft pre-open, infra soft pre-closed and infra soft pre-homeomorphism maps. We study their characterizations and establish main properties.

Definition 21. *A soft map $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is said to be infra soft pre-continuous at $\delta_\eta^x \in \widetilde{X}$ if for any infra soft pre-open set (Ψ, Δ) containing $E_\tau(\delta_\eta^x)$, there is an infra soft pre-open set (Ω, Σ) containing δ_η^x suchthat $E_\tau(\Omega, \Sigma)\widetilde{\subseteq}(\Psi, \Delta)$.*

If E_τ is infra soft pre-continuous at all soft points of the domain, then it is called infra soft pre-continuous.

Theorem 4. *Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ be an infra soft pre-continuous map. Then we have the following five equivalent statements:*

(i) *E_τ is an infra soft pre-continuous map;*

(ii) *The pre-image of each infra soft pre-closed set is infra soft pre-closed;*

(iii) *$pCl(E_\tau^{-1}(\Omega, \Delta))\widetilde{\subseteq}E_\tau^{-1}(pCl(\Omega, \Delta))$ for each $(\Omega, \Delta)\widetilde{\subseteq}\widetilde{\mathcal{S}}$;*

(iv) *$E_\tau(pCl(\Psi, \Sigma))\widetilde{\subseteq}pCl(E_\tau(\Psi, \Sigma))$ for each $(\Psi, \Sigma)\widetilde{\subseteq}\widetilde{X}$;*

(v) *$E_\tau^{-1}(pInt(\Omega, \Delta))\widetilde{\subseteq}pInt(E_\tau^{-1}(\Omega, \Delta))$ for each $(\Omega, \Delta)\widetilde{\subseteq}\widetilde{\mathcal{S}}$.*

Proof. $(i) \Rightarrow (ii)$: Let (Ω, Δ) be an infra soft pre-closed set in $(\mathcal{S}, \pi, \Delta)$. Then $E_\tau^{-1}(\Omega^c, \Delta)$ is an infrasoft pre-open subset of $\widetilde{X}$. Obviously, $E_\tau^{-1}(\Omega^c, \Delta) = \widetilde{X} - E_\tau^{-1}(\Omega, \Delta)$; hence, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft pre-closed subset of $\widetilde{X}$.
$(ii) \Rightarrow (iii)$: According to (ii), $E_\tau^{-1}(pCl(\Omega, \Delta))$ is an infra soft pre-closed subset of $\widetilde{X}$. Then $pCl(E_\tau^{-1}(\Omega, \Delta))\widetilde{\subseteq}pCl(E_\tau^{-1}(pCl(\Omega, \Delta))) = E_\tau^{-1}(pCl(\Omega, \Delta))$.
$(iii) \Rightarrow (vi)$: According to (iii), $pCl(E_\tau^{-1}(E_\tau(\Psi, \Sigma)))\widetilde{\subseteq}E_\tau^{-1}(pCl(E_\tau(\Psi, \Sigma)))$. Then $E_\tau(pCl(\Psi, \Sigma))\widetilde{\subseteq}E_\tau(E_\tau^{-1}(pCl(E_\tau(\Psi, \Sigma))))\widetilde{\subseteq}pCl(E_\tau(\Psi, \Sigma))$.
$(iv) \Rightarrow (v)$: According to (iv), $E_\tau(pCl(\widetilde{X} - E_\tau^{-1}(\Omega, \Delta)))\widetilde{\subseteq}pCl(E_\tau(\widetilde{X} - E_\tau^{-1}(\Omega, \Delta)))$. Therefore, $E_\tau(\widetilde{X} - pInt(E_\tau^{-1}(\Omega, \Delta))) = E_\tau(pCl(\widetilde{X} - E_\tau^{-1}(\Omega, \Delta))) \subseteq pCl(\widetilde{\mathcal{S}} - (\Omega, \Delta)) = \widetilde{\mathcal{S}} - pInt(\Omega, \Delta)$. Thus $\widetilde{X} - pInt(E_\tau^{-1}(\Omega, \Delta))\widetilde{\subseteq}E_\tau^{-1}(\widetilde{\mathcal{S}} - pInt(\Omega, \Delta)) = E_\tau^{-1}(\widetilde{\mathcal{S}}) - E_\tau^{-1}(pInt(\Omega, \Delta))$. Hence $E_\tau^{-1}(pInt(\Omega, \Delta))\widetilde{\subseteq}pInt(E_\tau^{-1}(\Omega, \Delta))$.
$(v) \Rightarrow (i)$: Let (Ω, Δ) be an infra soft open subset of $\widetilde{\mathcal{S}}$. According to (v), $E_\tau^{-1}(\Omega, \Delta)\widetilde{\subseteq}pInt(E_\tau^{-1}(\Omega, \Delta))$. This implies that $E_\tau^{-1}(\Omega, \Delta) = pInt(E_\tau^{-1}(\Omega, \Delta))$. Hence, E_τ is infra soft pre-continuous.

