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(Λ, sp)-open sets in topological spaces

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Abstract. This paper is concerned with the concepts of $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets and $b(\Lambda, sp)$-open sets. Some properties of $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets and $b(\Lambda, sp)$-open sets are discussed. In particular, the relationships between $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets, $b(\Lambda, sp)$-open sets and other related sets are established. Moreover, several characterizations of Λ_{sp}-extremally disconnected spaces are investigated.

2020 Mathematics Subject Classifications: 54A05, 54G05

Key Words and Phrases: (Λ, sp)-closed set, (Λ, sp)-open set, Λ_{sp}-extremally disconnected space

1. Introduction

Semi-open sets, preopen sets, α-open sets, b-open sets and β-open sets play an important for the study and investigation in topological spaces. In 1963, Levine [6] introduced the concept of semi-open sets in topological spaces. After the work of Levine on semi-open sets, several mathematicians turned their attention to the generalizations of various concepts of topology by considering semi-open sets instead of open sets. While open sets are replaced by semi-open sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited. In this direction, in 1975, Maheshwari and Prasad [7], used semi-open sets to define and investigate three new separation axioms called semi-T_0, semi-T_1 and semi-T_2. Later, in 1987, Bhattacharya and Lahiri [2] generalized the concept of closed sets to semi-generalized closed sets with the help of semi-openness. The notion of α-open sets (originally called α-sets) in topological spaces was introduced by Njåstad [10] in 1965. By using α-open sets, Mashhour et al. [9] defined and studied the notions of α-continuity and α-openness in topological spaces. In 1982, Mashhour et al. [8] introduced and investigated the concepts of preopen sets and precontinuous functions in topological spaces. In 1983, Abd El-Monsef et al. [4] introduced a weak form of open sets called β-open sets. The concept of β-open sets is equivalent to that

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of semi-preopen sets [1]. In 1996, Andrijević [1] introduced a class of generalized open sets in a topological space, the so-called b-open sets. The class of b-open sets is contained in the class of β-open sets and contains all semi-open sets and preopen sets.

The concept of extremally disconnected topological spaces was first introduced by Gillman and Jerison [5]. A topological space is called extremally disconnected if the closure of every open set is open. Sivaraj [13] investigated some characterizations of extremally disconnected spaces by utilizing semi-open sets due to Levine [6]. Noiri [11] obtained several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets. In 2004, Noiri and Hatir [12] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The purpose of the present paper is to investigate some properties of $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets and $b(\Lambda, sp)$-open sets. In particular, the relationships between $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets, $b(\Lambda, sp)$-open sets and other related sets are explored. Furthermore, some characterizations of Λ_{sp}-extremally disconnected spaces are discussed.

2. Preliminaries

Throughout the paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *β-open* [4] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set A is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [12] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$.

Lemma 1. [12] *For subsets A, B and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) If $U \in \beta(X, \tau)$, *then* $\Lambda_{sp}(U) = U$.

(5) $\Lambda_{sp}(\cap\{A_\alpha | \alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha | \alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [12] if $A = \Lambda_{sp}(A)$. The family of all Λ_{sp}-sets of a topological space (X, τ) is denoted by $\Lambda_{sp}(X, \tau)$ (or simply Λ_{sp}).

Lemma 2. [12] *For subsets A and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following hold:*

(1) $\Lambda_{sp}(A)$ is a Λ_{sp}-set.

(2) If A is β-open, then A is a Λ_{sp}-set.

(3) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cap_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

(4) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cup_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

A subset A of a topological space (X, τ) is called (Λ, sp)-*closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called (Λ, sp)-*open*. The family of all (Λ, sp)-open (resp. (Λ, sp)-closed) sets of a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$ (resp. $\Lambda_{sp}C(X, \tau)$). Let A be a subsets of a topological space (X, τ). A point $x \in X$ is called a (Λ, sp)-*cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-*closure* of A and is denoted by $A^{(\Lambda,sp)}$.

Lemma 3. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A^{(\Lambda,sp)} = A$.

Let A be a subset of a topological space (X, τ). The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-*interior* [3] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 4. [3] *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ and $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ is (Λ, sp)-open.

(4) A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

3. Generalized (Λ, sp)-open sets

In this section, we investigate some properties of $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets and $b(\Lambda, sp)$-open sets.

Definition 1. [3] *A subset A of a topological space (X, τ) is said to be:*

(i) $s(\Lambda, sp)$*-open if* $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$*;*

(ii) $p(\Lambda, sp)$*-open if* $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$*;*

(iii) $\alpha(\Lambda, sp)$*-open if* $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$*;*

(iv) $\beta(\Lambda, sp)$*-open if* $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

The family of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $p\Lambda_{sp}O(X, \tau)$, $\alpha\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$). The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open) set is called $s(\Lambda, sp)$*-closed* (resp. $p(\Lambda, sp)$*-closed*, $\alpha(\Lambda, sp)$*-closed*, $\beta(\Lambda, sp)$*-closed*). The family of all $s(\Lambda, sp)$-closed (resp. $p(\Lambda, sp)$-closed, $\alpha(\Lambda, sp)$-closed, $\beta(\Lambda, sp)$-closed) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}C(X, \tau)$ (resp. $p\Lambda_{sp}C(X, \tau)$, $\alpha\Lambda_{sp}C(X, \tau)$, $\beta\Lambda_{sp}C(X, \tau)$).

