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# Exact solution for nonlinear oscillators with coordinate-dependent mass

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**Abstract.** In this work, we aim to obtain an exact solution for a nonlinear oscillator with coordinate position- dependent mass. The equation of motion of the nonlinear oscillator under investigation becomes exact after making reduction of order. The obtained solution was expressed in terms of position and time. Initial conditions were applied, in addition to modified initial condition. Finally, fixed points where studied with their stability, and some plots describing the system where presented.

**2020 Mathematics Subject Classifications**: 34D10, 34L15, 34D20, 37C25, 31A15, 37M05,

**Key Words and Phrases**: Homotopoty Perturbation, Equilibrium (fixed) points, Simulation, stability, Conservative, potential function,Exact differential Equation, Reduction of order, Center point

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## 1. Introduction

Simple oscillating systems are modeled in general as a mass attached to a spring (i.e., simple oscillators). The equation of motion describing such systems are obtained either using Newtonian mechanics or Lagrangian method, and it can be solved exactly in some simple cases. Unfortunately, no such systems present in the macroscopic world and this is due to dissipative forces that are always present in nature. Dissipative forces can be ignored if they have small effects, but in many cases they lead to damping oscillators. Linear oscillators are those that oscillate with one frequency and its motion is sinusoidal and periodic, for more information related to oscillators (simple and damped) we advise interested people to refer to some classical mechanics texts [5, 9, 10].

Nonlinear oscillators result in complex motion and there are mainly two important features for such systems: as the amplitude increases then the non linearity motion becomes more important, and in some cases, the frequency will change with amplitude. In real

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world one can find many such nonlinear oscillators and one has to note that coupled nonlinear oscillators are a subject founded in many branches of science as: biology, physics, and many others. In literature there are a lot of efforts paid on studying these systems [13, 16, 20]. An important example is the van der Pol oscillator which is an oscillator with nonlinear damping introduced in the 1920's by Balthasar van der Pol (1889 - 1959).The van der Pol oscillator is considered as an example of an oscillator with nonlinear damping, energy being dissipated at large amplitudes and generated as low amplitude, and it attracts the attention of many researchers where many method have been applied in dealing with this oscillator either analytically using Homotopy analysis method (HAM) as in[4, 14]. Homotopy perturbation method (HPM) as in [18] or numerically using for example perturbation algorithm combining the method of Multiple Scales and Modified Lindstedt–Poincare Techniques as in [15], A domain decomposition method (ADM) as in [2, 8] and many other methods. Nonlinear oscillations have been of paramount importance in practical engineering, physics, applied mathematics, and several real-world requirements for many years. In literature, one can find many various analytical approaches for solving nonlinear systems, such as the iteration perturbation method [7], the homotopy perturbation method (HMP) [21], the variational method [17], and many other methods[6]. Interested researchers in this topic can refer to [12, 12] therein. In principle, the solution for such nonlinear oscillators is difficult to obtain analytically and researchers resort to use different numerical methods [1, 3, 21, 21]. In [21] the authors consider a nonlinear oscillator with coordinate-dependent mass, where they proposed a nonlinear oscillator with negative coefficient of linear term (see Eq. 3 in [21]) and apply the homotopy perturbation method to find an approximate period for their equation. In this paper we are going to find an exact solution for the above equation in section 2, while in section 3 the exact solution with modifying in the potential function will be presented and explained, an equilibrium points of the system and their stability with graphical simulation are given finally close the paper with a conclusion .