Theorem 5. *If $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is infra soft pre-continuous, then the restriction soft map $E_{\tau_{|\mathcal{M}}} : (\mathcal{M}, \xi_{\mathcal{M}}, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is infra soft pre-continuous provided that $\widetilde{\mathcal{M}}$ is an infra soft open set.*

Proof. Consider (Ω, Δ) is an infra soft pre-open set in $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is infra soft pre-open. Now, $E_{\tau_{|\mathcal{M}}}^{-1}(\Omega, \Delta) = E_\tau^{-1}(\Omega, \Delta)\widetilde{\bigcap}\widetilde{\mathcal{M}}$. Since $\widetilde{\mathcal{M}}$ is an infra soft open set, it follows from Proposition 9 that $E^{-1}_{\tau_{|\mathcal{M}}}(\Omega, \Delta)$ is infra soft pre-open. Hence, $E_{\tau_{|\mathcal{M}}}$ is an infra soft pre-continuous map.

Proposition 20. *Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \to (\mathcal{V}, \sigma, \Gamma)$ be infra soft pre-continuous. Then $F_\nu \circ E_\tau$ is infra soft pre-continuous.*

Proof. Straightforward.

Definition 22. *A soft map $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is said to be infra soft pre-open (resp., infra soft pre-closed) if the image of each infra soft pre-open (resp., infra soft pre-closed) set is infra soft pre-open (resp., infra soft pre-closed).*

Proposition 21. $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ *is an infra soft pre-open map iff* $E_\tau(pInt(\Omega, \Sigma))$ $\widetilde{\subseteq} pInt(E_\tau(\Omega, \Sigma))$ *for each subset of* (Ω, Σ) *of* $\widetilde{X}$.

Proof. $\Rightarrow$: Let (Ω, Σ) be a subset of $\widetilde{X}$. Now, $E_\tau(pInt(\Omega, \Sigma))\widetilde{\subseteq} E_\tau(\Omega, \Sigma)$ and $pInt(\Omega, \Sigma)$ is an infra soft pre-open set. By hypothesis, $E_\tau(pInt(\Omega, \Sigma))$ is infra soft pre-open. Therefore, $E_\tau(pInt(\Omega, \Sigma))\widetilde{\subseteq} pInt(E_\tau(\Omega, \Sigma))$.
$\Leftarrow$: Let (Λ, Σ) be an infra soft open subset of $\widetilde{X}$. Then $E_\tau(\Omega, \Sigma)\widetilde{\subseteq} pInt(E_\tau(\Omega, \Sigma))$. Therefore, $E_\tau(\Omega, \Sigma) = pInt(E_\tau(\Omega, \Sigma))$ which means that E_τ is an infra soft pre-open map.

Proposition 22. $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ *is an infra soft pre-closed map iff* $pCl(E_\tau(\Omega, \Sigma))$ $\widetilde{\subseteq} E_\tau(pCl(\Omega, \Sigma))$ *for each subset* (Ω, Σ) *of* $\widetilde{X}$.

