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On generalized (Λ, sp)-closed sets

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Abstract. This paper is concerned with the concept of generalized (Λ, sp)-closed sets. Some properties of generalized (Λ, sp)-closed sets and generalized (Λ, sp)-open sets are discussed. Moreover, several characterizations of Λ_{sp}-normal spaces are investigated.

2020 Mathematics Subject Classifications: 54A05, 54D10

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1. Introduction

General topology plays an important role in pure and applied sciences such as data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, engineering design, digital topology, information systems, quantum physics, high energy physics and superstring theory. The topological structures of set theories dealing with uncertainities were first introduced by Chang [3]. Lashin et al. [8] investigated topological spaces by generalizing rough set theory. The concept of soft topological spaces defined by Shabir and Naz [14] on an initial universe with a fixed set of parameters. Şenel and Çağman [5] extended the concept of bitopological spaces to soft bitopological spaces and obtained some relations between soft topology and soft bitopology. In [4], the present authors defined and studied the concepts of soft closed sets, soft α-closed sets, soft semi-closed sets, soft pre-closed sets, regular soft closed sets, soft g-closed sets and soft sg-closed sets in soft bitopological spaces. The notions of closed sets and open sets are fundamental with respect to the investigation of general topology. In 1970, Levine [10] introduced the concept of generalized closed sets in topological spaces and defined the notion of a $T_{\frac{1}{2}}$-space to be one in which the closed sets and the generalized closed sets coincide. Dunham and Levine [6] investigated the further properties of generalized closed sets. The concept of generalized closed sets has been modified and studied by using weaker forms of open sets such as α-open sets [12], semi-open sets [9], preopen

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sets [11] and semi-preopen sets [1]. In 1983, Abd El-Monsef et al. [7] introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets [1]. In 2004, Noiri and Hatir [13] introduced the notions of Λ_{sp}-sets, Λ_{sp}-closed sets and spg-closed sets and investigated properties of these sets. In [2], the author introduced and investigated the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. In the present paper, we introduce the concept of generalized (Λ, sp)-closed sets. Moreover, some properties of generalized (Λ, sp)-closed sets and generalized (Λ, sp)-open sets are discussed. Furthermore, several characterizations of Λ_{sp}-normal spaces are investigated.

2. Preliminaries

We begin with some definitions and known results which will be used throughout this paper. In the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *β-open* [7] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets in a topological space (X, τ) is denoted by $\beta(X, \tau)$. Let A be a subset of a topological space (X, τ). A subset $\Lambda_{sp}(A)$ [13] is defined as follows:

$$\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}.$$

Lemma 1. [13] *For subsets A, B and $A_\alpha (\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) *If* $A \subseteq B$, *then* $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) *If* $U \in \beta(X, \tau)$, *then* $\Lambda_{sp}(U) = U$.

(5) $\Lambda_{sp}(\cap\{A_\alpha | \alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha | \alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

A subset B of a topological space (X, τ) is called a *Λ_{sp}-set* [13] if $B = \Lambda_{sp}(B)$. The family of all Λ_{sp}-sets of a topological space (X, τ) is denoted by $\Lambda_{sp}(X, \tau)$ (or simply Λ_{sp}).

Lemma 2. [13] *For subsets A and $A_\alpha (\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{sp}(A)$ *is a* Λ_{sp}*-set.*

(2) *If A is β-open, then A is a Λ_{sp}-set.*

(3) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cap_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

(4) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cup_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

A subset A of a topological space (X, τ) is said to be (Λ, sp)-*closed* [2] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called (Λ, sp)-*open*. The family of all (Λ, sp)-open (resp. (Λ, sp)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$ (resp. $\Lambda_{sp}C(X, \tau)$). Let A be a subsets of a topological space (X, τ). A point $x \in X$ is called a (Λ, sp)-*cluster point* [2] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-*closure* [2] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-*interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 3. [2] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A^{(\Lambda,sp)} = A$.

Lemma 4. [2] *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ and $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ is (Λ, sp)-open.

(4) A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

3. Generalized (Λ, sp)-closed sets

We begin this section by introducing the concept of generalized (Λ, sp)-closed sets.

Definition 1. *A subset A of a topological space (X, τ) is said to be generalized (Λ, sp)-closed (briefly g-(Λ, sp)-closed) if $A^{(\Lambda,sp)} \subseteq U$ and U is (Λ, sp)-open in X.*

Definition 2. *A topological space (X, τ) is called Λ_{sp}-symmetric if, for each x and y in X, $x \in \{y\}^{(\Lambda,sp)}$ implies $y \in \{x\}^{(\Lambda,sp)}$.*

Theorem 1. *A topological space (X, τ) is Λ_{sp}-symmetric if and only if $\{x\}$ is g-(Λ, sp)-closed for each $x \in X$.*

Proof. Assume that $x \in \{y\}^{(\Lambda,sp)}$, but $y \notin \{x\}^{(\Lambda,sp)}$. This implies that the complement of $\{x\}^{(\Lambda,sp)}$ contains y. Therefore, the set $\{y\}$ is a subset of the complement of $\{x\}^{(\Lambda,sp)}$. This implies that $\{y\}^{(\Lambda,sp)}$ is a subset of the complement of $\{x\}^{(\Lambda,sp)}$. Now, the complement of $\{x\}^{(\Lambda,sp)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq V \in \Lambda_{sp}O(X, \tau)$, but $\{x\}^{(\Lambda,sp)}$ is not a subset of V. This means that $\{x\}^{(\Lambda,sp)}$ and the complement of V are not disjoint. Let y belongs to their intersection. Now, we have $x \in \{y\}^{(\Lambda,sp)}$ which is a subset of the complement of V and $x \notin V$. This is a contradiction.

