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On almost $\alpha(\Lambda, sp)$-continuous multifunctions

Chawalit Boonpok[1], Jeeranunt Khampakdee[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

Abstract. Our main purpose is to introduce the notion of almost $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, some characterizations of almost $\alpha(\Lambda, sp)$-continuous multifunctions are established.

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Key Words and Phrases: $\alpha(\Lambda, sp)$-open set, almost $\alpha(\Lambda, sp)$-continuous multifunction

1. Introduction

The notion of continuity is an important concept in topological spaces. Many mathematicians studied the various types of generalizations of continuity. In 1988, Noiri [6] introduced and studied the notion of almost α-continuity in topological spaces as a generalization of α-continuity due to Mashhour et al. [5]. In 1998, Popa and Noiri [8] extended the concept of almost α-continuous functions to multifunctions and defined almost α-continuous multifunctions and obtained several characterizations of almost α-continuous multifunctions. Abd El-Monsef et al. [4] introduced a weak form of open sets called β-open sets. This notion was also called semi-preopen sets in the sense of Andrijević [1]. In 2004, Noiri and Hatir [7] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. In particular, some characterizations of upper and lower (Λ, sp)-continuous multifunctions are investigated in [3]. The purpose of the present paper is to introduce the notion of almost $\alpha(\Lambda, sp)$-continuous multifunctions. Furthermore, several characterizations of almost $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

*Corresponding author.
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Email addresses: chawalit.b@msu.ac.th (C. Boonpok), jeeranunt.k@msu.ac.th (J. Khampakdee)

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2. Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [4] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [7] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [7] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*.

Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 1. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ *and* $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)} = \cap\{F | A \subseteq F \text{ and } F \text{ is } (\Lambda, sp)\text{-closed}\}$.

(4) $A^{(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

(5) A *is* (Λ, sp)*-closed if and only if* $A = A^{(\Lambda,sp)}$.

Lemma 2. [3] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-interior, the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A *is* (Λ, sp)*-open if and only if* $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

A subset A of a topological space (X,τ) is said to be $s(\Lambda,sp)$-*open* (resp. $p(\Lambda,sp)$-*open*, $r(\Lambda,sp)$-*open*, $\alpha(\Lambda,sp)$-*open*, $\beta(\Lambda,sp)$-*open*) if $A\subseteq[A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A\subseteq[A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A=[A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A\subseteq[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A\subseteq[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$) [3]. The complement of a $s(\Lambda,sp)$-open (resp. $p(\Lambda,sp)$-open, $r(\Lambda,sp)$-open, $\alpha(\Lambda,sp)$-open, $\beta(\Lambda,sp)$-open) set is said to be $s(\Lambda,sp)$-*closed* (resp. $p(\Lambda,sp)$-*closed*, $r(\Lambda,sp)$-*closed*, $\alpha(\Lambda,sp)$-*closed*, $\beta(\Lambda,sp)$-*closed*). The family of all $s(\Lambda,sp)$-open (resp. $p(\Lambda,sp)$-open, $r(\Lambda,sp)$-open, $\alpha(\Lambda,sp)$-open, $\beta(\Lambda,sp)$-open) sets in a topological space (X,τ) is denoted by $s\Lambda_{sp}O(X,\tau)$ (resp. $p\Lambda_{sp}O(X,\tau)$, $r\Lambda_{sp}O(X,\tau)$, $\alpha\Lambda_{sp}O(X,\tau)$, $\beta\Lambda_{sp}O(X,\tau)$). The intersection of all $\alpha(\Lambda,sp)$-closed (resp. $s(\Lambda,sp)$-closed) sets containing A is called the $\alpha(\Lambda,sp)$-*closure* (resp. $s(\Lambda,sp)$-*closure*) of A and is denoted by $A^{\alpha(\Lambda,sp)}$ (resp. $A^{s(\Lambda,sp)}$). The union of all $\alpha(\Lambda,sp)$-open (resp. $s(\Lambda,sp)$-open) sets contained in A is called the $\alpha(\Lambda,sp)$-*interior* (resp. $s(\Lambda,sp)$-*interior*) of A and is denoted by $A_{\alpha(\Lambda,sp)}$ (resp. $A_{s(\Lambda,sp)}$).