Proof. $\Rightarrow$: Let E_τ be an infra soft pre-closed map and (Ω, Σ) be a subset of $\widetilde{X}$. By hypothesis, $E_\tau(pCl(\Omega, \Sigma))$ is infra soft pre-closed. Since $E_\tau(\Omega, \Sigma)\widetilde{\subseteq} E_\tau(pCl(\Omega, \Sigma))$, $pCl(E_\tau(\Omega, \Sigma)) \widetilde{\subseteq} E_\tau(pCl(\Omega, \Sigma))$.
$\Leftarrow$: Suppose that (Ω, Σ) is an infra soft pre-closed subset of $\widetilde{X}$. By hypothesis, $E_\tau(\Omega, \Sigma)\widetilde{\subseteq}$ $pCl(E_\tau(\Omega, \Sigma))\widetilde{\subseteq} E_\tau(pCl(\Omega, \Sigma)) = E_\tau(\Omega, \Sigma)$. Therefore, $E_\tau(\Omega, \Sigma)$ is infra soft pre-closed. Hence, E_τ is an infra soft pre-closed map.

Proposition 23. *The concepts of infra soft pre-open and infra soft pre-closed maps are equivalent under bijectiveness.*

Proof. It comes from the fact that a bijective soft map $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ implies that $E_\tau(\Omega^c, \Sigma) = (E_\tau(\Omega, \Sigma))^c$.

Proposition 24. *Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \to (\mathcal{V}, \sigma, \Gamma)$ be two soft maps. Then:*

(i) *If E_τ and F_ν are infra soft pre-open maps, then $F_\nu \circ E_\tau$ is an infra soft pre-open map.*

(ii) *If $F_\nu \circ E_\tau$ is an infra soft pre-open map and E_τ is a surjective infra soft pre-continuous map, then F_ν is an infra soft pre-open map.*

(iii) *If $F_\nu \circ E_\tau$ is an infra soft pre-open map and F_ν is an injective infra soft pre-continuous map, then E_τ is an infra soft pre-open map.*

Proof.

(i) Straightforward.

(ii) Consider (Ω, Δ) as an infra soft pre-open subset of $(\mathcal{S}, \pi, \Delta)$. By hypothesis, $E_\tau^{-1}(\Omega, \Delta)$ is an infra soft pre-open subset of (X, ξ, Σ). Again, by hypothesis, $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta))$ is an infra soft pre-open subset of $(\mathcal{V}, \sigma, \Gamma)$. Since E_τ is surjective, then $(F_\nu \circ E_\tau)(E_\tau^{-1}(\Omega, \Delta)) = F_\nu(E_\tau(E_\tau^{-1}(\Omega, \Delta))) = F_\nu(\Omega, \Delta)$. Hence, F_ν is an infra soft pre-open map.

(iii) Consider (Ω, Σ) as an infra soft pre-open subset of (X, ξ, Σ). By hypothesis, $(F_\nu \circ E_\tau)(\Omega, \Sigma)$ is an infra soft pre-open subset of $(\mathcal{V}, \sigma, \Gamma)$. Again, by hypothesis, $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Sigma))$ is an infra soft pre-open subset of $(\mathcal{S}, \pi, \Delta)$. Since F_ν is injective, then $F_\nu^{-1}(F_\nu \circ E_\tau(\Omega, \Sigma)) = (F_\nu^{-1} F_\nu)(E_\tau(\Omega, \Sigma)) = E_\tau(\Omega, \Sigma)$. Hence, E_τ is an infra soft pre-open map.

In a similar way, one can prove the next proposition.

Proposition 25. *Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \to (\mathcal{V}, \sigma, \Gamma)$ be two infra soft maps. Then the following statements hold.*

(i) *If E_τ and F_ν are infra soft pre-closed maps, then $F_\nu \circ E_\tau$ is an infra soft pre-closed map.*

(ii) *If $F_\nu \circ E_\tau$ is an infra soft pre-closed map and E_τ is a surjective infra soft pre-continuous map, then F_ν is an infra soft pre-closed map.*

(iii) *If $F_\nu \circ E_\tau$ is an infra soft pre-closed map and F_ν is an injective infra soft pre-continuous map, then E_τ is an infra soft pre-closed map.*

Definition 23. *A bijective soft map $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is said to be an infra soft pre-homeomorphism if it is infra soft pre-continuous and infra soft pre-open.*

We cancel the proofs of the next two results because they are easy.

