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Some Properties of Weak Separation Axioms in Coc-Compact Sets

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Abstract. In this paper, we introduce some separation axioms in coc-compact set, namely coc-T_0-space, coc-$T_{\frac{1}{4}}$- space, coc-$T_{\frac{3}{8}}$- space, coc-$T_{\frac{1}{2}}$- space, coc-$T_{\frac{5}{8}}$- space, coc-D_i- space, coc-R_i- space for $i = 0, 1$, weak coc-D_1- space and weak coc-R_0- space, and we study some relations between them, also we prove that some of these separation axioms have "hereditary property".

2020 Mathematics Subject Classifications: 54B05, 54B10, 54D20.

Key Words and Phrases: coc-T_i- space for $i = 0, \frac{1}{4}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}$, coc-$D_i$- space, coc-$R_i$- space for $i = 0, 1$, weak coc-D_1- space and weak coc-R_0- space

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1. Introduction and Preliminaries

In [4], the authors defined a new type of open sets called coc-compact set as a generalizations of open sets. After this paper many papers in this concept were appeared, see [1–3].

Also many authors studied weak separation axioms in different types of open sets, for example [5, 8, 9] .

Definition 1. *[4] A subset A of a topological space (X, τ) is called co-compact open set (notation: coc-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset K of X such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space X will be denoted by τ^k.*

Theorem 1. *[4] Let (X, τ) be a topological space. Then*

(i) The collection τ^k forms a topology on X with $\tau \subseteq \tau^k$.

(ii) The set $\{U - K : U \in \tau$ and K is compact in $X\}$ forms a base for τ^k .

Lemma 1. *[4] Let (X, τ) be a topological space and A be a closed subset of X. Then $(\tau\mid_A)^k = \tau^k\mid_A$.*

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Throughout this paper, we use $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$ to denote the set of real numbers, rational numbers and natural numbers, respectively. The coc-closure of A and the coc- interior of A will be denoted by $\overline{A}^{coc}$and $int_{coc}(A)$, respectively.

Terms and notations not explained in this paper are taken from [4, 7].

2. Coc-T-spaces and Coc-D-spaces

Definition 2. *A space (X, τ) is called co-compact-T_0-space (coc-T_0-space) if for all $x \neq y \in X$, there exists a coc-open set U contains one point but not other.*

Definition 3. *A subset A of a topological space (X, τ) is called coc-D-set if $A = U - V$, for some $U, V \in \tau^k$.*

Definition 4. *A space (X, τ) is called co-compact-D_0-space (coc-D_0-space) if for all $x \neq y \in X$, there exists a coc-D-set U contains one point but not other.*

Theorem 2. *A coc-closed subspace of a coc-D_0-space (X, τ) is coc-D_0-space.*

Proof. Let A be a coc-closed subset X and let $x \neq y \in A$. So there exists a coc-D-set $D = U - V$ with $U, V \in \tau^k$ such that $x \in D$ and $y \notin D$. Now $x \in D \cap A = (U - V) \cap A = (A \cap U) - (A \cap V)$, so by Lemma 1 we have $A \cap U$ and $A \cap V \in (\tau \mid_A)^k = \tau^k \mid_A$, hence the result.

Theorem 3. *A space (X, τ) is coc-T_0-space if and only if it is coc-D_0-space.*

Proof. ($\Rightarrow$) It is clear since every proper coc-open subset of X is coc-D-set.
($\Leftarrow$) Let $x \neq y \in X$, so there exists a coc-D_0- set U contains x with $U = U_1 - U_2$ where $U_1, U_2 \in \tau^k$ i.e. $x \in U_1$ and $x \notin U_2$. For y, we have the following cases:(1) If $y \notin U_1$, we are done. (2) If $y \in U_1$ and $y \in U_2$, so U_2 contains y but not x.

Theorem 4. *A space (X, τ) is coc-T_0-space if and only if for all $x \neq y \in X$, we have $\overline{\{x\}}^{coc} = \overline{\{y\}}^{coc}$.*

Proof. ($\Rightarrow$) Let $x \neq y \in X$, there exists a coc- open set U contains one point but not other, say $x \in U$ and $y \notin U$. Then $X - U$ is a coc-closed set contains y and $\overline{\{y\}}^{coc} \subseteq X - U$, so $x \notin \overline{\{y\}}^{coc}$, hence $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$.
($\Leftarrow$) Let $x \neq y \in X$. Then it is clear that $X - \overline{\{y\}}^{coc}$ is coc-open set contains x but not y, hence X is coc-T_0-space.

Definition 5. *A space (X, τ) is called co-compact -T_1-space (coc-T_1-space) if for all $x \neq y \in X$, there exist coc-open sets U_x, V_y with $\{U_x, V_y\} \cap \tau \neq \phi$ such that $x \in U_x$, $y \in V_y$ and $y \notin U_x$, $x \notin V_y$.*

Definition 6. *[3] A space (X, τ) is called co-compact -T_2-space (coc-T_2-space) if for all $x \neq y \in X$, there exist coc-open sets U_x, V_y with $\{U_x, V_y\} \cap \tau \neq \phi$ such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \phi$.*

It is clear that if (X,τ) is coc-T_1-space, then (X,τ^k) is T_1-space. And every T_1-space is coc-T_1-space, but the converse need not be true, consider the following example.