Lemma 3. *Let A be a subset of a topological space (X,τ). Then, $x\in A^{s(\Lambda,sp)}$ if and only if $U\cap A\neq\emptyset$ for every $U\in s\Lambda_{sp}O(X,\tau)$ containing x.*

Lemma 4. *Let A be a subset of a topological space (X,τ). Then,*

$$A^{\alpha(\Lambda,sp)}=A\cup[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}.$$

By a multifunction $F:X\to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x)\neq\emptyset$ for all $x\in X$. For a multifunction $F:X\to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B)=\{x\in X\mid F(x)\subseteq B\}$ and $F^-(B)=\{x\in X\mid F(x)\cap B\neq\emptyset\}$. In particular, $F^-(y)=\{x\in X\mid y\in F(x)\}$ for each point $y\in Y$. For each $A\subseteq X$, $F(A)=\cup_{x\in A}F(x)$. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any (Λ,sp)-open set V of a topological space (Y,σ), we denote $V^+=\{B\in\mathcal{P}(Y)\mid B\subseteq V\}$ and $V^-=\{B\in\mathcal{P}(Y)\mid B\cap V\neq\emptyset\}$.

3. Almost $\alpha(\Lambda,sp)$-continuous multifunctions

In this section, we introduce the notion of almost $\alpha(\Lambda,sp)$-continuous multifunctions. Moreover, some characterizations of almost $\alpha(\Lambda,sp)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F:(X,\tau)\to(Y,\sigma)$ is said to be almost $\alpha(\Lambda,sp)$-continuous at $x\in X$ if, for any (Λ,sp)-open sets G_1,G_2 of Y such that $F(x)\in G_1^+\cap G_2^+$ and each $s(\Lambda,sp)$-open set U of X containing x, there exists a nonempty (Λ,sp)-open set G_U of X such that $G_U\subseteq U$, $F(G_U)\subseteq G_1^{s(\Lambda,sp)}$ and $F(z)\cap G_2^{s(\Lambda,sp)}\neq\emptyset$ for every $z\in G_U$. A multifunction $F:(X,\tau)\to(Y,\sigma)$ is said to be almost $\alpha(\Lambda,sp)$-continuous if F has this property at each point of X.*

Theorem 1. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F *is almost* $\alpha(\Lambda, sp)$*-continuous at a point* $x \in X$*;*

(2) for any (Λ, sp)*-open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$*, there exists an* $\alpha(\Lambda, sp)$*-open set* U *containing* x *such that* $F(U) \subseteq G_1^{s(\Lambda,sp)}$ *and* $F(z) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ *for every* $z \in U$*;*

(3) $x \in [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for any* (Λ, sp)*-open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$*;*

(4) $x \in [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ *for any* (Λ, sp)*-open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$*.*

Proof. (1) ⇒ (2): Let G_1, G_2 be any (Λ, sp)-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. For each $s(\Lambda, sp)$-open set H containing x, there exists a nonempty (Λ, sp)-open set G_H such that $G_H \subseteq H$, $F(G_H) \subseteq G_1^{s(\Lambda,sp)}$ and $F(z) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in G_H$. Let

$$W = \cup\{G_H \mid H \in s\Lambda_{sp}O(X, \tau) \text{ containing } x\}.$$

Then, W is (Λ, sp)-open in X, $x \in W^{s(\Lambda,sp)}$, $F(W) \subseteq G_1^{s(\Lambda,sp)}$ and $F(w) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq W^{s(\Lambda,sp)} = [W^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, U is an $\alpha(\Lambda, sp)$-open set containing x such that $F(U) \subseteq G_1^{s(\Lambda,sp)}$ and $F(u) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for every $u \in U$.