## 2. Exact solution for nonlinear oscillators with coordinate-dependent mass

Consider the following equation

$$(1 + \alpha x^2)\ddot{x} + \alpha x\dot{x}^2 - x(1 - x^2) = 0 \tag{1}$$

An important property of (1) is the frequency- amplitude relationship, where in principle frequency of a nonlinear oscillating system as (1) is nonlinearly related to its amplitude. This property was the aim of study of many researchers such as [12], where he suggests a direct frequency estimate method for nonlinear oscillators with arbitrary initial conditions and the method used is based on the work presented in [11], [19] shows that it can be greatly interesting for nonlinear oscillations.Firstly, we make reduction of order for the differential equation (1). Let $v = \frac{dx}{dt}$ then $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v\frac{dv}{dx}$,so we transform equation(1) into the following first order nonlinear differential equation

$$(1+\alpha x^2)v\frac{dv}{dx}+\alpha x v^2 - x(1-x^2)=0$$

which can be rewriiten as:

$$(1+\alpha x^2)vdv+\left(\alpha x v^2 - x(1-x^2)\right)dx=0 \tag{2}$$

Now, we will discuss the exactness solution of equation(2) (i.e., show if it is conservative or not). Letting $M(x,v)=(1+\alpha x^2)v$, $N(x,v)=\alpha x v^2 - x(1-x^2)$, then

$$\frac{\partial M}{\partial x}=2\alpha x v$$

and

$$\frac{\partial N}{\partial v}=2\alpha x v$$

This means that equation (2) is exact (conservative). According to differential equations anslysis there must exists a potential function $\psi(x,v)$ such that.

$$\begin{aligned}\psi(x,v) &= (1+\alpha x^2)vdv \\ &= \frac{1}{2}(1+\alpha x^2)v^2+f(x)\end{aligned}$$

where $f(x)$ is a constant function respect to $v$. In the following we aim to find $f(x)$

$$\frac{\partial\psi(x,v)}{\partial x}=N(x,v)$$

Thus

$$\alpha x v^2+\frac{df}{dx}=\alpha x v^2 - x(1-x^2).$$

This means that:

$$\begin{aligned}f(x) &= -x(1-x^2)dx \\ &= \frac{-x^2}{2}+\frac{x^4}{4}+c\end{aligned}$$

Therefore the potential function $\psi(x,v)$ becomes

$$\psi(x,v)=\frac{1}{2}(1+\alpha x^2)v^2-\frac{x^2}{2}+\frac{x^4}{4}+c$$

Set

$$\frac{1}{2}(1+\alpha x^2)v^2-\frac{x^2}{2}+\frac{x^4}{4}=c$$

Now substituting the initial conditions $x(0)=A$, and $\dot{x}(0)=0=v$, the potential function is

$$\psi(x,v)=\frac{1}{2}(1+\alpha x^2)v^2-\frac{x^2}{2}+\frac{x^4}{4}+\frac{A^2}{2}-\frac{A^4}{4}$$

Setting $c = 0$ ( arbitrary constant) in $\psi(x, v)$. Then we have

$$\begin{aligned} v^2 &= \frac{x^2 - \frac{x^4}{2}}{(1 + \alpha x^2)} \\ \frac{dx}{dt} &= v = \sqrt{\frac{2x^2 - x^4}{2(1 + \alpha x^2)}} \\ \frac{dt}{dx} &= \sqrt{\frac{2(1 + \alpha x^2)}{2x^2 - x^4}} \end{aligned}$$

Integrating the above equation to get

$$t = \sqrt{2}\frac{1}{x}\sqrt{\frac{(1 + \alpha x^2)}{2 - x^2}}dx$$

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$$\begin{aligned} t &= \sqrt{2}\frac{1}{x}\sqrt{\frac{(1 + \alpha x^2 + \alpha - \alpha)}{1 + 1 - x^2}}dx \\ t &= \sqrt{2}\frac{1}{x}\sqrt{\frac{1 + \alpha - \alpha(1 - x^2)}{1 + (1 - x^2)}}dx \end{aligned}$$

Now $y = 1 - x^2$, $dy = -2xdx$, and $dx = \frac{-dy}{\sqrt{1-y}}$

$$t = -\frac{1}{\sqrt{2}}\frac{1}{1 - y}\sqrt{\frac{1 + \alpha(1 - y)}{1 + y}}dy$$

We can write the integration as

$$t = -\frac{1}{\sqrt{2}}\frac{1}{1 - y}\sqrt{\frac{1 + 2\alpha}{1 + y} - \alpha}dy$$