Theorem 2. *A subset A of a topological space (X, τ) is g-(Λ, sp)-closed if and only if $A^{(\Lambda,sp)} - A$ contains no nonempty (Λ, sp)-closed set.*

Proof. Let F be a (Λ, sp)-closed subset of $A^{(\Lambda,sp)} - A$. Since $A \subseteq X - F$ and A is g-(Λ, sp)-closed, $A^{(\Lambda,sp)} \subseteq X - F$ and hence $F \subseteq X - A^{(\Lambda,sp)}$. Thus,

$$F \subseteq A^{(\Lambda,sp)} \cap [X - A^{(\Lambda,sp)}] = \emptyset$$

and F is empty.

Conversely, suppose that $A \subseteq U$ and U is (Λ, sp)-open. If $A^{(\Lambda,sp)} \not\subseteq U$, then

$$A^{(\Lambda,sp)} \cap (X - U)$$

is a nonempty (Λ, sp)-closed subset of $A^{(\Lambda,sp)} - A$.

Corollary 1. *Let A be a g-(Λ, sp)-closed subset of a topological space (X, τ). Then, A is (Λ, sp)-closed if and only if $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed.*

Proof. If A is a (Λ, sp)-closed set, then $A^{(\Lambda,sp)} - A = \emptyset$.

Conversely, suppose that $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed. Since A is g-(Λ, sp)-closed and $A^{(\Lambda,sp)} - A$ is a (Λ, sp)-closed subset of itself, by Theorem 2, $A^{(\Lambda,sp)} - A = \emptyset$ and hence $A^{(\Lambda,sp)} = A$.

Theorem 3. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is (Λ, sp)-closed, then A is g-(Λ, sp)-closed.

(2) If A is g-(Λ, sp)-closed and (Λ, sp)-open, then A is (Λ, sp)-closed.

(3) If A is g-(Λ, sp)-closed and $A \subseteq B \subseteq A^{(\Lambda,sp)}$, then B is g-(Λ, sp)-closed.

Proof. (1) Let A be (Λ, sp)-closed and $A \subseteq U \in \Lambda_{sp}O(X, \tau)$. Then, by Lemma 3, $A^{(\Lambda,sp)} = A \subseteq U$ and hence A is g-(Λ, sp)-closed.

(2) Let A be g-(Λ, sp)-closed and (Λ, sp)-open. Then, $A^{(\Lambda,sp)} = A$ and by Lemma 3, A is (Λ, sp)-closed.

(3) Let $B \subseteq U$ and $U \in \Lambda_{sp}O(X, \tau)$. Since $A \subseteq U$ and A is g-(Λ, sp)-closed, we have $A^{(\Lambda,sp)} \subseteq U$. Since $A \subseteq B \subseteq A^{(\Lambda,sp)}$, by Lemma 3, $A^{(\Lambda,sp)} = B^{(\Lambda,sp)}$ and hence $B^{(\Lambda,sp)} \subseteq U$. Thus, B is g-(Λ, sp)-closed.

Corollary 2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is (Λ, sp)-open, then A is g-(Λ, sp)-open.

(2) If A is g-(Λ, sp)-open and (Λ, sp)-closed, then A is (Λ, sp)-open.

(3) If A is g-(Λ, sp)-open and $A_{(\Lambda,sp)} \subseteq B \subseteq A$, then B is g-(Λ, sp)-open.

Proof. This follows from Theorem 3.

Definition 3. *Let A be a subset of a topological space (X, τ). The (Λ, sp)-frontier of A, $\Lambda_{sp}Fr(A)$, is defined as follows: $\Lambda_{sp}Fr(A) = A^{(\Lambda,sp)} \cap [X - A]^{(\Lambda,sp)}$.*

Theorem 4. *Let A be a subset of a topological space (X, τ). If A is g-(Λ, sp)-closed and $A \subseteq V \in \Lambda_{sp}O(X, \tau)$, then $\Lambda_{sp}Fr(V) \subseteq [X - A]_{(\Lambda,sp)}$.*

Proof. Let A be g-(Λ, sp)-closed and $A \subseteq V \in \Lambda_{sp}O(X, \tau)$. Then, $A^{(\Lambda,sp)} \subseteq V$. Let $x \in \Lambda_{sp}Fr(V)$. Since $V \in \Lambda_{sp}O(X, \tau)$, we have $\Lambda_{sp}Fr(V) = V^{(\Lambda,sp)} - V$. Thus, $x \notin V$ and hence $x \notin A^{(\Lambda,sp)}$. Therefore, $x \in [X - A]_{(\Lambda,sp)}$. This shows that $\Lambda_{sp}Fr(V) \subseteq [X - A]_{(\Lambda,sp)}$.

Theorem 5. *Let (X, τ) be a topological space. For each $x \in X$, either $\{x\}$ is (Λ, sp)-closed or g-(Λ, sp)-open.*

Proof. Suppose that $\{x\}$ is not (Λ, sp)-closed. Then, $X - \{x\}$ is not (Λ, sp)-open and the only (Λ, sp)-open set containing $X - \{x\}$ is X itself. Thus, $[X - \{x\}]^{(\Lambda,sp)} \subseteq X$ and hence $X - \{x\}$ is g-(Λ, sp)-closed. Therefore, $\{x\}$ is g-(Λ, sp)-open.