(2) ⇒ (3): Let G_1, G_2 be any (Λ, sp)-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq G_1^{s(\Lambda,sp)}$ and $F(z) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})$. Since $U \in \alpha\Lambda_{sp}O(X, \tau)$, we have $x \in U \subseteq [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$.

(3) ⇒ (4): Let G_1, G_2 be any (Λ, sp)-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Now, put $U = [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$. Then, U is an $\alpha(\Lambda, sp)$-open set and $x \in U \subseteq F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})$. Thus,

$$x \in U \subseteq [[U_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

(4) ⇒ (1): Let $U \in s\Lambda_{sp}O(X, \tau)$ containing x and let G_1, G_2 be any (Λ, sp)-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then, $x \in [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$, by Lemma 3,

$$\emptyset \neq U \cap [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}.$$

Put $G_U = [U \cap [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]_{(\Lambda,sp)}$, then G_U is a nonempty (Λ, sp)-open set of X such that $G_U \subseteq U$, $F(G_U) \subseteq G_1^{s(\Lambda,sp)}$ and $F(z) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for each $z \in G_U$. This shows that F is almost $\alpha(\Lambda, sp)$-continuous at x.

Theorem 2. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is almost $\alpha(\Lambda, sp)$-continuous at a point $x \in X$;

(2) for each $x \in X$ and any (Λ, sp)-open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists an $\alpha(\Lambda, sp)$-open set U containing x such that $F(U) \subseteq G_1^{s(\Lambda,sp)}$ and $F(z) \cap G_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$;

(3) for each $x \in X$ and any $r(\Lambda, sp)$-open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in \alpha\Lambda_{sp}O(X, \tau)$ containing x such that $F(U) \subseteq G_1$ and $F(z) \cap G_2 \neq \emptyset$ for every $z \in U$;

(4) $F^+(G_1) \cap F^-(G_2) \in \alpha\Lambda_{sp}O(X, \tau)$ for every $G_1, G_2 \in r\Lambda_{sp}O(Y, \sigma)$;

(5) $F^+(K_1) \cup F^-(K_2)$ is $\alpha(\Lambda, sp)$-closed in X for every $r(\Lambda, sp)$-closed sets K_1, K_2 of Y;

(6) $F^+(G_1) \cup F^-(G_2) \subseteq [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ for any (Λ, sp)-open sets G_1, G_2 of Y;

(7) $[F^-([K_1]_{s(\Lambda,sp)}) \cup F^+([K_2]_{s(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ for any (Λ, sp)-closed sets K_1, K_2 of Y;

(8) $[F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ for any (Λ, sp)-closed sets K_1, K_2 of Y;

(9) $[F^-([[B_1^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[B_2^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$ for any subsets B_1, B_2 of Y;

(10) $[[[F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ for any (Λ, sp)-closed sets K_1, K_2 of Y;

(11) $[[[F^-([K_1]_{s(\Lambda,sp)}) \cup F^+([K_2]_{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ for any (Λ, sp)-closed sets K_1, K_2 of Y;

(12) $F^+(G_1) \cap F^-(G_2) \subseteq [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for any (Λ, sp)-open sets G_1, G_2 of Y.

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem 1.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let $G_1, G_2 \in r\Lambda_{sp}O(Y, \sigma)$ and let $x \in F^+(G_1) \cap F^-(G_2)$. Then,

$$F(x) \in G_1^+ \cap G_2^-$$

and there exists $U \in \alpha\Lambda_{sp}O(X, \tau)$ containing x such that $F(U) \subseteq G_1$ and $F(z) \cap G_2 \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(G_1) \cap F^-(G_2)$ and hence

$$F^+(G_1) \cap F^-(G_2) \in \alpha\Lambda_{sp}O(X, \tau).$$

(4) ⇒ (5): This follows from the fact that $F^+(Y-B) = X - F^-(B)$ and $F^-(Y-B) = X - F^+(B)$ for every subset B of Y.