Proposition 1. *For a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{sp}O(X, \tau) \subseteq \alpha\Lambda_{sp}O(X, \tau) \subseteq s\Lambda_{sp}O(X, \tau) \subseteq \beta\Lambda_{sp}O(X, \tau)$.

(2) $\alpha\Lambda_{sp}O(X, \tau) \subseteq p\Lambda_{sp}O(X, \tau) \subseteq \beta\Lambda_{sp}O(X, \tau)$.

(3) $\alpha\Lambda_{sp}O(X, \tau) = s\Lambda_{sp}O(X, \tau) \cap p\Lambda_{sp}O(X, \tau)$.

Proof. (1) Let $V \in \Lambda_{sp}O(X, \tau)$. Then, we have $V = V_{(\Lambda,sp)} \subseteq [[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $\Lambda_{sp}O(X, \tau) \subseteq \alpha\Lambda_{sp}O(X, \tau) \subseteq s\Lambda_{sp}O(X, \tau) \subseteq \beta\Lambda_{sp}O(X, \tau)$.

(2) Let $V \in \alpha\Lambda_{sp}O(X, \tau)$. Then, $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $\alpha\Lambda_{sp}O(X, \tau) \subseteq p\Lambda_{sp}O(X, \tau) \subseteq \beta\Lambda_{sp}O(X, \tau)$.

(3) By (1) and (2), we have $\alpha\Lambda_{sp}O(X, \tau) \subseteq s\Lambda_{sp}O(X, \tau) \cap p\Lambda_{sp}O(X, \tau)$. Let

$$V \in s\Lambda_{sp}O(X, \tau) \cap p\Lambda_{sp}O(X, \tau).$$

Then, $V \in s\Lambda_{sp}O(X, \tau)$ and $V \in p\Lambda_{sp}O(X, \tau)$. Therefore, $V \subseteq [V_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $V \in \alpha\Lambda_{sp}O(X, \tau)$. Consequently, we obtain $s\Lambda_{sp}O(X, \tau) \cap p\Lambda_{sp}O(X, \tau) \subseteq \alpha\Lambda_{sp}O(X, \tau)$. This shows that $\alpha\Lambda_{sp}O(X, \tau) = s\Lambda_{sp}O(X, \tau) \cap p\Lambda_{sp}O(X, \tau)$.

Definition 2. *A subset A of a topological space (X, τ) is said to be $r(\Lambda, sp)$-open if $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. The complement of a $r(\Lambda, sp)$-open set is said to be $r(\Lambda, sp)$-closed.*

The family of all $r(\Lambda, sp)$-open (resp. $r(\Lambda, sp)$-closed) sets in a topological space (X, τ) is denoted by $r\Lambda_{sp}O(X, \tau)$ (resp. $r\Lambda_{sp}C(X, \tau)$).

Proposition 2. *Let A be a subset of a topological space (X, τ), the following properties hold:*

(1) A is $r(\Lambda, sp)$-open if and only if $A = F_{(\Lambda,sp)}$ for some (Λ, sp)-closed set F.

(2) A is $r(\Lambda, sp)$-closed if and only if $A = U^{(\Lambda,sp)}$ for some (Λ, sp)-open set U.

Proposition 3. *Let A be a subset of a topological space (X, τ), the following properties hold:*

(1) A is $s(\Lambda, sp)$-closed if and only if $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$.

(2) A is $p(\Lambda, sp)$-closed if and only if $[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$.

(3) A is $\alpha(\Lambda, sp)$-closed if and only if $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$.

(4) A is $\beta(\Lambda, sp)$-closed if and only if $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$.

Proposition 4. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $r(\Lambda, sp)$-open.

(2) A is (Λ, sp)-open and $s(\Lambda, sp)$-closed.

(3) A is $\alpha(\Lambda, sp)$-open and $s(\Lambda, sp)$-closed.

(4) A is $p(\Lambda, sp)$-open and $s(\Lambda, sp)$-closed.

(5) A is (Λ, sp)-open and $\beta(\Lambda, sp)$-closed.

(6) A is $\alpha(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4): Obvious.

(4) ⇒ (5): Let A be (Λ, sp)-open and $s(\Lambda, sp)$-closed. Then, $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$. This implies that $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, A is $r(\Lambda, sp)$-open and hence A is (Λ, sp)-open. Since every $s(\Lambda, sp)$-closed set is $\beta(\Lambda, sp)$-closed. Thus, A is (Λ, sp)-open and $\beta(\Lambda, sp)$-closed.

(5) ⇒ (6): The proof is obvious.

(6) ⇒ (1): Let A be $\alpha(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed. Then, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$. Thus, $A = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence

$$A_{(\Lambda,sp)} = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = A.$$

Therefore, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = A$. This shows that A is $r(\Lambda, sp)$-open.

Corollary 1. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $r(\Lambda, sp)$-closed.

(2) A is (Λ, sp)-closed and $s(\Lambda, sp)$-open.

(3) A is $\alpha(\Lambda, sp)$-closed and $s(\Lambda, sp)$-open.