Example 1. *Let $X=\mathbb{R}$ and $\tau=\{\phi\}\cup\{U\subseteq\mathbb{R}, 0\in U\}$.*

Proof.
A space (X,τ) is coc-T_1- space, to prove this let $x\neq y\in X$, so we have the following cases :

(i) For $x=0, y\neq 0$, let $U=\{0\}$ and $V=\{y,0\}-\{0\}$, then $U,V\in\tau^k$ and $\{U,V\}\cap\tau=\{V\}$ with $x\notin V$ and $y\notin U$.

(ii) For $y=0, x\neq 0$, same as (i).

(iii) For $x\neq 0, y\neq 0$, let $U=\{x,0\}, V=\{y,0\}-\{0\}$, then $U,V\in\tau^k$ and $\{U,V\}\cap\tau=\{V\}$, then $x\notin V$ and $y\notin U$.

But (X,τ) is not T_1- space, for instance take $x=0$, $y=1$, then there is no open set contains y but not x.

Theorem 5. *A space (X,τ) is coc-T_1-space if and only if every singleton is coc-closed.*

Definition 7. *A space (X,τ) is called co-compact-D_1-space (coc-D_1-space) if for all $x\neq y\in X$, there exist coc-D-sets U_x, V_y such that $x\in U_x$, $y\in V_y$ and $y\notin U_x$, $x\notin V_y$.*

Theorem 6. *A coc-closed subspace of a coc-D_1-space (X,τ) is coc-D_1-space.*

Definition 8. *A space (X,τ) is called co-compact-D_2-space (coc-D_2-space) if for all $x\neq y\in X$, there exist disjoint coc-D-sets U_x, V_y such that $x\in U_x$, $y\in V_y$ and $y\notin U_x$, $x\notin V_y$.*

Theorem 7. *Let (X,τ) be a topological space. Then:*

(i) If X is coc-T_i-space, then X is coc-T_{i-1}-space for $i=1,2$.

(ii) If X is coc-T_i-space, then X is coc-D_i-space for $i=1,2$.

(iii) If X is coc-D_i-space, then X is coc-D_{i-1}-space for $i=1,2$.

(iv) If X is coc-D_1-space, then X is coc-T_0-space.

(v) X is coc-D_1-space if and only if X is coc-D_2-space .

Proof. We will prove (v) only.
$(\Leftarrow)$ Obvious.
$(\Rightarrow)$ For $x\neq y\in X$, there exist coc-D-sets U_1, U_2 with $x\in U_1, y\notin U_1$ and $y\in U_2, x\notin U_1$, assume $U_1=V_1-W_1, U_2=V_2-W_2$ where $V_1,W_1,V_2,W_2\in\tau^k$. Then for $x\notin U_2$, we have the following cases:
(1) $x\notin V_2$ (2) $x\in V_2$ and $x\in W_2$. For (1) If $x\notin V_2$, we have: (i) If $y\notin V_1$, $x\in V_1-W_1$, then $x\in V_1-(V_2\cup W_1)$ and $y\in V_2-W_2$, so $y\in V_2-(V_1\cup W_2)$ and

$\big(V_1-(V_2\cup W_1)\big)\cap\big(V_2-(V_1\cup W_2)\big)=\phi$. (ii) If $y\in V_1$ and $y\in W_1$, we have $x\in U_1-U_2$, $y\in U_2$ and $\big(U_1-U_2\big)\cap U_2=\phi$.
For (2) If $y\in U_2=V_2-W_2$, then $x\in W_2$ and $\big(V_2-W_2\big)\cap W_2=\phi$. From (1) and (2), X is coc-D_2-space .

The following theorem gives improvement of Theorem 7(iv).

Theorem 8. *A space (X,τ) is coc-D_1-space if and only if X is coc-T_0-space and $int_{coc}(A_x)\neq X$ for all $x\in A_x\subseteq X$.*

Proof. ($\Rightarrow$) For $x\in X$, there exists a coc-D-set $O_x=U-V$ with $U,V\in\tau^k$ and $x\in O_x$, but $U\neq X$, so $int_{coc}(U)\neq X$, hence the result.
($\Leftarrow$) For $x\neq y\in X$, with out loss of generality there exists a coc-open set U contains x but not y and there exists coc-open set V contains y and $int_{coc}(V)\neq X$, hence $y\in V-U$, therefore X is coc-D_1-space.

3. Coc-R_0 and Coc-R_1-spaces

Definition 9. *A space (X,τ) is called co-compact-R_0-space (coc-R_0-space) if every coc-open set contains the coc-closure of its singletons, i.e. for each coc-open set O we have $\overline{\{x\}}^{coc}\subseteq O$ for all $x\in O$.*

Definition 10. *A space (X,τ) is called co-compact-R_1-space (coc-R_1-space) if for $x\neq y\in X$ with $\overline{\{x\}}^{coc}\neq\overline{\{y\}}^{coc}$, then there exist disjoint coc-open sets U,V with $\overline{\{x\}}^{coc}\subseteq U$, $\overline{\{y\}}^{coc}\subseteq V$.*

The following theorem is obvious.