(5) ⇒ (6): Let G_1, G_2 be any (Λ, sp)-open sets of Y and let $x \in F^+(G_1) \cap F^-(G_2)$. Then, $F(x) \subseteq G_1 \subseteq G_1^{s(\Lambda,sp)}$ and $\emptyset \neq F(x) \cap G_2 \subseteq F(x) \cap G_2^{s(\Lambda,sp)}$. Thus,

$$x \in F^+(G_1^{s(\Lambda,sp)}) = X - F^-(Y - G_1^{s(\Lambda,sp)})$$

and $x \in F^-(G_2^{s(\Lambda,sp)}) = X - F^-(Y - G_2^{s(\Lambda,sp)})$. Since $Y - G_1^{s(\Lambda,sp)}$ and $Y - G_2^{s(\Lambda,sp)}$ are $r(\Lambda, sp)$-closed, $F^-(Y - G_1^{s(\Lambda,sp)}) \cup F^+(Y - G_2^{s(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-closed in X. Since

$$\begin{aligned} F^-(Y - G_1^{s(\Lambda,sp)}) \cup F^+(Y - G_2^{s(\Lambda,sp)}) &= [X - F^+(G_1^{s(\Lambda,sp)})] \cup [X - F^-(G_2^{s(\Lambda,sp)})] \\ &= X - [F^+(G_1^{s(\Lambda,sp)}) \cup F^-(G_2^{s(\Lambda,sp)})], \end{aligned}$$

we have $F^+(G_1^{s(\Lambda,sp)}) \cup F^-(G_2^{s(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-open in X and hence

$$x \in [F^+(G_1^{s(\Lambda,sp)}) \cup F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}.$$

Thus, $F^+(G_1) \cup F^-(G_2) \subseteq [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$.

(6) ⇒ (7): Let K_1, K_2 be any (Λ, sp)-closed sets of Y. Then, $Y - K_1$ and $Y - K_2$ are (Λ, sp)-open, by (6),

$$\begin{aligned} X - [F^-(K_1) \cup F^+(K_2)] &= [X - F^-(K_1)] \cap [X - F^+(K_2)] \\ &= F^+(Y - K_1) \cap F^-(Y - K_2) \\ &\subseteq [F^+([Y - K_1]^{s(\Lambda,sp)}) \cap F^-([Y - K_2]^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)} \\ &= [F^+(Y - [K_1]_{s(\Lambda,sp)}) \cap F^-(Y - [K_2]_{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)} \\ &= [[X - F^-([K_1]_{s(\Lambda,sp)})] \cap [X - F^+([K_2]_{s(\Lambda,sp)})]]_{\alpha(\Lambda,sp)} \\ &= X - [F^-([K_1]_{s(\Lambda,sp)}) \cup F^+([K_2]_{s(\Lambda,sp)})]^{\alpha(\Lambda,sp)}. \end{aligned}$$

Thus, $[F^-([K_1]_{s(\Lambda,sp)}) \cup F^+([K_2]_{s(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$.

(7) ⇒ (8): The proof is obvious since $K_{s(\Lambda,sp)} = [K_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-closed set K.

(8) ⇒ (9): The proof is obvious.

(9) ⇒ (10): Let K_1, K_2 be any (Λ, sp)-closed sets of Y. Thus, by (9) and Lemma 4,

$$\begin{aligned} &[[[F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &\subseteq [F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &= [F^-([[K_1^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &\subseteq F^-(K_1) \cup F^+(K_2). \end{aligned}$$

(10) ⇒ (11): The proof is obvious since $K_{s(\Lambda,sp)} = [K_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-closed set K.