(4) A is $p(\Lambda, sp)$-closed and $s(\Lambda, sp)$-open.

(5) A is (Λ, sp)-closed and $\beta(\Lambda, sp)$-open.

(6) A is $\alpha(\Lambda, sp)$-closed and $\beta(\Lambda, sp)$-open.

Proposition 5. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $[[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

(2) $[[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

Definition 3. *A subset A of a topological space (X, τ) is called (Λ, sp)-clopen if A is both (Λ, sp)-open and (Λ, sp)-closed.*

Proposition 6. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is (Λ, sp)-clopen.

(2) A is $r(\Lambda, sp)$-open and $r(\Lambda, sp)$-closed.

(3) A is (Λ, sp)-open and $\alpha(\Lambda, sp)$-closed.

(4) A is (Λ, sp)-open and $p(\Lambda, sp)$-closed.

(5) A is $\alpha(\Lambda, sp)$-open and $p(\Lambda, sp)$-closed.

(6) A is $\alpha(\Lambda, sp)$-open and (Λ, sp)-closed.

(7) A is $p(\Lambda, sp)$-open and (Λ, sp)-closed.

(8) A is $\beta(\Lambda, sp)$-open and $\alpha(\Lambda, sp)$-closed.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (6): Let A be $\alpha(\Lambda, sp)$-open and $p(\Lambda, sp)$-closed. Then, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$. Thus, $A = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence

$$A^{(\Lambda,sp)} = [[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}.$$

By Proposition 5, $A^{(\Lambda,sp)} = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$, we have $A^{(\Lambda,sp)} \subseteq A$ and hence $A^{(\Lambda,sp)} = A$. This shows that A is (Λ, sp)-closed.

(6) ⇒ (7) ⇒ (8): Obvious.

(8) ⇒ (1): Let A be $\beta(\Lambda, sp)$-open and $\alpha(\Lambda, sp)$-closed. Then, $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$. Thus, $A^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$ and hence $A^{(\Lambda,sp)} \subseteq A$. Therefore, A is (Λ, sp)-closed. Since $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$, we have

$$[[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)},$$

by Proposition 5, $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}$ and hence $A \subseteq A_{(\Lambda,sp)}$. Thus, A is (Λ, sp)-open. Therefore, A is (Λ, sp)-clopen.

Definition 4. *A subset A of a topological space (X, τ) is said to be:*

(i) $\alpha(\Lambda, sp)$*-regular if* $A = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$*;*

(ii) $\beta(\Lambda, sp)$*-regular if* $A = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$*.*

Proposition 7. *Let A be a subset of a topological space (X, τ). Then, A is $r(\Lambda, sp)$-open if and only if A is $\alpha(\Lambda, sp)$-regular.*

Proof. Suppose that A is a $r(\Lambda, sp)$-open set. Then, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This implies that A is (Λ, sp)-open and so $A = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, A is $\alpha(\Lambda, sp)$-regular.

Conversely, suppose that A is an $\alpha(\Lambda, sp)$-regular set. Then, $A = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, $A = [[[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = A$ and hence $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, A is (Λ, sp)-open.

Proposition 8. *Let A be a subset of a topological space (X, τ). Then, A is $r(\Lambda, sp)$-closed if and only if A is $\beta(\Lambda, sp)$-regular.*

Proof. Suppose that A is a $r(\Lambda, sp)$-closed set. Then, we have $A = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and so A is (Λ, sp)-closed. Therefore, $A = [A_{(\Lambda,sp)}]^{(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. This shows that A is $\beta(\Lambda, sp)$-regular.

Conversely, suppose that A is a $\beta(\Lambda, sp)$-regular set. Then, $A = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, A is $r(\Lambda, sp)$-closed.

Proposition 9. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $\beta(\Lambda, sp)$-regular.

(2) A is $\beta(\Lambda, sp)$-open and (Λ, sp)-closed.

(3) A is $\beta(\Lambda, sp)$-open and $\alpha(\Lambda, sp)$-closed.

Proposition 10. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $\alpha(\Lambda, sp)$-regular.

(2) A is $\alpha(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed.

Definition 5. *A subset A of a topological space (X,τ) is said to be $b(\Lambda, sp)$-open if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. The complement of a $b(\Lambda, sp)$-open set is said to be $b(\Lambda, sp)$-closed.*

The family of all $b(\Lambda, sp)$-open (resp. $b(\Lambda, sp)$-closed) sets in a topological space (X,τ) is denoted by $b\Lambda_{sp}O(X,\tau)$ (resp. $b\Lambda_{sp}C(X,\tau)$).

Remark 1. *It is easy to see that for a topological space (X,τ),*

$$s\Lambda_{sp}O(X,\tau) \cup p\Lambda_{sp}O(X,\tau) \subseteq b\Lambda_{sp}O(X,\tau) \subseteq \beta\Lambda_{sp}O(X,\tau).$$

Proposition 11. *Let A be a subset of a topological space (X,τ). If $A = B \cup C$, where A is $s(\Lambda, sp)$-open and C is $p(\Lambda, sp)$-open, then A is $b(\Lambda, sp)$-open.*

The following result is an immediate consequence of Proposition 5 and Remark 1.

Corollary 2. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $r(\Lambda, sp)$-open.