Theorem 9. *Let (X,τ) be a topological space. Then:*

(i) A coc-closed subspace of a coc-R_0-space X is coc-R_0-space.

(ii) A coc-closed subspace of a coc-R_1-space X is coc-R_1-space.

Theorem 10. *Every coc-R_1-space (X,τ) is coc-R_0-space.*

Proof. Let U be a coc-open set in X with $x\in U$. For $y\notin U$, we have $x\notin\overline{\{y\}}^{coc}$, thus $\overline{\{x\}}^{coc}\neq\overline{\{y\}}^{coc}$, but X is coc-R_1-space, so there exits a coc-open set V_y contains y such that $\overline{\{y\}}^{coc}\subseteq V_y$ and $x\notin V_y$, hence $\overline{\{x\}}^{coc}\subseteq U$, thus X is coc-R_0-space.

Theorem 11. *A space (X,τ) is coc-T_1-space if and only if it is coc-T_0-space and coc-R_0-space.*

Proof. ($\Rightarrow$) Notes that $\{x\}$ is coc-closed subset of X for all $x\in X$.
($\Leftarrow$) Let $x\neq y\in X$, with out loss of generality there exists a coc-open set O with $x\in O\subseteq X-\{y\}$. Thus $x\notin\overline{\{y\}}^{coc}$, so $y\notin\overline{\{x\}}^{coc}$, hence $X-\overline{\{x\}}^{coc}$ is coc-open set contains y but not x.

Corollary 1. *Let* (X,τ) *be a coc-*R_0*-space. Then the following are equivalent:*

(i) X *is coc-*T_2*-space,*

(ii) X *is coc-*T_1*-space,*

(iii) X *is coc-*T_0*-space.*

Definition 11. *Let* (X,τ) *be a topological space and* $A \subseteq X$*. Then the coc-Kernal of* A *define by:*

$$coc\text{-}ker(A) = \cap\{U \in \tau^k : A \subseteq U\},$$

if there no coc-open set contains A*, then* $coc\text{-}ker(A) = X$*.*

Lemma 2. *If* (X,τ) *is a topological space and* A *is a subset of* X*, then* $coc\text{-}ker(A) = \{x \in X : \overline{\{x\}}^{coc} \cap A \neq \phi\}$*.*

Proof. For $x \notin coc\text{-}ker(A)$, there exists a coc-open set U contains A and $x \notin U$, then $\overline{\{x\}}^{coc} \cap U = \phi$. For $\overline{\{x\}}^{coc} \cap U = \phi$, we have $x \notin X - \overline{\{x\}}^{coc}$, thus $x \notin coc\text{-}ker(A)$.

Lemma 3. *Let* (X,τ) *be a topological space and* $x \in X$*. Then* $y \in coc\text{-}ker(\{x\})$ *if and only if* $x \in \overline{\{y\}}^{coc}$*.*

Theorem 12. *Let* (X,τ) *be a topological space and* $x \neq y \in X$*. Then* $coc\text{-}ker(\{x\}) \neq coc\text{-}ker(\{y\})$ *if and only if* $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$*.*

Proof. $(\Rightarrow)$ Let $w \in coc\text{-}ker(\{x\})$ and $w \notin coc\text{-}ker(\{y\})$. Then $\overline{\{w\}}^{coc} \cap \{x\} \neq \phi$ and $\overline{\{w\}}^{coc} \cap \{y\} = \phi$, so $x \in \overline{\{w\}}^{coc}$, and hence $\overline{\{x\}}^{coc} \subseteq \overline{\{w\}}^{coc}$, therefore $\overline{\{w\}}^{coc} \cap \{y\} = \phi$ and hence $y \notin \overline{\{x\}}^{coc}$.
$(\Leftarrow)$ Since $coc\text{-}ker(\{x\}) \neq coc\text{-}ker(\{y\})$, there is $z \in \overline{\{x\}}^{coc}$ and $z \notin \overline{\{y\}}^{coc}$, hence there exists a coc-open set U_z with $x \in U_z$ and $y \notin U_z$, so $y \notin coc\text{-}ker(\{x\})$.

Theorem 13. *A space* (X,τ) *is coc-*R_0*-space if and only if for* $x \neq y \in X$*,* $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$ *gives* $\overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc} = \phi$*.*

Proof. $(\Leftarrow)$ Let $x \in O_x \in \tau^k$ and assume that $y \notin O_x$. Then $x \notin \overline{\{y\}}^{coc}$, hence $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$, so $\overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc} = \phi$, therefore $y \notin \overline{\{x\}}^{coc}$ and $\overline{\{x\}}^{coc} \subseteq O_x$, so X is coc-R_0-space.

$(\Rightarrow)$ Let $x \neq y \in X$ with $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$. So there exists $z \in \overline{\{x\}}^{coc}$ and $z \notin \overline{\{y\}}^{coc}$, then $z \in X - \overline{\{y\}}^{coc}$, so there exists a coc-open set U contains z but not y, but $z \in \overline{\{x\}}^{coc}$, so $x \in U$ and $x \notin \overline{\{y\}}^{coc}$, hence $\overline{\{x\}}^{coc} \subseteq X - \overline{\{y\}}^{coc}$, therefore $\overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc} = \phi$.