Again, let $w = \frac{1+2\alpha}{1+y} - \alpha$,then $y = \frac{1+2\alpha}{w+\alpha} - 1$, and $dy = \frac{-(1+2\alpha)}{(w+\alpha)^2}dw$, the equation integral reads

$$\begin{aligned} t &= \frac{1}{\sqrt{2}}\frac{1}{\frac{2w-1}{w+\alpha}}\sqrt{w}\frac{(1 + 2\alpha)}{(w + \alpha)^2}dw \\ t &= \frac{(1 + 2\alpha)}{\sqrt{2}}\frac{\sqrt{w}}{(2w - 1)(w + \alpha)}dw \end{aligned}$$

Substituting $u = \sqrt{w}$, $du = \frac{1}{2u}dw$, then

$$t = \frac{(1 + 2\alpha)}{\sqrt{2}}\frac{2u^2}{(2u^2 - 1)(u^2 + \alpha)}du$$

$$= (1+2\alpha)\sqrt{2}\frac{u^2}{(2u^2-1)(u^2+\alpha)}du$$

Making use of partial fraction

$$t = \frac{1}{2}\ln\left(\frac{\sqrt{2}u-1}{\sqrt{2}u+1}\right) + \sqrt{\frac{2}{\alpha}}\tan^{-1}(\frac{u}{\sqrt{\alpha}}) + c$$

But $u = \sqrt{w} = \sqrt{\frac{1+2\alpha}{1+y} - \alpha} = \sqrt{\frac{1+\alpha x^2}{2-x^2}}$, which imply to the solution

$$t = \frac{1}{2}\ln\left(\frac{\sqrt{\frac{2(1+\alpha x^2)}{2-x^2}}-1}{\sqrt{\frac{2(1+\alpha x^2)}{2-x^2}}+1}\right) + \sqrt{\frac{2}{\alpha}}\tan^{-1}\left(\sqrt{\frac{2\left(1+\alpha x^2\right)}{\alpha\left(2-x^2\right)}}\right) + c \tag{3}$$

using $x(0) = A$, we get

$$c = \frac{1}{2}\ln\left(\frac{\sqrt{\frac{2(1+\alpha A^2)}{2-A^2}}+1}{\sqrt{\frac{2(1+\alpha A^2)}{2-A^2}}-1}\right) - \sqrt{\frac{2}{\alpha}}\tan^{-1}\left(\sqrt{\frac{2\left(1+\alpha A^2\right)}{\alpha\left(2-A^2\right)}}\right)$$

The solution of equation (3) has a discrete two motional oscillation when $2 - x^2 > 0$ ,$|x| < \sqrt{2}$ Otherwise we have the two motional oscillation intersect, provided that $\alpha$ is positive in the term $\sqrt{\frac{2}{\alpha}}$.

## 3. The exact solution with modifying in the potential function

In section 2, the following scalar function was obtained

$$\psi(x,v) = \frac{1}{2}(1+\alpha x^2)v^2 - \frac{x^2}{2} + \frac{x^4}{4} + \frac{A^2}{2} - \frac{A^4}{4} \tag{4}$$

If we replace the constant position term from the potintial function $\psi(x,v)$ to be propotinal by $\frac{A^2}{2} - \frac{A^4}{4} = \beta x^2$,where $\beta \geq 0$,then equation (4) becomes

$$\psi(x,v) = \frac{1}{2}(1+\alpha x^2)v^2 - \frac{x^2}{2} + \frac{x^4}{4} + \beta x^2$$

So

$$\frac{1}{2}(1+\alpha x^2)v^2 - \frac{x^2}{2} + \frac{x^4}{4} + \beta x^2 = 0$$

$$t = \sqrt{2}\frac{1}{x}\sqrt{\frac{(1+\alpha x^2)}{2(1-2\beta)-x^2}}dx$$

Let $x^2 = u$, and $2(1-2\beta) = \lambda$, then

$$t = \frac{1}{\sqrt{2}}\frac{1}{u}\sqrt{\frac{1+\alpha u}{\lambda - u}}du \tag{5}$$