Proposition 26. *Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ and $F_\nu : (\mathcal{S}, \pi, \Delta) \to (\mathcal{V}, \sigma, \Gamma)$ be infra soft pre-homeomorphism maps. Then $F_\nu \circ E_\tau$ is an infra soft pre-homeomorphism map.*

Proposition 27. *If $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is a bijective soft map, then the following statements are equivalent.*

(i) *E_τ is an infra soft pre-homeomorphism.*

(ii) *E_τ and E_τ^{-1} is infra soft pre-continuous.*

(iii) *E_τ is infra soft pre-closed and infra soft pre-continuous.*

Proposition 28. *If $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ is an infra soft pre-homeomorphism map, then the following statements hold for each $(\Omega, \Sigma) \in S(X)_A$.*

(i) $E_\tau(pInt(\Omega, \Sigma)) = pInt(E_\tau(\Omega, \Sigma))$.

(ii) $E_\tau(pCl(\Omega, \Sigma)) = pCl(E_\tau(\Omega, \Sigma))$.

Proof. (i): According to Proposition 21 (i), we obtain $E_\tau(pInt(\Omega, \Sigma))\widetilde{\subseteq}pInt(E_\tau(\Omega, \Sigma))$. Conversely, let $\delta^s_\kappa \in pInt(E_\tau(\Omega, \Sigma)$. Then there is an infra soft pre-open set (Ψ, Δ) such that $\delta^s_\kappa \in (\Psi, \Delta)\widetilde{\subseteq}E_\tau(\Omega, \Sigma)$. By hypothesis, $\delta^x_\eta = E^{-1}_\tau(\delta^s_\kappa) \in E^{-1}_\tau(\Psi, \Delta)\widetilde{\subseteq}(\Omega, \Sigma)$ such that $E^{-1}_\tau(\Psi, \Delta)$ is an infra soft pre-open set. So that, $\delta^x_\eta \in pInt(\Omega, \Sigma)$ which means that $\delta^s_\kappa \in E_\tau(pInt(\Omega, \Sigma))$.
One can achieve item (ii) following similar arguments.

Theorem 6. *The property of an infra soft pre-dense set is an infra soft topological invariant.*

Proof. Let $E_\tau : (X, \xi, \Sigma) \to (\mathcal{S}, \pi, \Delta)$ be an infra soft pre-homeomorphism map and consider (Ω, Σ) as an infra soft pre-dense subset of (X, ξ, Σ), i.e. $pCl(\Omega, \Sigma) = \widetilde{X}$. It comes from Proposition 28 (ii) that $pCl(E_\tau(\Omega, \Sigma)) = E_\tau(pCl(\Omega, \Sigma)) = E_\tau(\widetilde{X}) = pCl(\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}$. Thus, $E_\tau(\Omega, \Sigma)$ is an infra soft pre-dense set in $(\mathcal{S}, \pi, \Delta)$, as required.

We complete this section by studying the concept of fixed soft points with respect to infra soft pre-open sets.

Definition 24. *We say that (X, ξ, Σ) has a pre-fixed soft point property provided that for every infra soft pre-continuous map $E_\tau : (X, \xi, \Sigma) \to (X, \xi, \Sigma)$ there exists $\delta^s_\eta \in X$ such that $E_\tau(\delta^s_\eta) = \delta^s_\eta$.*

Proposition 29. *The property of being a pre-fixed soft point is preserved under an infra soft pre-homeomorphism.*