Theorem 14. *A space* (X,τ) *is coc-*R_0*-space if and only if for* $x \neq y \in X$*,* $coc\text{-}ker(\{x\}) \neq coc\text{-}ker(\{y\})$ *gives* $coc\text{-}ker(\{x\}) \cap coc\text{-}ker(\{y\}) = \phi$*.*

Proof. ($\Rightarrow$) Let X be a coc-R_0-space and for $x \neq y \in X$ with $coc\text{-}ker(\{x\}) \neq coc\text{-}ker(\{y\})$. Let $w \in coc\text{-}ker(\{x\}) \cap coc\text{-}ker(\{y\})$. Then $w \in coc\text{-}ker(\{x\})$ so $x \in \overline{\{w\}}^{coc}$, and then by Lemma 3 $\overline{\{x\}}^{coc} = \overline{\{w\}}^{coc}$, in same method we have $\overline{\{y\}}^{coc} = \overline{\{w\}}^{coc}$, and this is a contradiction which completes the proof.

($\Leftarrow$) Assume $\overline{\{x\}}^{coc} \neq \overline{\{y\}}^{coc}$, then $coc\text{-}ker(\{x\}) \neq coc\text{-}ker(\{y\})$, so $coc\text{-}ker(\{x\}) \cap coc\text{-}ker(\{y\}) = \phi$. If $z \in \overline{\{x\}}^{coc}$, then $x \in coc\text{-}ker(\{z\})$ and $coc\text{-}ker(\{x\}) \cap coc\text{-}ker(\{z\}) = \phi$, so $coc\text{-}ker(\{x\}) = coc\text{-}ker(\{z\})$. Now for $z \in \overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc}$, we have $coc\text{-}ker(\{x\}) = coc\text{-}ker(\{y\}) = coc\text{-}ker(\{z\})$, and this is a contradiction, hence $\overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc} = \phi$.

Theorem 15. *For a topological space (X, τ). The following are equivalent:*

(i) X is a coc-R_0-space,

(ii) For a subset A of X and G coc-open set of X such that $A \cap G \neq \phi$, there exists a coc-closed subset F of X such that $A \cap F \neq \phi$ and $F \subseteq G$,

(iii) For any coc-open set G of X, $G = \cup\{F : F \text{ is coc-closed subset with } F \subseteq G\}$,

(iv) For any coc-closed subset F of X, $F = coc\text{-}ker(F)$,

(v) For any $x \in X$, $\overline{\{x\}}^{coc} \subseteq coc\text{-}ker(\{x\})$.

Proof. $(iii) \Rightarrow (iv)$, $(v) \Rightarrow (i)$ Obvious.
$(i) \Rightarrow (ii)$ Let $A \subseteq X$ and G is a coc-open set and let $x \in A \cap G$. Then the needed coc-closed subset F is $\overline{\{x\}}^{coc}$.
$(ii) \Rightarrow (iii)$ For a coc-open set $G \supseteq \cup\{F : F \text{ is a coc-closed with } F \subseteq G\}$, let $x \in G$, then there exists a coc-closed set F such that $x \in F$ and $F \subseteq G$, so $x \in F \subseteq \cup\{F : F \text{ is a coc-closed}, F \subseteq G\}$, hence the result.
$(iv) \Rightarrow (v)$ Let $x \in X$ and $y \notin coc\text{-}ker(\{x\})$, there exists a coc-open set U_x contains x with $y \notin U$, so $\overline{\{y\}}^{coc} \cap U = \phi$ and hence $coc\text{-}ker\big(\overline{\{y\}}^{coc}\big) \cap U = \phi$, therefore there exists a coc-open set O_y such that $x \notin O_y$ and $\overline{\{y\}}^{coc} \subseteq O_y$, so $\overline{\{x\}}^{coc} \cap O_y = \phi$ and $y \notin \overline{\{x\}}^{coc}$, hence the result.

Lemma 4. *A topological space (X, τ) is coc-R_0-space if and only if for each $x \neq y \in X$ with $x \in \overline{\{y\}}^{coc}$ gives $y \in \overline{\{x\}}^{coc}$*

Proof. ($\Rightarrow$) Let X be a coc-R_0-space and $x \in \overline{\{y\}}^{coc}$. If U is any coc-open set with $y \in U$, then $x \in U$ and any coc-open set contains y must contains x, hence $y \in \overline{\{x\}}^{coc}$.
($\Leftarrow$) Let U be a coc-open set with $x \in U$. For $x \in \overline{\{y\}}^{coc}$, we have $y \in \overline{\{x\}}^{coc}$, therefore $\overline{\{x\}}^{coc} \subseteq U$, hence X is coc-R_0-space.