Integrating the equation (5) as previouse calculation we get

$$t = \frac{1}{\sqrt{2\lambda}}\ln\left(\frac{\sqrt{\lambda}u-1}{\sqrt{\lambda}u+1}\right) + \frac{1}{\sqrt{2\lambda\alpha}}\tan^{-1}(\frac{u}{\sqrt{\alpha}}) + c$$

$$t = \frac{1}{\sqrt{4(1-2\beta)}}\ln\left(\frac{\sqrt{\frac{2(1-2\beta)(1+\alpha x^2)}{2(1-2\beta)-x^2}}-1}{\sqrt{\frac{2(1-2\beta)(1+\alpha x^2)}{2(1-2\beta)-x^2}}+1}\right) + \frac{1}{\sqrt{4\alpha(1-2\beta)}}\tan^{-1}(\sqrt{\frac{2(1-2\beta)\,(1+\alpha x^2)}{2(1-2\beta)-x^2}}) + c$$

Applying the boundary condition $x(0) = A$,we obtain $c = \frac{1}{\sqrt{4(1-2\beta)}}\ln\left(\frac{\sqrt{\frac{2(1-2\beta)\left(1+\alpha A^2\right)}{2(1-2\beta)-A^2}}+1}{\sqrt{\frac{2(1-2\beta)\left(1+\alpha A^2\right)}{2(1-2\beta)-A^2}}-1}\right) - \frac{1}{\sqrt{4\alpha(1-2\beta)}}\tan^{-1}(\sqrt{\frac{2(1-2\beta)(1+\alpha A^2)}{2(1-2\beta)-A^2}})$

It is important to notice that $|x| < \sqrt{4(1-2\beta)} < \sqrt{2}, 0 \leq \beta < \frac{1}{2}$,and $\alpha > 0$ due to the present the following term

$$\sqrt{\frac{2(1-2\beta)\,(1+\alpha x^2)}{2(1-2\beta)-x^2}} > 1$$

Here we have a solution for this equation to make a two discrete oscillate motion (see Figs. 13- 14, when $|x| < \sqrt{2(1-2\beta)} < \sqrt{2}, 0 \leq \beta < \frac{1}{2}$ otherwise we have overlapping by two oscillate motion (see Figs. 2- 4).

## 4. An equilibrium points and their stability with graphical simulation

In this section, we are going to find the equilibrium points and analyse their stability. For this issue convert equation(1) to a system of nonlinear equation by $y_1 = x, y_2 = \dot{x}$,then we get the following two nonlinear system of equation.

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -(\alpha y_1 y_2^2 - y_1(1-y_1^2))/(1+\alpha y_1^2) \end{aligned} \tag{6}$$

Now Let

$$\left[ \begin{array}{c} y_2 \\ -(\alpha y_1 y_2^2 - y_1(1-y_1^2))/(1+\alpha y_1^2) \end{array} = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \right] \tag{7}$$

Solving equations(7) one gets three equilibrium points $(y_1^*, y_2^*) = (0,0)$ ,$(1,0), (-1,0)$. To determine their stability of the equilibrium points we find the Jacobian matrix for the system(6)

$$J = \begin{bmatrix} 0 & 1 \\ -\left[(1+\alpha y_1^2)(\alpha y_2^2 + 3y_1^2 - 1) - (\alpha y_1 y_2^2 - y_1(1-y_1^2))2\alpha y_1\right]/(1+\alpha y_1^2)^2 & -2\alpha y_1 y_2/(1+\alpha y_1^2) \end{bmatrix}$$

At the point $(0,0)$the Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since the charactrestic equation is $\lambda^2 - 1 = 0$ so the eigenvalues are $\lambda_{1,2} = \pm 1$ so the fixed point is unstable as shown in the graphical analysis (see Figs. 2- 3).