Theorem 6. *Let A be a subset of a topological space (X, τ). Then, A is g-(Λ, sp)-open if and only if $F \subseteq A_{(\Lambda,sp)}$ whenever $F \subseteq A$ and F is (Λ, sp)-closed.*

Proof. Suppose that A is a g-(Λ, sp)-open set. Let F be a (Λ, sp)-closed set and $F \subseteq A$. Then, $X - A \subseteq X - F \in \Lambda_{sp}O(X, \tau)$ and $X - A$ is g-(Λ, sp)-closed. Thus,

$$X - A_{(\Lambda,sp)} = [X - A]^{(\Lambda,sp)} \subseteq X - F$$

and hence $F \subseteq A_{(\Lambda,sp)}$.

Conversely, let $X - A \subseteq U$ and $U \in \Lambda_{sp}O(X, \tau)$. Then, $X - U \subseteq A$ and $X - U$ is (Λ, sp)-closed. By the hypothesis, $X - U \subseteq A_{(\Lambda,sp)}$ and hence $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)} \subseteq U$. Thus, $X - A$ is g-(Λ, sp)-closed. This shows that A is g-(Λ, sp)-open.

Lemma 5. *Let A be a subset of a topological space (X, τ). If $G \in \Lambda_{sp}O(X, \tau)$ and $A \cap G = \emptyset$, then $A^{(\Lambda,sp)} \cap G = \emptyset$.*

Theorem 7. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-(Λ, sp)-closed.

(2) $A^{(\Lambda,sp)} - A$ contains no nonempty (Λ, sp)-closed set.

(3) $A^{(\Lambda,sp)} - A$ is g-(Λ, sp)-open.

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 2.

(2) $\Rightarrow$ (3): Let F be a (Λ, sp)-closed set and $F \subseteq A^{(\Lambda,sp)} - A$. By (2), we have $F = \emptyset$ and $F \subseteq [A^{(\Lambda,sp)} - A]_{(\Lambda,sp)}$. It follows from Theorem 6 that $A^{(\Lambda,sp)} - A$ is g-(Λ, sp)-open.

(3) $\Rightarrow$ (1): Suppose that $A \subseteq U$ and $U \in \Lambda_{sp}O(X, \tau)$. Then, $A^{(\Lambda,sp)} - U \subseteq A^{(\Lambda,sp)} - A$. By (3), we have $A^{(\Lambda,sp)} - A$ is g-(Λ, sp)-open. Since $A^{(\Lambda,sp)} - U$ is (Λ, sp)-closed, by Theorem 6, $A^{(\Lambda,sp)} - U \subseteq [A^{(\Lambda,sp)} - A]_{(\Lambda,sp)} = \emptyset$. Thus, $A^{(\Lambda,sp)} \subseteq U$ and hence A is g-(Λ, sp)-closed. Now, the proof of $[A^{(\Lambda,sp)} - A]_{(\Lambda,sp)} = \emptyset$ is given as follows. Suppose that $[A^{(\Lambda,sp)} - A]_{(\Lambda,sp)} \neq \emptyset$. Then, there exists $x \in [A^{(\Lambda,sp)} - A]_{(\Lambda,sp)}$ and hence there exists $G \in \Lambda_{sp}O(X, \tau)$ such that $x \in G \subseteq A^{(\Lambda,sp)} - A$. Since $G \subseteq X - A$, we have $G \cap A = \emptyset$, by Lemma 5, $G \cap A^{(\Lambda,sp)} = \emptyset$ and hence $G \subseteq X - A^{(\Lambda,sp)}$. Thus, $G \subseteq [X - A^{(\Lambda,sp)}] \cap A^{(\Lambda,sp)} = \emptyset$. This is a contradiction.

Theorem 8. *A subset A of a topological space (X, τ) is g-(Λ, sp)-closed if and only if $F \cap A^{(\Lambda,sp)} = \emptyset$ whenever $A \cap F = \emptyset$ and F is (Λ, sp)-closed.*

Proof. Suppose that A is a (Λ, sp)-closed set. Let F be a (Λ, sp)-closed set and $A \cap F = \emptyset$. Then, $A \subseteq X - F \in \Lambda_{sp}O(X, \tau)$ and $A^{(\Lambda,sp)} \subseteq X - F$. Thus, $F \cap A^{(\Lambda,sp)} = \emptyset$.

Conversely, let $A \subseteq U$ and $U \in \Lambda_{sp}O(X, \tau)$. Then, $A \cap (X - U) = \emptyset$ and $X - U$ is (Λ, sp)-closed. By the hypothesis, $(X - U) \cap A^{(\Lambda,sp)} = \emptyset$ and hence $A^{(\Lambda,sp)} \subseteq U$. Thus, A is g-(Λ, sp)-closed.