Theorem 16. *For a topological space (X, τ). The following are equivalent:*

(i) X is a coc-R_0-space,

(ii) If F is a coc-closed subset of X with $x \in F$, then $coc\text{-}ker(\{x\}) \subseteq F$,

(iii) If $x \in X$, then coc-ker$(\{x\}) \subseteq \overline{\{x\}}^{coc}$.

Proof. $(ii) \Rightarrow (iii)$ Obvious .
$(i) \Rightarrow (ii)$ Let F be a coc-closed and $x \in F$. So *coc-ker*$(\{x\}) \subseteq$ *coc-ker*(F), then by Theorem 15 we have coc-*ker*$(\{x\}) \subseteq F$.
$(iii) \Rightarrow (i)$ Let $x \in \overline{\{y\}}^{coc}$. So $y \in$ *coc-ker*$(\{x\})$, therefore by (iii) $y \in \overline{\{x\}}^{coc}$, and the result comes from Lemma 4.

Corollary 2. *A topological space (X, τ) is coc-R_0-space if and only if coc-ker$(\{x\}) = \overline{\{x\}}^{coc}$ for all $x \in X$.*

4. Coc-$T_{\frac{1}{2}}$-space, Coc-$T_{\frac{3}{8}}$-space and Coc-$T_{\frac{1}{4}}$-space

In this section we define more weak separation axioms in coc-open set, but before this we need some definitions and lemmas.

Definition 12. *Let A be a subset of a topological space (X, τ). Then A is called coc-g-closed if $\overline{\{A\}}^{coc} \subseteq U$, whenever $A \subseteq U$ and U is coc-open set. A is called coc-g-open if $X - A$ is coc-g-closed.*

Clearly, A is a coc-g-closed of (X, τ) if $F \subseteq int_{coc}(A)$, whenever $F \subseteq A$ and F is coc-closed set of X.

Definition 13. *Let A be a subset of a topological space (X, τ). Then coc-$A^{\vee} = \cup\{F : X - F \in \tau^k : F \subseteq A\}$, if there is no coc-closed set contains in A, then coc-$A^{\vee} = \phi$.*

Lemma 5. *Let A be a subset of a topological space (X, τ). Then A is coc-g-closed (coc-g-open) if and only if $\overline{\{A\}}^{coc} \subseteq$ coc-ker(A) (coc-$A^{\vee} \subseteq int_{coc}(A)$).*

Definition 14. *Let A be a subset of a topological space (X, τ) . Then A is called coc-$\wedge$-set (coc-$\vee$-set) if $A =$ coc-ker$(A)(A =$ coc-$A^{\vee})$, or equivalently, A is the intersection of coc-open sets or $A = X$(A is the union of coc-closed sets or $A = \phi$).*

Lemma 6. *Let A, B are subsets of a topological space (X, τ). Then :*

- *(i) coc-ker$\{\phi\} = \phi$, coc-$\phi^{\vee} = \phi$, coc-ker$\{X\} = X$, coc-$X^{\vee} = X$.*
- *(ii) $A \subseteq$ coc-ker(A), coc-$A^{\vee} \subseteq A$.*
- *(iii) coc-ker(coc-ker(A))=coc-ker(A), coc-$\left(\text{coc-}A^{\vee}\right)^{\vee} =$ coc-$A^{\vee}$.*
- *(iv) If $A \subseteq B$, then coc-ker$(A) \subseteq$ coc-ker(B).*
- *(v) If $A \subseteq B$, then coc-$A^{\vee} \subseteq$ coc-$B^{\vee}$.*

Lemma 7. *Let (X, τ) be a topological space. Then the following are hold :*

- *(i) If A is coc-$\wedge$-set (coc-$A^{\vee}$-set), then A is coc-g-closed (coc-g-open) if and only if A is coc-closed (coc-open).*

(ii) For $A \subseteq X$, if coc-ker(A) is coc-g-closed set (coc-$A^{\vee}$ is coc-g-open set), then A is coc-g-closed (coc-g-open).

Proof. (i) Obvious.
(ii) From Lemma 5 and Lemma 6.

The following definition gives a weaker form of coc-$\wedge$-set.

Definition 15. *A subset A of a space (X, τ) is called generalized coc-kernal set (g-coc-$\wedge$-set) if coc-ker(A) $\subseteq \overline{\{A\}}^{coc}$, or equivalently coc-ker(A) $\subseteq F$, whenever $A \subseteq F$ and F is coc-closed. A subset A of a space (X, τ) is called generalized coc-$\vee$-set (g-coc-$\vee$-set) if $X - A$ is g-coc-$\wedge$-set, or equivalently $int_{coc}(A) \subseteq coc - A^{\vee}$.*

Lemma 8. *Let A be subset of a topological space (X, τ). If A is coc-$\wedge$-set (coc-$A^{\vee}$-set), then it is g-coc$-\wedge$-set (g-coc-$\vee$-set).*

Theorem 17. *Let (X, τ) be a topological space. Then for $x \in X$, $\{x\}$ is either coc-open or g-coc-$\vee$-set.*

Proof. Let $x \in X$ and $\{x\}$ is not coc-open subset of X. Hence $X - \{x\}$ is not coc-closed subset of X and $\overline{\{X - \{x\}\}}^{coc} = X$, so coc-ker $(X - \{x\}) \subseteq \overline{\{X - \{x\}\}}^{coc}$, therefore $X - \{x\}$ is g-coc-$\wedge$-set, i.e. $\{x\}$ is g-coc-$\vee$-set.