(2) A is (Λ, sp)-open and $b(\Lambda, sp)$-closed.

(3) A is $\alpha(\Lambda, sp)$-open and $b(\Lambda, sp)$-closed.

Lemma 5. *Let A be a subset of a topological space (X,τ). If A is $s(\Lambda, sp)$-closed and $\beta(\Lambda, sp)$-open, then A is $s(\Lambda, sp)$-open.*

Proof. Since A is $s(\Lambda, sp)$-closed, it follows from Proposition 3 that $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A$. Since A is $\beta(\Lambda, sp)$-open, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}$. Therefore, $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence A is $s(\Lambda, sp)$-open.

Proposition 12. *Let A be a subset of a topological space (X,τ). If A is $b(\Lambda, sp)$-open, then $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed.*

Proof. Since A is $b(\Lambda, sp)$-open, we have $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence

$$\begin{aligned} A^{(\Lambda,sp)} &\subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &\subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)} \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}. \end{aligned}$$

Thus, $A^{(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. This shows that $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed.

Corollary 3. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is $s(\Lambda, sp)$-open, then $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed.

(2) If A is $p(\Lambda, sp)$-open, then $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed.

(3) If A is $\alpha(\Lambda, sp)$-open, then $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed.

Proposition 13. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_{sp}O(X, \tau)$.

(2) $A^{(\Lambda,sp)} \in r\Lambda_{sp}C(X, \tau)$.

(3) $A^{(\Lambda,sp)} \in \beta\Lambda_{sp}O(X, \tau)$.

(4) $A^{(\Lambda,sp)} \in s\Lambda_{sp}O(X, \tau)$.

(5) $A^{(\Lambda,sp)} \in b\Lambda_{sp}O(X, \tau)$.

Proof. (1) $\Rightarrow$ (2): Let $A \in \beta\Lambda_{sp}O(X, \tau)$. Then, we have $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $A^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}$. Thus, $A^{(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Therefore, $A^{(\Lambda,sp)} \in r\Lambda_{sp}C(X, \tau)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (1): Let $A^{(\Lambda,sp)} \in b\Lambda_{sp}O(X, \tau)$. Then,

$$\begin{aligned} A^{(\Lambda,sp)} &\subseteq [[A^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \end{aligned}$$

and hence $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $A \in \beta\Lambda_{sp}O(X, \tau)$.

Corollary 4. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_{sp}C(X, \tau)$.

(2) $A_{(\Lambda,sp)} \in r\Lambda_{sp}O(X, \tau)$.

(3) $A_{(\Lambda,sp)} \in \beta\Lambda_{sp}C(X, \tau)$.

(4) $A_{(\Lambda,sp)} \in s\Lambda_{sp}C(X, \tau)$.

(5) $A_{(\Lambda,sp)} \in b\Lambda_{sp}C(X, \tau)$.

Definition 6. *A subset A of a topological space (X, τ) is called $rs(\Lambda, sp)$-open if there exists a $r(\Lambda, sp)$-open set U such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$. The complement of a $rs(\Lambda, sp)$-open set is called $rs(\Lambda, sp)$-closed.*

The family of all $rs(\Lambda, sp)$-open (resp. $rs(\Lambda, sp)$-closed) sets in a topological space (X, τ) is denoted by $rs\Lambda_{sp}O(X, \tau)$ (resp. $rs\Lambda_{sp}C(X, \tau)$).

Remark 2. *It is clear that every $r(\Lambda, sp)$-open set is $rs(\Lambda, sp)$-open.*

Proposition 14. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $rs(\Lambda, sp)$-open.

(2) A is $s(\Lambda, sp)$-open and $s(\Lambda, sp)$-closed.

(3) A is $b(\Lambda, sp)$-open and $s(\Lambda, sp)$-closed.

(4) A is $\beta(\Lambda, sp)$-open and $s(\Lambda, sp)$-closed.

(5) A is $s(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed.

(6) A is $s(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed.

Proof. (1) ⇒ (2): Suppose that A is a $rs(\Lambda, sp)$-open set. There exists a $r(\Lambda, sp)$-open set U such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$. Then, $U \subseteq A_{(\Lambda,sp)}$ and hence $A \subseteq U^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, A is $s(\Lambda, sp)$-open. On the other hand, since $U^{(\Lambda,sp)} = A^{(\Lambda,sp)}$ and U is $r(\Lambda, sp)$-open, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} = [U^{(\Lambda,sp)}]_{(\Lambda,sp)} = U \subseteq A$. Thus, by Proposition 3, A is $s(\Lambda, sp)$-closed.

(2) ⇒ (3) and (3) ⇒ (4): The proofs are obvious.

(4) ⇒ (5): Follows from Lemma 5 and since $s\Lambda_{sp}O(X, \tau) \subseteq b\Lambda_{sp}O(X, \tau)$.

(5) ⇒ (6): The proof is obvious.

(6) ⇒ (1): Since A is $s(\Lambda, sp)$-open and $\beta(\Lambda, sp)$-closed, it follows from Lemma 5 that A is $s(\Lambda, sp)$-closed. Thus, by Proposition 3, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Let $U = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Then, U is $r(\Lambda, sp)$-open and hence $U \subseteq A \subseteq U^{(\Lambda,sp)}$. Thus, A is $rs(\Lambda, sp)$-open.