Proof. Consider (X_1, ξ_1, Σ_1) and (X_2, ξ_2, Σ_2) as two infra soft pre-homeomorphism. This means that there exists a bijective soft map $E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_2, \xi_2, \Sigma_2)$ suchthat E_τ and E^{-1}_τ are infra soft pre-continuous. Suppose that (X_1, ξ_1, Σ_1) has the property of pre-fixed soft point. That is any infra soft pre-continuous map $E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_1, \xi_1, \Sigma_1)$ has a pre-fixed soft point. Now, consider $C_\tau : (X_2, \xi_2, \Sigma_2) \to (X_2, \xi_2, \Sigma_2)$ is infra soft pre-continuous. It is clear that $C_\tau \circ E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_2, \xi_2, \Sigma_2)$ is infra soft pre-continuous. Therefore, $E^{-1}_\tau \circ C_\tau \circ E_\tau : (X_1, \xi_1, \Sigma_1) \to (X_1, \xi_1, \Sigma_1)$ is infra soft pre-continuous. Since (X_1, ξ_1, Σ_1) has a pre-fixed soft point property, $E^{-1}_\tau(h_\tau(E_\tau(\delta^s_\eta))) = \delta^s_\eta$ for some $\delta^s_\eta \in \widetilde{X}$. Thus, $E_\tau(E^{-1}_\tau(h_\tau(E_\tau(\delta^s_\eta)))) = E_\tau(\delta^s_\eta)$. This implies that $h_\tau(E_\tau(\delta^s_\eta)) = E_\tau(\delta^s_\eta)$. Hence, $E_\tau(\delta^s_\eta)$ is a pre-fixed soft point of C_τ which means that (X_2, ξ_2, Σ_2) has a pre-fixed soft point property.

6. Concluding remark and further work

In this paper, we contribute to the area of infra soft topologies. We have generalized infra soft open and infra soft closed sets by introducing the concepts of infra soft pre-open and infra soft pre-closed sets. Then, we have applied them to define new kinds of soft operators and soft maps. To validate and illustrate the obtained findings and relationships, we have constructed some examples.

As we have noted, most of soft topological properties of initiated concepts are kept via infra soft topologies. This means the absence of some topology's stipulations does not effected in the behaviours and properties of topological concepts which considers an advantage of studying infra soft topological spaces. However, there is a few properties of some topological concept are partially losing such as the those given in Proposition 6 and Proposition 21.

In the upcoming works, we will apply infra soft pre-open sets to introduce the some topological concepts like separation axioms, compactness and connectedness. Also, we will present the concepts and results given in this paper using new generalizations of infra soft open sets such as infra soft α-open and infra soft b-open sets. Furthermore, we shall define new rough set models using infra soft pre-open sets to improve the accuracy measures of sets following a similar technique what was given in [8].

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgements

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant code 22UQU4330052DSR01.

References

[1] H Aktaş and N Çağman. Soft sets and soft groups. *Information Sciences*, 177, 2007.

[2] S Al-Ghour. Strong form of soft pre-open sets in soft topological spaces. *International Journal of Fuzzy Logic and Intelligent Systems*, 21(2):159–168, 2021.

[3] S Al-Ghour and W Hamed. On two classes of soft sets in soft topological spaces. *Symmetry*, 12(2):265, 2020.

[4] T M Al-shami. Soft somewhere dense sets on soft topological spaces. *Communications of the Korean Mathematical Society*, 33(4):1341–1356, 2018.

[5] T M Al-shami. Bipolar soft sets: relations between them and ordinary points and their applications. *Complexity*, Volume 2021, Article ID 6621854, 2021.

[6] T M Al-shami. Compactness on soft topological ordered spaces and its application on the information system. *Journal of Mathematics*, Volume 2021, Article ID 6699092, 2021.

[7] T M Al-shami. Homeomorphism and quotient mappings in infra soft topological spaces. *Journal of Mathematics*, Volume 2021, Article ID 3388288, 2021.

[8] T M Al-shami. Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets. *Soft Computing*, 25(23):14449–14460, 2021.

[9] T M Al-shami. Infra soft compact spaces and application to fixed point theorem. *Journal of Function Spaces*, Volume 2021, Article ID 3417096, 2021.

[10] T M Al-shami. New soft structure: infra soft topological spaces. *Mathematical Problems in Engineering*, Volume 2021, Article ID 3361604, 2021.

[11] T M Al-shami and E A Abo-Tabl. Connectedness and local connectedness on infra soft topological spaces. *Mathematics*, 9(15):1759, 2021.