Definition 16. *A topological space (X, τ) is called coc-$T_{\frac{1}{2}}$-space if every coc-g-closed subset of X is coc-closed.*

Lemma 9. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is coc-g-closed subset if and only if $\overline{A}^{coc} - A$ contains no coc-closed subset of X.*

Proof. $(\Leftarrow)$ Obvious.
$(\Rightarrow)$ Let A be coc-g-closed and assume there exists a coc-closed subset F with $A \subseteq X - F$. Since A is coc-g-closed set, we have $\overline{A}^{coc} \subseteq X - F$, hence $F \subseteq X - \overline{A}^{coc}$ and this is a contradiction which completes the proof.

Theorem 18. *A topological space (X, τ) is coc-$T_{\frac{1}{2}}$-space if and only if every singleton of X is coc-open or coc-closed.*

Proof. $(\Rightarrow)$ Let $x \in X$ and $\{x\}$ is not coc-closed set. Hence $X - \{x\}$ is not coc-open, therefore X is the only coc-open set with $X - \{x\} \subseteq X$, that is mean $X - \{x\}$ is coc-g-closed, so $X - \{x\}$ is coc-closed, i.e. $\{x\}$ is coc-open.
$(\Leftarrow)$ Let $x \in X$ and A is coc-g-closed subset of X with $x \in \overline{A}^{coc}$. If $\{x\}$ is a coc-open set, then $\{x\} \cap A \neq \phi$ and hence $x \in A$. If $\{x\}$ is a coc-closed, then by Lemma 9, $x \notin \overline{A}^{coc} - A$, hence $x \in A$ and $A = \overline{A}^{coc}$, therefore X is coc-$T_{\frac{1}{2}}$-space.

Corollary 3. *Every coc-T_1-space is coc-$T_{\frac{1}{2}}$-space .*

Theorem 19. *For a topological space (X, τ). The following are equivalent:*

(i) X *is coc-*$T_{\frac{1}{2}}$*-space,*

(ii) *Every g-coc-*$\wedge$*- set is coc-*$\wedge$*-set,*

(iii) *Every g-coc-*$\vee$*-set is coc-*$\vee$*-set.*

Proof.
$(iii) \Rightarrow (ii)$ Obvious.
$(ii) \Rightarrow (i)$ Let $x \in X$. If $\{x\}$ is not coc-open, then $X - \{x\}$ is not coc-closed, so the only coc-open set contains $X - \{x\}$ is X, but $X - \{x\}$ is g-coc-$\wedge$-set, so $X - \{x\}$ is coc-$\wedge$-set, therefore $X - \{x\}$ is coc-open, hence $\{x\}$ is coc-closed set, that's complete the proof.

$(i) \Rightarrow (ii)$ Assume that a subset A of X is g-coc-$\wedge$-set which is not coc-$\wedge$-set, then coc-ker$(A) \nsubseteq A$, so there exists $x \in$ *coc-ker*(A) and $x \notin A$, but X is a coc-$T_{\frac{1}{2}}$-space, so $\{x\}$ is a coc-open or coc-closed set, we need to discuss the following two cases:
(1) If $\{x\}$ is a coc-closed, then $X - \{x\}$ is a coc-open set contains A, but $x \in coc - ker(A)$, so $x \in X - \{x\}$ and this is a contradiction. (2) If $\{x\}$ is coc-open set, then $X - \{x\}$ is a coc-open set contains A, by assumption coc-ker $(A) \subseteq X - \{x\}$, i.e. $x \notin$coc-ker(A) and this is a contradiction, hence A is coc-$\wedge$-set.

Definition 17. *A subset A of a topological space* (X, τ) *is called coc-*λ*-closed if* $A = L \cap F$, *where* L *is coc-*$\wedge$*-set and* F *is coc-closed set. A subset A is coc-*λ*-open if* $X - A$ *is coc-*λ*-closed.*

Lemma 10. *For a subset A of* (X, τ)*. The following are equivalent :*

(i) A *is coc-*λ*-closed,*

(ii) $A = L \cap \overline{A}^{coc}$, *where* L *is coc-*$\wedge$*-set,*

(iii) A*=coc-ker*$(A) \cap \overline{A}^{coc}$.

Theorem 20. *A topological space* (X, τ) *is coc-*$T_{\frac{1}{2}}$*-space if and only if every subset of* X *is coc-*λ*-closed.*

Proof. $(\Leftarrow)$ Let $x \in X$. Assume that $\{x\}$ is not coc-open, then $A = X - \{x\}$ is not coc-closed, but A is coc-λ-closed, so A is coc-$\wedge$-set, thus A is coc-open set, then A is coc-open, that is $\{x\}$ is coc-closed, which is complete the proof.
$(\Rightarrow)$ Let $A \subseteq X$ and $x \in X - A$. Then $\{x\}$ is coc-open or coc-closed subset of X. Define $B = \{x \in X - A, \{x\} \in \tau^k\}$, $C = \{x \in X - A, X - \{x\} \in \tau^k\}$. Also define $F = \bigcap_{x \in B} \big(X - \{x\}\big) = X - B$, and $L = \bigcap_{x \in C} \big(X - \{x\}\big) = X - C$, then F is coc-closed set and L is coc-$\wedge$-set with $L \cap F = A$, hence A is coc-λ-set.