Theorem 9. *A subset A of a topological space (X, τ) is g-(Λ, sp)-closed if and only if $A \cap \{x\}^{(\Lambda,sp)} \neq \emptyset$ for every $x \in A^{(\Lambda,sp)}$.*

Proof. Let A be a g-(Λ, sp)-closed set and suppose that there exists $x \in A^{(\Lambda,sp)}$ such that $A \cap \{x\}^{(\Lambda,sp)} = \emptyset$. Thus, $A \subseteq X - \{x\}^{(\Lambda,sp)}$ and hence $A^{(\Lambda,sp)} \subseteq X - \{x\}^{(\Lambda,sp)}$. Therefore, $x \notin A^{(\Lambda,sp)}$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any (Λ, sp)-open set containing A. Let $x \in A^{(\Lambda,sp)}$. By the hypothesis, $A \cap A^{(\Lambda,sp)} \neq \emptyset$, so there exists $y \in A \cap \{x\}^{(\Lambda,sp)}$ and hence $y \in A \subseteq U$. Thus, $\{x\} \cap U \neq \emptyset$. Therefore, $x \in U$, which implies that $A^{(\Lambda,sp)} \subseteq U$. This shows that A is g-(Λ, sp)-closed.

Corollary 3. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-(Λ, sp)-open.

(2) $A - A_{(\Lambda,sp)}$ *does not contain any nonempty* (Λ, sp)*-closed set.*

(3) $(X - A) \cap \{x\}^{(\Lambda,sp)} \neq \emptyset$ *for every* $x \in A - A_{(\Lambda,sp)}$.

Theorem 10. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) For every (Λ, sp)*-open set* U *of* X, $U^{(\Lambda,sp)} \subseteq U$.

(2) Every subset of X *is g-*(Λ, sp)*-closed.*

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $A \subseteq U \in \Lambda_{sp}O(X, \tau)$. By (1), $U^{(\Lambda,sp)} \subseteq U$ and hence $A^{(\Lambda,sp)} \subseteq U^{(\Lambda,sp)} \subseteq U$. Thus, A is g-(Λ, sp)-closed.

(2) $\Rightarrow$ (1): Let $U \in \Lambda_{sp}O(X, \tau)$. By (2), U is g-(Λ, sp)-closed and hence $U^{(\Lambda,sp)} \subseteq U$.

Theorem 11. *A subset* A *of a topological space* (X, τ) *is g-*(Λ, sp)*-open if and only if* $U = X$ *whenever* U *is* (Λ, sp)*-open and* $(X - A) \cap A_{(\Lambda,sp)} \subseteq U$.

Proof. Suppose that A is g-(Λ, sp)-open and $U \in \Lambda_{sp}O(X, \tau)$ such that

$$(X - A) \cap A_{(\Lambda,sp)} \subseteq U.$$

Thus, $X - U \subseteq [X - A_{(\Lambda,sp)}] \cap A$ and hence $X - U \subseteq [X - A]^{(\Lambda,sp)} - (X - A)$. Since $X - A$ is g-(Λ, sp)-closed and $X - U$ is (Λ, sp)-closed, by Theorem 2, $X - U = \emptyset$. This shows that $X = U$.

Conversely, suppose that $F \subseteq A$ and F is (Λ, sp)-closed. By Lemma 4,

$$(X - A) \cup A_{(\Lambda,sp)} \subseteq (X - F) \cup A_{(\Lambda,sp)} \in \Lambda_{sp}O(X, \tau).$$

By the hypothesis, we have $X = (X - F) \cup A_{(\Lambda,sp)}$ and hence

$$F = F \cap [(X - F) \cup A_{(\Lambda,sp)}] = F \cap A_{(\Lambda,sp)} \subseteq A_{(\Lambda,sp)}.$$

It follows from Theorem 6 that A is g-(Λ, sp)-open.

Theorem 12. *Let* A *be a subset of a topological space* (X, τ)*. If* A *is g-*(Λ, sp)*-open and* $A_{(\Lambda,sp)} \subseteq B \subseteq A$*, then* B *is g-*(Λ, sp)*-open.*

Proof. We have $X - A \subseteq X - B \subseteq X - A_{(\Lambda,sp)} = [X - A]^{(\Lambda,sp)}$. Since $X - A$ is g-(Λ, sp)-closed, it follows from Theorem 3 that $X - B$ is g-(Λ, sp)-closed and hence B is g-(Λ, sp)-open.

Definition 4. [2] *A subset* A *of a topological space* (X, τ) *is said to be locally* (Λ, sp)*-closed if* $A = U \cap F$*, where* $U \in \Lambda_{sp}O(X, \tau)$ *and* F *is a* (Λ, sp)*-closed set.*

Lemma 6. [2] *For a subset* A *of a topological space* (X, τ)*, the following properties are equivalent:*

(1) A *is locally* (Λ, sp)*-closed;*

(2) $A = U \cap A^{(\Lambda,sp)}$ *for some* $U \in \Lambda_{sp}O(X, \tau)$*;*

(3) $A^{(\Lambda,sp)} - A$ *is* (Λ, sp)*-closed;*

(4) $A \cup [X - A^{(\Lambda,sp)}] \in \Lambda_{sp}O(X, \tau)$*;*

(5) $A \subseteq [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)}$.

Theorem 13. *A subset* A *of a topological space* (X, τ) *is* (Λ, sp)*-closed if and only if* A *is locally* (Λ, sp)*-closed and g-*(Λ, sp)*-closed.*

Proof. Let A be a (Λ, sp)-closed set. By Theorem 3, A is g-(Λ, sp)-closed. Since X is (Λ, sp)-open and $A = X \cap A$, A is locally (Λ, sp)-closed.