[12] T M Al-shami and A A Azzam. Infra soft semiopen sets and infra soft semicontinuity. *Journal of Function Spaces*, Volume 2021, Article ID 5716876, 2021.

[13] T M Al-shami and L D R Kočinac. Nearly soft menger spaces. *Journal of Mathematics*, Volume 2020, Article ID 3807418, 2020.

[14] T M Al-shami and L D R Kočinac. Almost soft menger and weakly soft menger spaces. *Applied and Computational Mathematics*, 21(1), 2022.

[15] T M Al-shami and M E El-Shafei. T-soft equality relation. *Turkish Journal of Mathematics*, 44(8):1427–1441, 2020.

[16] T M Al-shami and J B Liu. Two classes of infrasoft separation axioms. *Journal of Mathematics*, Volume 2021, Article ID 4816893, 2021.

[17] T M Al-shami and A Mhemdi. Two families of separation axioms on infra soft topological spaces. *Filomat*, 2022.

[18] J C R Alcantud. Soft open bases and a novel construction of soft topologies from bases for topologies. *Mathematics*, 8(5):672, 2020.

[19] M I Ali, F Feng, X Liu, W K Min, and M Shabir. On some new operations in soft set theory. *Computers and Mathematics with Applications*, 57, 2009.

[20] B A Asaad, T M Al-shami, and A Mhemdi. Bioperators on soft topological spaces. *AIMS Mathematics*, 6(11):12471–12490, 2021.

[21] A Aygünoğlu and H Aygün. Some notes on soft topological spaces. *Neural Computing and Applications*, 21, 2012.

[22] N Çağman and S Enginoğlu. Soft matrix theory and its decision making. *Computers and Mathematics with Applications*, 59:3308–3314, 2010.

[23] N Çağman, S Karataş, and S Enginoglu. Soft topology. *Computers and Mathematics with Applications*, 62, 2011.

[24] B Chen. Soft semi-open sets and related properties in soft topological spaces. *Appl. Math. Inf. Sci.*, 7(1):287–294, 2013.

[25] L D R. Kočinac, T M Al-shami, and V Çetkin. Selection principles in the context of soft sets: Menger spaces. *Soft computing*, 25:12693–12702, 2021.

[26] S Das and S K Samanta. Soft metric. *Annals of Fuzzy Mathematics and Informatics*, 6(1):77–94, 2013.

[27] M E El-Shafei, M Abo-Elhamayel, and T M Al-shami. Partial soft separation axioms and soft compact spaces. *Filomat*, 32(13):4755–4771, 2018.

[28] M E El-Shafei and T M Al-shami. Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem. *Computational and Applied Mathematics*, 39(3):138, 2020.

[29] F Feng, Y M Li, B Davvaz, and M I Ali. Soft sets combined with fuzzy sets and rough sets: a tentative approach. *Soft Computing*, 14, 2010.

[30] G Ilango and M Ravindran. On soft preopen sets in soft topological spaces. *International Journal of Mathematics Research*, 5(4):399–409, 2013.

[31] A Kharal and B Ahmed. Mappings on soft classes. *New Mathematic Natural Computing*, 7(3):471–481, 2011.

[32] F Lin. Soft connected spaces and soft paracompact spaces. *International Journal of Mathematical Science and Engineering*, 7(2):1–7, 2013.

[33] P K Maji, R Biswas, and R Roy. Soft set theory. *Computers & Mathematics with Applications*, 45, 2003.

[34] D Molodtsov. Soft set theory-first results. *Computers & Mathematics with Applications*, 37:19–31, 1999.

[35] S Nazmul and S K Samanta. Neighbourhood properties of soft topological spaces. *Annals of Fuzzy Mathematics and Informatics*, 6(1):1–15, 2013.

[36] K Qin and Z Hong. On soft equality. *Journal of Computational and Applied Mathematics*, 234, 2010.

[37] M Shabir and M Naz. On soft topological spaces. *Computers & Mathematics with Applications*, 61, 2011.

[38] J Yang and Y Yao. Semantics of soft sets and three-way decision with soft sets. *Knowledge-Based Systems*, Article ID 105538, 2020.