Remark 3. *It is clear from Proposition 14 that if A is a $rs(\Lambda, sp)$-open set of a topological space (X, τ), then $X - A$ is $rs(\Lambda, sp)$-open.*

Proposition 15. *Let (X, τ) be a topological space and $x \in X$. Then, $\{x\}$ is (Λ, sp)-open if and only if $\{x\}$ is $s(\Lambda, sp)$-open.*

Proof. The necessity is clear. Suppose that $\{x\}$ is $s(\Lambda, sp)$-open. Then, $\{x\} \subseteq [\{x\}_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Now $\{x\}_{(\Lambda,sp)}$ is either $\{x\}$ or $\emptyset$. Since $\emptyset^{(\Lambda,sp)} = \emptyset$ and $\{x\} \subseteq [\{x\}_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $\{x\}_{(\Lambda,sp)} \neq \emptyset$. Therefore, $\{x\}_{(\Lambda,sp)} = \{x\}$ and by Lemma 4, $\{x\}$ is (Λ, sp)-open.

Lemma 6. *Let (X, τ) be a topological space and $A \subseteq X$. If $U \in \Lambda_{sp}O(X, \tau)$ and $U \cap A = \emptyset$, then $U \cap A^{(\Lambda,sp)} = \emptyset$.*

Proposition 16. *Let (X, τ) be a topological space and $x \in X$. Then, the following properties are equivalent:*

(1) $\{x\}$ *is* $p(\Lambda, sp)$*-open.*

(2) $\{x\}$ *is* $b(\Lambda, sp)$*-open.*

(3) $\{x\}$ *is* $\beta(\Lambda, sp)$*-open.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) follows from Remark 1.

(3) ⇒ (1): Let $\{x\}$ be $\beta(\Lambda, sp)$-open. Assume that $\{x\}$ is not $p(\Lambda, sp)$-open. Then, $\{x\} \nsubseteq [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}$, that is $\{x\} \cap [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$. Since $[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ, sp)-open, it follows from Lemma 6 that $\{x\}^{(\Lambda,sp)} \cap [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$. Thus, $[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$ and hence $[[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = \emptyset^{(\Lambda,sp)} = \emptyset$. This is a contradiction.

Proposition 17. *Let* (X, τ) *be a topological space and* $x \in X$*. Then,* $\{x\}$ *is* $p(\Lambda, sp)$*-open or* $\{x\}$ *is* $\alpha(\Lambda, sp)$*-closed.*

Proof. Assume that $\{x\}$ is not $p(\Lambda, sp)$-open. Then, $\{x\} \nsubseteq [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $\{x\} \cap [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$. Since $[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ, sp)-open, it follows from Lemma 6 that $\{x\}^{(\Lambda,sp)} \cap [\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$. Therefore, $[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$. This implies that $[[\{x\}^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = \emptyset^{(\Lambda,sp)} = \emptyset$. Thus, by Proposition 3, $\{x\}$ is $\alpha(\Lambda, sp)$-closed.

Proposition 18. *Let* A *be a subset of a topological space* (X, τ)*. Then,* A *is* $s(\Lambda, sp)$*-open if and only if there exists a* (Λ, sp)*-open set* U *such that* $U \subseteq A \subseteq U^{(\Lambda,sp)}$*.*

Proof. Suppose that A is $s(\Lambda, sp)$-open. Then, $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Let $U = A_{(\Lambda,sp)}$. Thus, we obtain $U \subseteq A \subseteq U^{(\Lambda,sp)}$.

Conversely, assume that there exists a (Λ, sp)-open set U such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$. Then, $U \subseteq A_{(\Lambda,sp)}$ and hence $U^{(\Lambda,sp)} \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $A \subseteq U^{(\Lambda,sp)}$, $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, A is $s(\Lambda, sp)$-open.

Proposition 19. *Let* A *be a subset of a topological space* (X, τ)*. If there exists a* $p(\Lambda, sp)$*-open set* U *such that* $U \subseteq A \subseteq U^{(\Lambda,sp)}$*, then* A *is* $\beta(\Lambda, sp)$*-open.*

Proof. Since $U \subseteq A \subseteq U^{(\Lambda,sp)}$, we have $A^{(\Lambda,sp)} = U^{(\Lambda,sp)}$ and hence

$$[A^{(\Lambda,sp)}]_{(\Lambda,sp)} = [U^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Since U is $p(\Lambda, sp)$-open, we have $U \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, $A \subseteq U^{(\Lambda,sp)}$ and hence $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. This shows that A is $\beta(\Lambda, sp)$-open.

A subset D of a topological space (X, τ) is called Λ_{sp}*-dense* [3] if $D^{(\Lambda,sp)} = X$. D is called Λ_{sp}*-codense* [3] if $X - D$ is Λ_{sp}-dense.

Proposition 20. *Let* (X, τ) *be a topological space and* $D \subseteq X$*. Then, the following properties are equivalent:*

(1) D *is* Λ_{sp}*-dense.*

(2) If F is any (Λ, sp)-closed set and $D \subseteq F$, then $F = X$.

(3) Each nonempty (Λ, sp)-open set contains an element of D.