Conversely, suppose that A is locally (Λ, sp)-closed and g-(Λ, sp)-closed. Since A is locally (Λ, sp)-closed, by Lemma 6, $A \subseteq [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)}$. Since

$$[A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)} \in \Lambda_{sp}O(X, \tau)$$

and A is g-(Λ, sp)-closed, we have $A^{(\Lambda,sp)} \subseteq [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)} \subseteq A \cup [X - A^{(\Lambda,sp)}]$ and hence $A^{(\Lambda,sp)} = A$. Thus, by Lemma 3, A is (Λ, sp)-closed.

Definition 5. *Let* A *be a subset of a topological space* (X, τ)*. A subset* $\Lambda_{(\Lambda,sp)}(A)$ *is defined as follows:* $\Lambda_{(\Lambda,sp)}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda_{sp}O(X, \tau)\}$.

Lemma 7. *For subsets* A, B *of a topological space* (X, τ)*, the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,sp)}(A)$.

(2) If $A \subseteq B$*, then* $\Lambda_{(\Lambda,sp)}(A) \subseteq \Lambda_{(\Lambda,sp)}(B)$.

(3) $\Lambda_{(\Lambda,sp)}[\Lambda_{(\Lambda,sp)}(A)] = \Lambda_{(\Lambda,sp)}(A)$.

(4) If A *is* (Λ, sp)*-open,* $\Lambda_{(\Lambda,sp)}(A) = A$.

A subset N_x of a topological space (X, τ) is said to be (Λ, sp)-*neighbourhood* of a point $x \in X$ if there exists a (Λ, sp)-open set U such that $x \in U \subseteq N_x$.

Lemma 8. *A subset* A *of a topological space* (X, τ) *is* (Λ, sp)*-open in* X *if and only if* A *is a* (Λ, sp)*-neighbourhood of each point of* A*.*

Definition 6. *Let* (X, τ) *be a topological space and* $x \in X$*. A subset* $\langle x\rangle_{sp}$ *is defined as follows:* $\langle x\rangle_{sp} = \Lambda_{(\Lambda,sp)}(\{x\}) \cap \{x\}^{(\Lambda,sp)}$.

Theorem 14. *Let* (X, τ) *be a topological space. Then, the following properties hold:*

(1) $\Lambda_{(\Lambda,sp)}(A) = \{x \in X \mid A \cap \{x\}^{(\Lambda,sp)} \neq \emptyset\}$ *for each subset* A *of* X.

(2) For each $x \in X$, $\Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp}) = \Lambda_{(\Lambda,sp)}(\{x\})$.

(3) For each $x \in X$, $(\langle x\rangle_{sp})^{(\Lambda,sp)} = \{x\}^{(\Lambda,sp)}$.

(4) If U is (Λ, sp)-open in X and $x \in U$, then $\langle x\rangle_{sp} \subseteq U$.

(5) If F is (Λ, sp)-closed in X and $x \in F$, then $\langle x\rangle_{sp} \subseteq F$.

Proof. (1) Suppose that $A \cap \{x\}^{(\Lambda,sp)} = \emptyset$. Then, $x \notin X - \{x\}^{(\Lambda,sp)}$ which is a (Λ, sp)-open set containing A. Thus, $x \notin \Lambda_{(\Lambda,sp)}(A)$ and hence

$$\Lambda_{(\Lambda,sp)}(A) \subseteq \{x \in X \mid A \cap \{x\}^{(\Lambda,sp)} \neq \emptyset\}.$$

Next, let $x \in X$ such that $A \cap \{x\}^{(\Lambda,sp)} \neq \emptyset$ and suppose that $x \notin \Lambda_{(\Lambda,sp)}(A)$. Then, there exists a (Λ, sp)-open set U containing A and $x \notin U$. Let $y \in A \cap \{x\}^{(\Lambda,sp)}$. Therefore, U is a (Λ, sp)-neighbourhood of y which does not contain x. By this contradiction $x \in \Lambda_{(\Lambda,sp)}(A)$.

(2) Let $x \in X$. Then, we have $\{x\} \subseteq \{x\}^{(\Lambda,sp)} \cap \Lambda_{(\Lambda,sp)}(\{x\}) = \langle x\rangle_{sp}$. By Lemma 7, we obtain $\Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp})$. Next, we show the opposite implication. Suppose that $y \notin \Lambda_{(\Lambda,sp)}(\{x\})$. Then, there exists a (Λ, sp)-open set V such that $x \in V$ and $y \notin V$. Since $\langle x\rangle_{sp} \subseteq \Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \Lambda_{(\Lambda,sp)}(V) = V$, we have $\Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp}) \subseteq V$. Since $y \notin V, y \notin \Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp})$. Consequently, we have $\Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp}) \subseteq \Lambda_{(\Lambda,sp)}(\{x\})$ and hence $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\langle x\rangle_{sp})$.

(3) By the definition of $\langle x\rangle_{sp}$, we have $\{x\} \subseteq \langle x\rangle_{sp}$ and $\{x\}^{(\Lambda,sp)} \subseteq (\langle x\rangle_{sp})^{(\Lambda,sp)}$ by Lemma 3. On the other hand, we have $\langle x\rangle_{sp} \subseteq \{x\}^{(\Lambda,sp)}$ and

$$(\langle x\rangle_{sp})^{(\Lambda,sp)} \subseteq (\{x\}^{(\Lambda,sp)})^{(\Lambda,sp)} = \{x\}^{(\Lambda,sp)}.$$

Thus, $(\langle x\rangle_{sp})^{(\Lambda,sp)} \subseteq \{x\}^{(\Lambda,sp)}$.