(11) ⇒ (12): Let G_1, G_2 be any (Λ, sp)-open sets of Y. Then, $Y - G_1$ and $Y - G_2$ are (Λ, sp)-closed sets of Y, by (11),

$$
\begin{aligned}
&[[[F^-([Y - G_1]_{s(\Lambda,sp)}) \cup F^+([Y - G_2]_{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&\subseteq F^-(Y - G_1) \cup F^+(Y - G_2) \\
&= [X - F^+(G_1)] \cup [X - F^-(G_2)] \\
&= X - [F^+(G_1) \cap F^-(G_2)].
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&[[[F^-([Y - G_1]_{s(\Lambda,sp)}) \cup F^+([Y - G_2]_{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&= [[[F^-(Y - G_1^{s(\Lambda,sp)}) \cup F^+(Y - G_2^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&= [[[[X - [F^+(G_1^{s(\Lambda,sp)})]] \cup [X - [F^-(G_2^{s(\Lambda,sp)})]]]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&= [[[X - [F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]]^{s(\Lambda,sp)}]_{s(\Lambda,sp)}]^{s(\Lambda,sp)} \\
&= X - [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.
\end{aligned}
$$

Thus, $F^+(G_1) \cap F^-(G_2) \subseteq [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(12) ⇒ (1): Let $x \in X$ and let G_1, G_2 be any (Λ, sp)-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then,

$$x \in F^+(G_1) \cap F^-(G_2) \subseteq [[[F^+(G_1^{s(\Lambda,sp)}) \cap F^-(G_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$$

and hence F is almost $\alpha(\Lambda, sp)$-continuous at x by Theorem 1. This shows that F is almost $\alpha(\Lambda, sp)$-continuous.

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost $\alpha(\Lambda, sp)$-continuous if $f^{-1}(V) \in \alpha\Lambda_{sp}O(X, \tau)$ for every $V \in r\Lambda_{sp}O(Y, \sigma)$.*

Corollary 1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is almost $\alpha(\Lambda, sp)$-continuous;

(2) for each $x \in X$ and any (Λ, sp)-open set G of Y containing $f(x)$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $f(U) \subseteq G^{s(\Lambda,sp)}$;

(3) for each $x \in X$ and any $r(\Lambda, sp)$-open set G of Y containing $f(x)$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $f(U) \subseteq G$;

(4) $f^{-1}(G) \in \alpha\Lambda_{sp}O(X, \tau)$ for every $G \in r\Lambda_{sp}O(Y, \sigma)$;

(5) $f^{-1}(K) \in \alpha\Lambda_{sp}C(X, \tau)$ for every $K \in r\Lambda_{sp}C(Y, \sigma)$;

(6) $f^{-1}(G) \subseteq [f^{-1}(G^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ for any (Λ, sp)-open set G of Y;

(7) $[f^{-1}(K_{s(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(K)$ *for any* (Λ, sp)*-closed set* K *of* Y*;*

(8) $[f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(K)$ *for any* (Λ, sp)*-closed set* K *of* Y*;*

(9) $[f^{-1}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ *for any subset* B *of* Y*;*

(10) $[[[f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq f^{-1}(K)$ *for any* (Λ, sp)*-closed set* K *of* Y*;*

(11) $[[[f^{-1}(K_{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq f^{-}(K)$ *for any* (Λ, sp)*-closed set* K *of* Y*;*

(12) $f^{-1}(G) \subseteq [[[f^{-1}(G^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ *for any* (Λ, sp)*-open set* G *of* Y*.*

Theorem 3. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is almost* $\alpha(\Lambda, sp)$*-continuous;*

(2) $[F^{-}(G_1) \cup F^{+}(G_2)]^{\alpha(\Lambda,sp)} \subseteq F^{-}(G_1^{(\Lambda,sp)}) \cup F^{+}(G_2^{(\Lambda,sp)})$ *for any* $G_1, G_2 \in \beta\Lambda_{sp}O(Y, \sigma)$*;*

(3) $[F^{-}(G_1) \cup F^{+}(G_2)]^{\alpha(\Lambda,sp)} \subseteq F^{-}(G_1^{(\Lambda,sp)}) \cup F^{+}(G_2^{(\Lambda,sp)})$ *for any* $G_1, G_2 \in s\Lambda_{sp}O(Y, \sigma)$*;*

(4) $F^{+}(G_1) \cap F^{-}(G_2) \subseteq [F^{+}(G_1^{s(\Lambda,sp)}) \cap F^{-}(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for any* $G_1, G_2 \in p\Lambda_{sp}O(Y, \sigma)$*.*