Definition 18. *A topological space* (X, τ) *is called coc-*$T_{\frac{1}{4}}$*-space if every finite subset* F *of* X *and every* $y \in X - F$*, there exists a set* A_y *with* $F \subseteq A_y$ *such that* $\{y\} \cap A_y = \phi$ *and* A_y *is either coc-open or coc-closed.*

Theorem 21. *A topological space* (X, τ) *is coc-*$T_{\frac{1}{4}}$*-space if and only if every finite subset of* X *is coc-*λ*-closed.*

Proof. ($\Rightarrow$) Let F be any finite subset of X and $y \in X - F$. So there exist a set A_y such that $A_y \cap \{x\} = \phi$, and A_y is either coc-open or coc-closed. Let C be the intersection of all coc-open sets A_y and let L be the intersection of all coc-closed sets A_y, clearly $F = C \cap L$, C is coc-$\wedge$-set and L is coc-closed set, hence F is coc-λ-closed set.
($\Leftarrow$) Let $F = L \cap C$ and $y \in X - F$ where C is coc-$\wedge$-set and L coc-closed set. If $y \notin C$, we are done. If $y \in C$, then $y \notin L$, so there exists a coc-open set U_y with $y \in U_y$, hence X is coc-$T_{\frac{1}{4}}$-space.

Definition 19. *A topological space* (X, τ) *is called coc-*$T_{\frac{3}{8}}$*-space if every countable subset* F *of* X *and every* $y \in X - F$*, there exists a set* A_y *with* $F \subseteq A_y$ *such that* $\{y\} \cap A_y = \phi$ *and* A_y *is coc-open or coc-closed.*

Clearly every coc-$T_{\frac{1}{2}}$-space is coc-$T_{\frac{3}{8}}$-space and hence coc-$T_{\frac{1}{4}}$-space.

Theorem 22. *A topological space* (X, τ) *is coc-*$T_{\frac{3}{8}}$*-space if and only if every countable subset of* X *is coc-*λ*-closed.*

Proof. Same as Theorem 21.

In the end of this section, we give weak forms of coc-D_1-space and coc-R_0-space.

Definition 20. *A topological space* (X, τ) *is called weak coc-*D_1*-space if* $\bigcap_{x \in X} \overline{\{x\}}^{coc} = \phi$.

Theorem 23. *A coc-closed subspace of weak coc-*D_1*-space* (X, τ) *is weak coc-*D_1*-space.*

Theorem 24. *A topological space* (X, τ) *is weak coc-*D_1*-space if and only if* $int_{coc}(A_x) \neq X$ *for all* $x \in A_x \subseteq X$.

Proof. ($\Rightarrow$) Assume that there exists $y \in X$ with $int_{coc}(\{A_y\}) = X$, then $y \in \overline{\{x\}}^{coc}$ for each $x \in X$, this is a contradiction, hence the result.
($\Leftarrow$) Let $y \in \bigcap_{x \in X} \overline{\{x\}}^{coc}$, then the coc-open set contains y must be X, so $int_{coc}(\{A_y\}) = X$, this is a contradiction, hence the result.

Corollary 4. *A topological space* (X, τ) *is coc-*D_1*-space if and only if* (X, τ) *is coc-*T_0*-space and weak coc-*D_1*-space.*

Theorem 25. *A topological space* (X, τ) *is weak coc-*D_1*-space if and only if coc-*$ker(\{x\}) \neq X$ *for all* $x \in X$.

Proof. ($\Rightarrow$) Obvious.
($\Leftarrow$) From Theorem 24.

Definition 21. *A topological space (X,τ) is called weak coc-R_0-space if every coc-λ-closed singleton is a coc-$\wedge$-set.*

Theorem 26. *Every coc-R_0-space (X,τ) is weak coc-R_0-space.*

Proof. Let $x \in X$ with $\{x\}$ is coc$-\lambda-$closed. By Lemma 10 $\{x\} = coc\text{-}ker(\{x\}) \cap \overline{\{x\}}^{coc}$. If $\{x\}$ is not coc-ker-set, then there exists $y \in coc-ker(\{x\})-\{x\}$ with $y \notin \overline{\{x\}}^{coc}$, but X is coc-R_0-space, so $\overline{\{x\}}^{coc} \cap \overline{\{y\}}^{coc} = \phi$ and $x \in \overline{\{y\}}^{coc}$, therefore there exists a coc-open set U_x contains x but not y, thus $y \notin coc - ker(\{x\})$, and this is a contradiction which completes the proof.

The following theorems are easily to prove.