(4) Since $x \in U$ and U is a (Λ, sp)-open set, we have $\Lambda_{(\Lambda,sp)}(\{x\}) \subseteq U$. Thus, $\langle x\rangle_{sp} \subseteq U$.

(5) Since $x \in F$ and F is a (Λ, sp)-closed set,

$$\langle x\rangle_{sp} = \{x\}^{(\Lambda,sp)} \cap \Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \{x\}^{(\Lambda,sp)} \subseteq F^{(\Lambda,sp)} = F.$$

Theorem 15. *For any points x and y in a topological space (X, τ), the following properties are equivalent:*

(1) $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$.

(2) $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$. Then, there exists a point $z \in X$ such that $z \in \Lambda_{(\Lambda,sp)}(\{x\})$ and $z \notin \Lambda_{(\Lambda,sp)}(\{y\})$ or $z \in \Lambda_{(\Lambda,sp)}(\{y\})$ and $z \notin \Lambda_{(\Lambda,sp)}(\{x\})$. We prove only the first case being the second analogous. From $z \in \Lambda_{(\Lambda,sp)}(\{x\})$ it follows that $\{x\} \cap \{z\}^{(\Lambda,sp)} \neq \emptyset$ which implies $x \in \{z\}^{(\Lambda,sp)}$. By $z \notin \Lambda_{(\Lambda,sp)}(\{y\})$, we have $\{y\} \cap \{z\}^{(\Lambda,sp)} = \emptyset$. Since $x \in \{z\}^{(\Lambda,sp)}$, $\{x\}^{(\Lambda,sp)} \subseteq \{z\}^{(\Lambda,sp)}$ and

$\{y\} \cap \{x\}^{(\Lambda,sp)} = \emptyset$. Therefore, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Thus, $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$ and hence $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$. Then, there exists a point $z \in X$ such that $z \in \{x\}^{(\Lambda,sp)}$ and $z \notin \{y\}^{(\Lambda,sp)}$ or $z \in \{y\}^{(\Lambda,sp)}$ and $z \notin \{x\}^{(\Lambda,sp)}$. We prove only the first case being the second analogous. It follows that there exists a (Λ, sp)-open set containing z and therefore x but not y, namely, $y \notin \Lambda_{(\Lambda,sp)}(\{x\})$ and thus $\Lambda_{(\Lambda,sp)}(\{x\}) \neq \Lambda_{(\Lambda,sp)}(\{y\})$.

Theorem 16. *Let (X, τ) be a topological space and $x, y \in X$. Then, the following properties hold:*

(1) $y \in \Lambda_{(\Lambda,sp)}(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,sp)}$.

(2) $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\})$ if and only if $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$.

Proof. (1) Let $x \notin \{y\}^{(\Lambda,sp)}$. Then, there exists $U \in \Lambda_{sp}O(X, \tau)$ such that $x \in U$ and $y \notin U$. Thus, $y \notin \Lambda_{(\Lambda,sp)}(\{x\})$. The converse is similarly shown.

(2) Suppose that $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\})$ for any $x, y \in X$. Since $x \in \Lambda_{(\Lambda,sp)}(\{x\})$, $x \in \Lambda_{(\Lambda,sp)}(\{y\})$, by (1), $y \in \{x\}^{(\Lambda,sp)}$. By Lemma 3, $\{y\}^{(\Lambda,sp)} \subseteq \{x\}^{(\Lambda,sp)}$. Similarly, we have $\{x\}^{(\Lambda,sp)} \subseteq \{y\}^{(\Lambda,sp)}$ and hence $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$.

Conversely, suppose that $\{x\}^{(\Lambda,sp)} = \{y\}^{(\Lambda,sp)}$. Since $x \in \{x\}^{(\Lambda,sp)}$, $x \in \{y\}^{(\Lambda,sp)}$, by (1), $y \in \Lambda_{(\Lambda,sp)}(\{x\})$. By Lemma 7, $\Lambda_{(\Lambda,sp)}(\{y\}) \subseteq \Lambda_{(\Lambda,sp)}(\Lambda_{(\Lambda,sp)}(\{x\})) = \Lambda_{(\Lambda,sp)}(\{x\})$. Similarly, we have $\Lambda_{(\Lambda,sp)}(\{x\}) \subseteq \Lambda_{(\Lambda,sp)}(\{y\})$ and hence $\Lambda_{(\Lambda,sp)}(\{x\}) = \Lambda_{(\Lambda,sp)}(\{y\})$.

Definition 7. *A subset A of a topological space (X, τ) is called a $\Lambda_{(\Lambda,sp)}$-set if $A = \Lambda_{(\Lambda,sp)}(A)$.*

The family of all $\Lambda_{(\Lambda,sp)}$-sets of a topological space (X, τ) is denoted by $\Lambda_{(\Lambda,sp)}(X, \tau)$ (or simply $\Lambda_{(\Lambda,sp)}$).

Definition 8. *A subset A of a topological space (X, τ) is called a generalized $\Lambda_{(\Lambda,sp)}$-set (briefly g-$\Lambda_{(\Lambda,sp)}$-set) if $\Lambda_{(\Lambda,sp)}(A) \subseteq F$ whenever $A \subseteq F$ and F is a (Λ, sp)-closed set.*

Definition 9. *A topological space (X, τ) is called a Λ_{sp}-$T_{\frac{1}{2}}$-space if every g-(Λ, sp)-closed set of X is (Λ, sp)-closed.*

Lemma 9. *For a topological space (X, τ), the following properties hold:*

(1) For each $x \in X$, the singleton $\{x\}$ is (Λ, sp)-closed or $X - \{x\}$ is g-(Λ, sp)-closed.