(4) The complement of D has empty (Λ, sp)-interior.

Proof. (1) $\Rightarrow$ (2): Let F be a (Λ, sp)-closed set such that $D \subseteq F$. Then, $X = D^{(\Lambda,sp)} \subseteq F^{(\Lambda,sp)} = F$.

(2) $\Rightarrow$ (3): Let U be a nonempty (Λ, sp)-open set such that $U \cap D = \emptyset$; then $D \subseteq X - U \neq X$, which contradicts (2), since $X - U$ is (Λ, sp)-closed.

(3) $\Rightarrow$ (4): Assume that $[X - D]_{(\Lambda,sp)} \neq \emptyset$; since $[X - D]_{(\Lambda,sp)}$ is (Λ, sp)-open, there is a nonempty (Λ, sp)-open set U such that $U \subseteq [X - D]_{(\Lambda,sp)}$, and since $[X - D]_{(\Lambda,sp)} \subseteq X - D$, U contains no point of D.

(3) $\Rightarrow$ (4): $[X - D]_{(\Lambda,sp)} = X - D^{(\Lambda,sp)} = \emptyset$ so that $D^{(\Lambda,sp)} = X$.

Remark 4. *Let A be a subset of a topological space (X, τ). If A is Λ_{sp}-dense, then A is $p(\Lambda, sp)$-open.*

Proposition 21. *Let A be a subset of a topological space (X, τ). If A is $p(\Lambda, sp)$-open, then A is the intersection of a $r(\Lambda, sp)$-open set and a Λ_{sp}-dense set.*

Proof. Suppose that A is a $p(\Lambda, sp)$-open set. Then, we have $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $A = [A \cup [X - A^{(\Lambda,sp)}]] \cap [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Let $C = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $D = A \cup [X - A^{(\Lambda,sp)}]$. Then, C is $r(\Lambda, sp)$-open, by Proposition 2, $A^{(\Lambda,sp)} \subseteq D^{(\Lambda,sp)}$ since $A \subseteq D$ and $X - A^{(\Lambda,sp)} \subseteq D \subseteq D^{(\Lambda,sp)}$. Thus, $D^{(\Lambda,sp)} = X$.

Corollary 5. *Let A be a subset of a topological space (X, τ). If A is $p(\Lambda, sp)$-closed, then A is the union of a $r(\Lambda, sp)$-closed set and a set has empty (Λ, sp)-interior.*

Proposition 22. *Let A be a subset of a topological space (X, τ). If A is $s(\Lambda, sp)$-open, then A is the intersection of a $r(\Lambda, sp)$-closed set F and a set C such that $C_{(\Lambda,sp)}$ is Λ_{sp}-dense.*

Proof. Suppose that A is $s(\Lambda, sp)$-open. Then, we have $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $A = [A \cup [X - [A_{(\Lambda,sp)}]^{(\Lambda,sp)}]] \cap [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Let $F = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and

$$C = A \cup [X - [A_{(\Lambda,sp)}]^{(\Lambda,sp)}].$$

Then, F is $r(\Lambda, sp)$-closed, by Proposition 2, we have $[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [C_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $X - [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq C$ and $X - [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is (Λ, sp)-open, $X - [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq C_{(\Lambda,sp)} \subseteq [C_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $[C_{(\Lambda,sp)}]^{(\Lambda,sp)} = X$.

Corollary 6. *Let A be a subset of a topological space (X, τ). If A is $s(\Lambda, sp)$-closed, then A is the union of a $r(\Lambda, sp)$-open set and a set whose (Λ, sp)-closure has empty (Λ, sp)-interior.*

Proposition 23. *Let A be a subset of a topological space (X, τ). If A is $\beta(\Lambda, sp)$-open, then A is the intersection of a $r(\Lambda, sp)$-closed set F and a Λ_{sp}-dense set D.*

Proof. Suppose that A is $\beta(\Lambda, sp)$-open. Then, we have $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $A = [A \cup [X - A^{(\Lambda,sp)}]] \cap [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Let $F = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and $D = A \cup [X - A^{(\Lambda,sp)}]$. Then, F is $r(\Lambda, sp)$-closed by Proposition 2, also $A^{(\Lambda,sp)} \subseteq D^{(\Lambda,sp)}$. Since $X - A^{(\Lambda,sp)} \subseteq D \subseteq D^{(\Lambda,sp)}$, we have $D^{(\Lambda,sp)} = X$.

Corollary 7. *Let A be a subset of a topological space (X, τ). If A is $\beta(\Lambda, sp)$-closed, then A is the union of a $r(\Lambda, sp)$-open set and a set has empty (Λ, sp)-interior.*

Lemma 7. *Let A be a subset of a topological space (X, τ). If A is (Λ, sp)-closed and $p(\Lambda, sp)$-open, then A is (Λ, sp)-open.*

Theorem 1. *For a topological space (X, τ), the following properties are equivalent:*

(1) Every $s(\Lambda, sp)$-open set of X is $\alpha(\Lambda, sp)$-open.

(2) Every $s(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-open.

(3) Every $\beta(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-open.

(4) Every $b(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-open.

(5) Every $rs(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-open.

(6) Every $rs(\Lambda, sp)$-open set of X is $r(\Lambda, sp)$-open.