Theorem 27. *For a topological space (X,τ). The following are equivalent:*

(i) *X is coc-T_1-space,*

(ii) *Every subset of X is coc-$\wedge$-set,*

(iii) *Every singleton of X is coc-$\wedge$-set.*

Theorem 28. *For a topological space (X,τ). The following are equivalent:*

(i) *X is coc-T_1-space,*

(ii) *X is coc-T_0-space and coc-R_0-space,*

(iii) *X is coc-T_0-space and weak coc-R_0-space.*

Corollary 5. *For a weak coc-R_0-space (X,τ). The following are equivalent:*

(i) *X is coc-T_0-space,*

(ii) *X is coc-$T_{\frac{1}{4}}$-space,*

(iii) *X is coc-$T_{\frac{3}{8}}$-space,*

(iv) *X is coc-$T_{\frac{1}{2}}$-space,*

(v) *X is coc-T_1-space.*

5. Hereditary Property for Weak Coc-compact Separation Axioms

In this section, we discuss the known problem that appeared by Arenas [6] "If every subspace of a topological space X has a property, then the space X has this property" in weak separation axioms via coc-open sets.

Theorem 29. *If every proper subspace of a topological space (X,τ) is coc-$T_{\frac{1}{2}}$-space, then X is coc-$T_{\frac{1}{2}}$-space with $|X| \geq 4$.*

Proof. Let $x \in X$ and let $z_1 \neq z_2 \neq z_3 \in X - \{x\}$ and $Z_i = X - \{z_i\}$ for $i = 1, 2, 3$. So $\{x\}$ is either coc-open or coc-closed in Z_i, therefore either $\{x\}$ is coc-open in at least two of Z_1, Z_2, Z_3, and hence $\{x\}$ is coc-open in X, or $\{x\}$ is coc-closed in at least two of Z_1, Z_2, Z_3, and hence $\{x\}$ is coc-closed in X, hence the result.

Theorem 30. *Let (X, τ) be infinite topological space. If every proper subspace of a topological space X is coc-$T_{\frac{1}{4}}$-space, then X is coc-$T_{\frac{1}{4}}$-space.*

Proof. Let F be a finite set and $y \notin F$ and let $z \in X - \big(F \cup \{y\}\big)$. So there exists a set A contains F and $y \notin A$ which is either coc-open or coc-closed in $X - \{z\}$, therefore there exists a set B which is either coc-open or coc-closed in X with $A = B \cap (X - \{x\})$, hence X is coc-$T_{\frac{1}{4}}$-space.

Theorem 31. *Let (X, τ) be infinite topological space. If every proper subspace of a topological space X is coc-$T_{\frac{3}{8}}$-space, then X is coc-$T_{\frac{3}{8}}$-space.*

Proof. Same as Theorem 30.

Theorem 32. *If every proper subspace of a topological space (X, τ) is coc-R_0-space, then X is coc-R_0-space with $|X| \geq 3$.*

Proof. Assume that all proper subspaces of X are coc-R_0-space. Let U be coc-open subset of X. If $X = U$ we are done, so we may assume $X \neq U$. Let $x \notin U$ and $p \in U$ with $y \in X - \{p, x\}$. So we have the following cases :
(1) If $y \in U$, so $X - \{y\}$ is coc-R_0-space, so by Theorem 15 (iii) there is a coc-closed set G_y in $X - \{y\}$ such that $p \in G_y \subseteq U - \{y\}$ and also there exists a coc-closed set G in X such that $G_y = G \cap (X - \{y\})$, then $p \in G \subseteq G_y \cup \{y\} \subseteq (U - \{y\}) \cup \{y\} = U$, hence X is coc-R_0-space.
(2) If $y \notin U$, then $X - \{x\}$ and $X - \{y\}$ are proper subspaces of X, so there exist coc-closed subsets G_x, G_y in $X - \{x\}$ and $X - \{y\}$, respectively such that $p \in G_x \subseteq U$ and $p \in G_y \subseteq U$, also there exist coc-closed sets G_1, G_2 in X such that $G_x = G_1 \cap (X - \{x\})$ and $G_y = G_2 \cap (X - \{y\})$. Define $G = G_1 \cap G_2$, so $p \in G \subseteq (G_x \cup \{x\}) \cap (G_y \cup \{y\}) \subseteq U$, hence X is coc-R_0-space.

Theorem 33. *If every proper subspace of a topological space (X, τ) is coc-T_1-space, then X is coc-T_1-space with $|X| \geq 3$.*

Proof. Suppose that X is not a coc-T_1-space, so there exists $x \in X$ such that $\{x\}$ is not coc-closed in X. Let $z \in X - \{x\}$. Then $X - \{z\}$ is a coc-T_1-space, so $\{x\}$ is coc-closed is $X - \{z\}$ and $\overline{\{x\}}^{coc} = \{x, z\}$. Now let $y \in X - \{x, z\}$ and $B = X - \{y\}$, then $\overline{\{x\}}^{coc(B)} = \{x, z\}$ that means $\{x\}$ is not coc-closed in B which is a contradiction, hence X is coc-T_1-space .

The following theorem can be proved as the previous one.

Theorem 34. *If every proper subspace of a topological space (X, τ) is coc-T_2-space, then X is coc-T_2-space with $|X| \geq 3$.*

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