(2) For each $x \in X$, the singleton $\{x\}$ is (Λ, sp)-open or $X - \{x\}$ is a g-$\Lambda_{(\Lambda,sp)}$-set.

Proof. (1) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, sp)-closed. Then, $X - \{x\}$ is not (Λ, sp)-open and X is the only (Λ, sp)-open set which contains $X - \{x\}$ and hence $X - \{x\}$ is g-(Λ, sp)-closed.

(2) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, sp)-open. Then, $X - \{x\}$ is not (Λ, sp)-closed and X is the only (Λ, sp)-closed set which contains $X - \{x\}$ and hence $X - \{x\}$ is a g-$\Lambda_{(\Lambda,sp)}$-set.

Theorem 17. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a Λ_{sp}-$T_{\frac{1}{2}}$-space.

(2) For each $x \in X$, the singleton $\{x\}$ is (Λ, sp)-open or (Λ, sp)-closed.

(3) Every g-$\Lambda_{(\Lambda,sp)}$-set is a $\Lambda_{(\Lambda,sp)}$-set.

Proof. (1) $\Rightarrow$ (2): By Lemma 9, for each $x \in X$, the singleton $\{x\}$ is (Λ, sp)-closed or $X - \{x\}$ is g-(Λ, sp)-closed. Since (X, τ) is a Λ_{sp}-$T_{\frac{1}{2}}$-space, $X - \{x\}$ is (Λ, sp)-closed and hence $\{x\}$ is (Λ, sp)-open in the latter case. Thus, the singleton $\{x\}$ is (Λ, sp)-open or (Λ, sp)-closed.

(2) $\Rightarrow$ (3): Suppose that there exists a g-$\Lambda_{(\Lambda,sp)}$-set A which is not a $\Lambda_{(\Lambda,sp)}$-set. There exists $x \in \Lambda_{(\Lambda,sp)}(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is (Λ, sp)-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, sp)-closed. Since A is a g-$\Lambda_{(\Lambda,sp)}$-set, $\Lambda_{(\Lambda,sp)}(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is (Λ, sp)-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, sp)-open. By Lemma 7, $\Lambda_{(\Lambda,sp)}(A) \subseteq \Lambda_{(\Lambda,sp)}(X - \{x\}) = X - \{x\}$. This is a contradiction. Thus, every g-$\Lambda_{(\Lambda,sp)}$-set is a $\Lambda_{(\Lambda,sp)}$-set.

(3) $\Rightarrow$ (1): Suppose that (X, τ) is not a Λ_{sp}-$T_{\frac{1}{2}}$-space. Then, there exists a g-(Λ, sp)-closed set A which is not (Λ, sp)-closed. Since A is not (Λ, sp)-closed, there exists a point $x \in A^{(\Lambda,sp)}$ such that $x \notin A$. By Lemma 9, the singleton $\{x\}$ is (Λ, sp)-open or $X - \{x\}$ is a $\Lambda_{(\Lambda,sp)}$-set. (a) In case $\{x\}$ is (Λ, sp)-open, since $x \in A^{(\Lambda,sp)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda_{(\Lambda,sp)}$-set, if $\{x\}$ is not (Λ, sp)-closed, $X - \{x\}$ is not (Λ, sp)-open and $\Lambda_{(\Lambda,sp)}(X - \{x\}) = X$. Thus, $X - \{x\}$ is not a $\Lambda_{(\Lambda,sp)}$-set. This contradicts (3). If $\{x\}$ is (Λ, sp)-closed, $A \subseteq X - \{x\} \in \Lambda_{sp}O(X, \tau)$ and A is g-(Λ, sp)-closed. Hence, we have $A^{(\Lambda,sp)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda,sp)}$. This shows that (X, τ) is a Λ_{sp}-$T_{\frac{1}{2}}$-space.

4. An application of generalized (Λ, sp)-closed sets

In this section, we introduce the notion of Λ_{sp}-normal spaces and investigate several characterizations of Λ_{sp}-normal spaces.

Definition 10. *A topological space (X, τ) is said to be Λ_{sp}-normal if, for any pair of disjoint (Λ, sp)-closed sets F and H, there exist disjoint (Λ, sp)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.*

Lemma 10. *Let (X, τ) be a topological space. If U is (Λ, sp)-open in X, then*

$$U^{(\Lambda,sp)} \cap A \subseteq [U \cap A]^{(\Lambda,sp)}$$

for every subset A of X.

Theorem 18. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-normal.