(7) Every $r(\Lambda, sp)$-closed set of X is $p(\Lambda, sp)$-open.

(8) Every $r(\Lambda, sp)$-closed set of X is (Λ, sp)-open.

Proof. (1) $\Rightarrow$ (2): This is obvious since $\alpha\Lambda_{sp}O(X, \tau) \subseteq s\Lambda_{sp}O(X, \tau)$.

(2) $\Rightarrow$ (3): Let A be a $\beta(\Lambda, sp)$-open set. Then, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. It follows from Proposition 2 that $B = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed and thus $s(\Lambda, sp)$-open. By (2), B is $p(\Lambda, sp)$-open and hence $A \subseteq B \subseteq [B^{(\Lambda,sp)}]_{(\Lambda,sp)} = B_{(\Lambda,sp)}$. Also it is clear that $B \subseteq A^{(\Lambda,sp)}$ and thus $B_{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that A is $p(\Lambda, sp)$-open.

(3) $\Rightarrow$ (4): This is obvious since $b\Lambda_{sp}O(X, \tau) \subseteq \beta\Lambda_{sp}O(X, \tau)$.

(4) $\Rightarrow$ (5): It follows from Proposition 14 that $rs\Lambda_{sp}O(X, \tau) \subseteq s\Lambda_{sp}O(X, \tau)$. Since $s\Lambda_{sp}O(X, \tau) \subseteq b\Lambda_{sp}O(X, \tau)$, $rs\Lambda_{sp}O(X, \tau) \subseteq b\Lambda_{sp}O(X, \tau)$. Thus, the result follows from (3).

(5) $\Rightarrow$ (6): Since every $rs(\Lambda, sp)$-open set is $s(\Lambda, sp)$-closed, it follows from (4) that a $rs(\Lambda, sp)$-open set is both $s(\Lambda, sp)$-closed and $p(\Lambda, sp)$-open. Thus, by Proposition 4, follows.

(6) $\Rightarrow$ (7): Follows from Proposition 4 and Remark 3.

(7) $\Rightarrow$ (8): Follows from Lemma 7.

(8) $\Rightarrow$ (1): Let A be a $s(\Lambda, sp)$-open set. Then, by Corollary 3, $A^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed. By (8), $A^{(\Lambda,sp)}$ is (Λ, sp)-open and hence $A^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, A is $p(\Lambda, sp)$-open. Since $A \in s\Lambda_{sp}O(X,\tau) \cap p\Lambda_{sp}O(X,\tau) = \alpha\Lambda_{sp}O(X,\tau)$, (1) follows.

Corollary 8. *For a topological space (X,τ), the following properties are equivalent:*

(1) $\alpha\Lambda_{sp}O(X,\tau) = s\Lambda_{sp}O(X,\tau)$.

(2) Every $rs(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-closed.

(3) Every $rs(\Lambda, sp)$-open set of X is $r(\Lambda, sp)$-closed.

Proof. Follows from Remark 3 and Theorem 1.

Definition 7. *A subset A of a topological space (X,τ) is called $p(\Lambda, sp)$-clopen if A is both $p(\Lambda, sp)$-open and $p(\Lambda, sp)$-closed.*

Corollary 9. *For a topological space (X,τ), the following properties are equivalent:*

(1) $\alpha\Lambda_{sp}O(X,\tau) = s\Lambda_{sp}O(X,\tau)$.

(2) Every $rs(\Lambda, sp)$-open set of X is $p(\Lambda, sp)$-clopen.

(3) Every $rs(\Lambda, sp)$-open set of X is (Λ, sp)-clopen.

Proof. Follows from Theorem 1 and Corollary 8.

Proposition 24. *For a topological space (X,τ), the following properties are equivalent:*

(1) Every $p(\Lambda, sp)$-open set of X is $\alpha(\Lambda, sp)$-open.

(2) Every $p(\Lambda, sp)$-open set of X is $s(\Lambda, sp)$-open.

Proof. Follows from Proposition 1.

4. Some characterizations of Λ_{sp}-extremally disconnected spaces

In this section, we investigate some characterizations of Λ_{sp}-extremally disconnected spaces.

Definition 8. [3] *A topological space (X,τ) is called Λ_{sp}-extremally disconnected if $U^{(\Lambda,sp)}$ is (Λ, sp)-open in X for every (Λ, sp)-open set U of X.*

Theorem 2. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-extremally disconnected.

(2) For each $V \in \beta\Lambda_{sp}O(X,\tau)$, $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$.

(3) For each $V \in b\Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

(4) For each $V \in s\Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

(5) For each $V \in \alpha\Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

(6) For each $V \in \Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

(7) For each $V \in r\Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

(8) For each $V \in p\Lambda_{sp}O(X,\tau)$*,* $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$*.*

Proof. (1) $\Rightarrow$ (2): Let $V \in \beta\Lambda_{sp}O(X,\tau)$. By (1) and Proposition 13, we have $V^{(\Lambda,sp)} = [[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [[[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $V^{(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[V^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7): Obvious.