Proof. (1) ⇒ (2): Let G_1, G_2 be any $\beta(\Lambda, sp)$-open sets of Y. Since $G_1^{(\Lambda,sp)}$ and $G_2^{(\Lambda,sp)}$ are $r(\Lambda, sp)$-closed, by Theorem 2, $F^{-}(G_1^{(\Lambda,sp)}) \cup F^{+}(G_2^{(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-closed in X and $F^{-}(G_1) \cup F^{+}(G_2) \subseteq F^{-}(G_1^{(\Lambda,sp)}) \cup F^{+}(G_2^{(\Lambda,sp)})$. Thus,

$$[F^{-}(G_1) \cup F^{+}(G_2)]^{\alpha(\Lambda,sp)} \subseteq F^{-}(G_1^{(\Lambda,sp)}) \cup F^{+}(G_2^{(\Lambda,sp)}).$$

(2) ⇒ (3): This is obvious since $s\Lambda_{sp}O(Y, \sigma) \subseteq \beta\Lambda_{sp}O(Y, \sigma)$.

(3) ⇒ (1): Let $K_1, K_2 \in r\Lambda_{sp}C(Y, \sigma)$. Then, $K_1, K_2 \in s\Lambda_{sp}O(Y, \sigma)$ and hence $[F^{-}(K_1) \cup F^{+}(K_2)]^{\alpha(\Lambda,sp)} \subseteq F^{-}(K_1) \cup F^{+}(K_2)$. Thus, we have $F^{-}(K_1) \cup F^{+}(K_2)$ is $\alpha(\Lambda, sp)$-closed in X and hence F is almost $\alpha(\Lambda, sp)$-continuous by Theorem 2.

(1) ⇒ (4): Let G_1, G_2 be any $p(\Lambda, sp)$-open sets of Y. Since $[G_1^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[G_2^{(\Lambda,sp)}]_{(\Lambda,sp)}$ are $r(\Lambda, sp)$-open in Y, we have $[G_1^{(\Lambda,sp)}]_{(\Lambda,sp)} = G_1^{s(\Lambda,sp)}$ and

$$[G_1^{(\Lambda,sp)}]_{(\Lambda,sp)} = G_2^{s(\Lambda,sp)},$$

by Theorem 2, $F^{+}(G_1^{s(\Lambda,sp)}) \cap F^{-}(G_2^{s(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-open in X. Thus,

$$\begin{aligned} F^{+}(G_1) \cap F^{-}(G_2) &\subseteq F^{+}(G_1^{s(\Lambda,sp)}) \cap F^{-}(G_2^{s(\Lambda,sp)}) \\ &= [F^{+}(G_1^{s(\Lambda,sp)}) \cap F^{-}(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}. \end{aligned}$$

(4) ⇒ (1): Let G_1, G_2 be any $r(\Lambda, sp)$-open sets of Y. Since $G_1, G_2 \in p\Lambda_{sp}O(Y, \sigma)$, we have $F^{+}(G_1) \cap F^{-}(G_2) \subseteq [F^{+}(G_1^{s(\Lambda,sp)}) \cap F^{-}(G_2^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)} = [F^{+}(G_1) \cap F^{-}(G_2)]_{\alpha(\Lambda,sp)}$ and hence $F^{+}(G_1) \cap F^{-}(G_2) \in \alpha\Lambda_{sp}O(X, \tau)$. It follows from Theorem 2 that F is almost $\alpha(\Lambda, sp)$-continuous.

Corollary 2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f *is almost* $\alpha(\Lambda, sp)$*-continuous;*

(2) $[f^{-1}(V)]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for any* $V \in \beta\Lambda_{sp}O(Y, \sigma)$*;*

(3) $[f^{-1}(V)]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ *for any* $V \in s\Lambda_{sp}O(Y, \sigma)$*;*

(4) $f^{-1}(V) \subseteq [f^{-1}(V^{s(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for any* $V \in p\Lambda_{sp}O(Y, \sigma)$*.*

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