(2) For every pair of (Λ, sp)-open sets U and V whose union is X, there exist (Λ, sp)-closed sets F and H such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(3) For every (Λ, sp)-closed set F and every (Λ, sp)-open set G containing F, there exists a (Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(4) For every pair of disjoint (Λ, sp)-closed sets F and H, there exist disjoint (Λ, sp)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$ and $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let U and V be any pair of (Λ, sp)-open sets in X such that $X = U \cup V$. Then, $X - U$ and $X - V$ are disjoint (Λ, sp)-closed sets. Since (X, τ) is Λ_{sp}-normal, there exist disjoint (Λ, sp)-open sets G and W such that $X - U \subseteq G$ and $X - V \subseteq W$. Put $F = X - G$ and $H = X - W$. Then, F and H are (Λ, sp)-closed sets such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(2) $\Rightarrow$ (3): Let F be a (Λ, sp)-closed set and let G be a (Λ, sp)-open set containing F. Then, $X - F$ and G are (Λ, sp)-open sets whose union is X. Then by (2), there exist (Λ, sp)-closed sets M and N such that $M \subseteq X - F$, $N \subseteq G$ and $M \cup N = X$. Then, $F \subseteq X - M$, $X - G \subseteq X - N$ and $(X - M) \cap (X - N) = \emptyset$. Put $U = X - M$ and $V = X - N$. Then, U and V are disjoint (Λ, sp)-open sets such that $F \subseteq U \subseteq X - V \subseteq G$. As $X - V$ is a (Λ, sp)-closed set, we have $U^{(\Lambda,sp)} \subseteq X - V$ and $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(3) $\Rightarrow$ (4): Let F and H be two disjoint (Λ, sp)-closed sets of X. Then, $F \subseteq X - H$ and $X - H$ is (Λ, sp)-open, by (3), there exists a (Λ, sp)-open set U of X such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq X - H$. Put $V = X - U^{(\Lambda,sp)}$. Then, U and V are disjoint (Λ, sp)-open sets of X such that $F \subseteq U$, $H \subseteq V$ and $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.

(4) $\Rightarrow$ (1): The proof is obvious.

Theorem 19. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-normal.

(2) For any pair of disjoint (Λ, sp)-closed sets F and H, there exist disjoint g-(Λ, sp)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.

(3) For each (Λ, sp)-closed set F and each (Λ, sp)-open set G containing F, there exists a g-(Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(4) For each (Λ, sp)-closed set F and each g-(Λ, sp)-open set G containing F, there exists a (Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G_{(\Lambda,sp)}$.

(5) For each (Λ, sp)-closed set F and each g-(Λ, sp)-open set G containing F, there exists a g-(Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G_{(\Lambda,sp)}$.

(6) For each g-(Λ, sp)-closed set F and each (Λ, sp)-open set G containing F, there exists a (Λ, sp)-open set U such that $F^{(\Lambda,sp)} \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(7) For each g-(Λ, sp)-closed set F and each (Λ, sp)-open set G containing F, there exists a g-(Λ, sp)-open set U such that $F^{(\Lambda,sp)} \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let F be a (Λ, sp)-closed set and let G be a (Λ, sp)-open set containing F. Then, F and $X-G$ are two disjoint (Λ, sp)-closed sets. Hence by (2), there exist disjoint g-(Λ, sp)-open sets U and V of X such that $F \subseteq U$ and $X-G \subseteq V$. Since V is g-(Λ, sp)-open and $X-G$ is (Λ, sp)-closed, by Theorem 6, $X-G \subseteq V_{(\Lambda,sp)}$. Thus, $[X-V]^{(\Lambda,sp)} = X - V_{(\Lambda,sp)} \subseteq G$ and hence $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$.

(3) $\Rightarrow$ (1): Let F and H be two disjoint (Λ, sp)-closed sets of X. Then, F is a (Λ, sp)-closed set and $X-H$ is a (Λ, sp)-open set containing F, by (3), there exists a g-(Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq X-H$. Thus, by Theorem 6, $F \subseteq U_{(\Lambda,sp)}$, $H \subseteq X - U^{(\Lambda,sp)}$, where $U_{(\Lambda,sp)}$ and $X - U^{(\Lambda,sp)}$ are two disjoint (Λ, sp)-open sets.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2): The proofs are obvious.

(6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (5): Let F be a (Λ, sp)-closed set and let G be a g-(Λ, sp)-open set containing F. Since G is g-(Λ, sp)-open and F is (Λ, sp)-closed, by Theorem 6, $F \subseteq G_{(\Lambda,sp)}$. Thus, by (3), there exists a g-(Λ, sp)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G_{(\Lambda,sp)}$.

(5) $\Rightarrow$ (6): Let F be a g-(Λ, sp)-closed set and let G be a (Λ, sp)-open set containing F. Then, we have $F^{(\Lambda,sp)} \subseteq G$. Since G is g-(Λ, sp)-open and $F^{(\Lambda,sp)}$ is (Λ, sp)-closed, by (5), there exists a g-(Λ, sp)-open set U such that $F^{(\Lambda,sp)} \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G$. Since U is g-(Λ, sp)-open and $F^{(\Lambda,sp)} \subseteq U$, by Theorem 6, $F^{(\Lambda,sp)} \subseteq U_{(\Lambda,sp)}$. Put $V = U_{(\Lambda,sp)}$. Then, V is (Λ, sp)-open and $F^{(\Lambda,sp)} \subseteq V \subseteq V^{(\Lambda,sp)} = [U_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq U^{(\Lambda,sp)} \subseteq G$.

(6) $\Rightarrow$ (4): Let F be a (Λ, sp)-closed set and let G be a g-(Λ, sp)-open set containing F. Thus, by Theorem 6, $F^{(\Lambda,sp)} = F \subseteq G_{(\Lambda,sp)}$. Since F is g-(Λ, sp)-closed and $G_{(\Lambda,sp)}$ is (Λ, sp)-open, by (6), there exists a (Λ, sp)-open set U such that

$$F^{(\Lambda,sp)} \subseteq U \subseteq U^{(\Lambda,sp)} \subseteq G_{(\Lambda,sp)}.$$

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