(7) $\Rightarrow$ (8): Let $V \in p\Lambda_{sp}O(X,\tau)$. Then, we have $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda,sp)$-open, by (7), $[[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $r(\Lambda,sp)$-open. Therefore,

$$\begin{aligned} V^{(\Lambda,sp)} &= [[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[[[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &= [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &= [[V^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}. \end{aligned}$$

Thus, $V^{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$.

(8) $\Rightarrow$ (1): The proof is obvious.

Theorem 3. *For a topological space* (X,τ)*, the following properties are equivalent:*

(1) (X,τ) *is* Λ_{sp}*-extremally disconnected.*

(2) $r\Lambda_{sp}C(X,\tau) \subseteq \Lambda_{sp}O(X,\tau)$*.*

(3) $r\Lambda_{sp}C(X,\tau) \subseteq \alpha\Lambda_{sp}O(X,\tau)$*.*

(4) $r\Lambda_{sp}C(X,\tau) \subseteq p\Lambda_{sp}O(X,\tau)$*.*

(5) $s\Lambda_{sp}O(X,\tau) \subseteq \alpha\Lambda_{sp}O(X,\tau)$*.*

(6) $s\Lambda_{sp}C(X,\tau) \subseteq \alpha\Lambda_{sp}C(X,\tau)$*.*

(7) $s\Lambda_{sp}C(X,\tau) \subseteq p\Lambda_{sp}C(X,\tau)$*.*

(8) $s\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}O(X,\tau)$*.*

(9) $\beta\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}O(X,\tau)$*.*

(10) $\beta\Lambda_{sp}C(X,\tau) \subseteq p\Lambda_{sp}C(X,\tau)$.

(11) $b\Lambda_{sp}C(X,\tau) \subseteq p\Lambda_{sp}C(X,\tau)$.

(12) $b\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}O(X,\tau)$.

(13) $r\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}C(X,\tau)$.

(14) $r\Lambda_{sp}O(X,\tau) \subseteq \Lambda_{sp}C(X,\tau)$.

(15) $r\Lambda_{sp}O(X,\tau) \subseteq \alpha\Lambda_{sp}C(X,\tau)$.

Proof. (1) $\Rightarrow$ (2): Let $V \in r\Lambda_{sp}C(X,\tau)$. Then, we have $V = [V_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since (X,τ) is Λ_{sp}-extremally disconnected, $V_{(\Lambda,sp)} = [[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [V_{(\Lambda,sp)}]^{(\Lambda,sp)} = V$ and hence $V \in \Lambda_{sp}O(X,\tau)$. Consequently, we obtain $r\Lambda_{sp}C(X,\tau) \subseteq \Lambda_{sp}O(X,\tau)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (5): Let $V \in s\Lambda_{sp}O(X,\tau)$. Then, we have $V \subseteq [V_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $[V_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $r(\Lambda,sp)$-closed, by (4), $[V_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $p(\Lambda,sp)$-open and hence

$$[V_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [[[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

This implies that $V \subseteq [[V_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $V \in \alpha\Lambda_{sp}O(X,\tau)$. Thus,

$$s\Lambda_{sp}O(X,\tau) \subseteq \alpha\Lambda_{sp}O(X,\tau).$$

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8): Obvious.

(8) $\Rightarrow$ (9): Let $V \in \beta\Lambda_{sp}O(X,\tau)$. By Proposition 13, $V^{(\Lambda,sp)}$ is $s(\Lambda,sp)$-open, by (8), $V^{(\Lambda,sp)}$ is $p(\Lambda,sp)$-open. Thus, $V^{(\Lambda,sp)} \subseteq [[V^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore, $V \in p\Lambda_{sp}O(X,\tau)$. This shows that $\beta\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}O(X,\tau)$.

(9) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12): Obvious.

(12) $\Rightarrow$ (13): Let $V \in r\Lambda_{sp}O(X,\tau)$. Then, V is $\beta(\Lambda,sp)$-open, by Proposition 13, $V^{(\Lambda,sp)}$ is $b(\Lambda,sp)$-open and by (12), $V^{(\Lambda,sp)}$ is $p(\Lambda,sp)$-open. Thus,

$$V^{(\Lambda,sp)} \subseteq [[V^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = V.$$

Therefore, V is $p(\Lambda,sp)$-closed and hence $r\Lambda_{sp}O(X,\tau) \subseteq p\Lambda_{sp}C(X,\tau)$.

(13) $\Rightarrow$ (14): Let $V \in r\Lambda_{sp}O(X,\tau)$. By (13), we have V is $p(\Lambda,sp)$-closed and hence $[V_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq V$. Since V is (Λ,sp)-open, $V^{(\Lambda,sp)} \subseteq V$. Thus, $V \in \Lambda_{sp}C(X,\tau)$. Consequently, we obtain $r\Lambda_{sp}O(X,\tau) \subseteq \Lambda_{sp}C(X,\tau)$.

(14) $\Rightarrow$ (15): The proof is obvious.

(15) $\Rightarrow$ (1): Let V be a (Λ,sp)-open set. Then, $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda,sp)$-open, by (15), $[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $\alpha(\Lambda,sp)$-closed. Therefore,

$$V^{(\Lambda,sp)} \subseteq [[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [[[[V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Thus, $V^{(\Lambda,sp)}$ is (Λ,sp)-open. This shows that (X,τ) is Λ_{sp}-extremally disconnected.

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