For the other two equilibrium points $(\pm 1, 0)$, the Jacobian takes the following form:

$$J = \begin{bmatrix} 0 & 1 \\ \frac{-2}{1+\alpha} & 0 \end{bmatrix}.$$

So $\lambda_{1,2} = \pm\sqrt{\frac{2}{1+\alpha}}i$ This means that the points $(\pm 1, 0)$ are center points when $\alpha \in (-\infty, -1)$ goes to limit cycle and unstable when $\alpha \in (-1, \infty)$ since one of the eigenvalues is positive and the other is negative. Since the fixed points does not depend on the value of $\alpha$ so we can take it any real value so we can make simulation for the solution graphically. Below, we plot the oscillator equation of motion with the following cases.

**Case I**. For fixed point $(0,0)$, where $\alpha$ any real

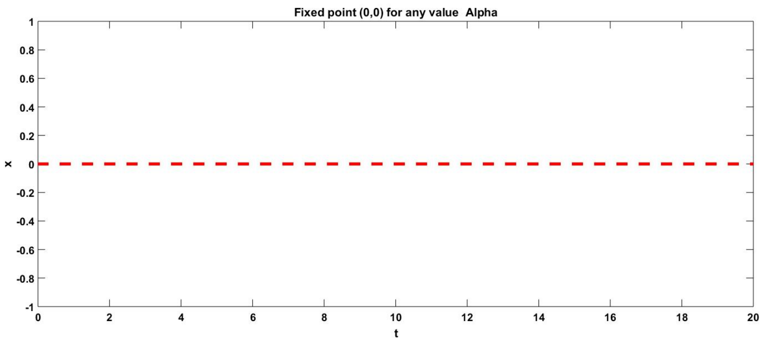


Figure 1: $(0,0)$ $\alpha \in R$.

We see that the fixed point still $(0,0)$despite the changes in $\alpha$ (see Fig. 1) so the fixed point does not depend in $\alpha$, while if we move a little bet from this fixed point oscillation intersect as in (Figs. 2-3) this is show how the fixed point $(0,0)$ unstable.

**CaseII**. We see that after this little change of initial value away from the fixed point the equation will depend in the changes of $\alpha$.

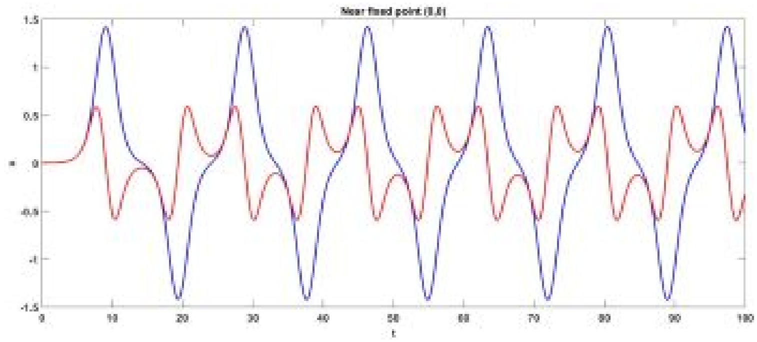


Figure 2: $(0.001, 0)$ $\alpha = 0.5$.

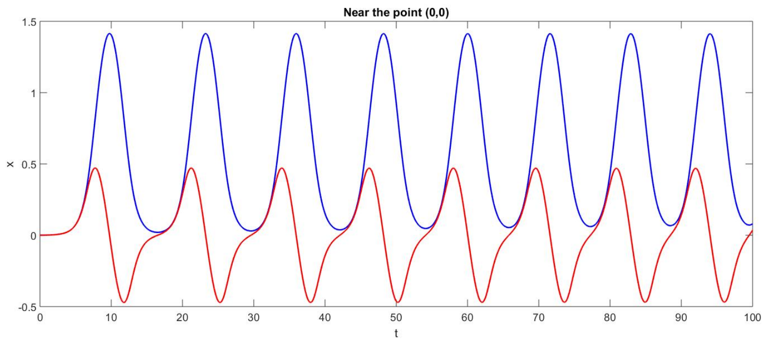


Figure 3: $(0.001, 0)$ $\alpha = 1.5$.

**CaseIII**.For changing $0.5 < \alpha < 4.72$ we have the change of oscillation while $\alpha$ increase as shown in Fig.4.

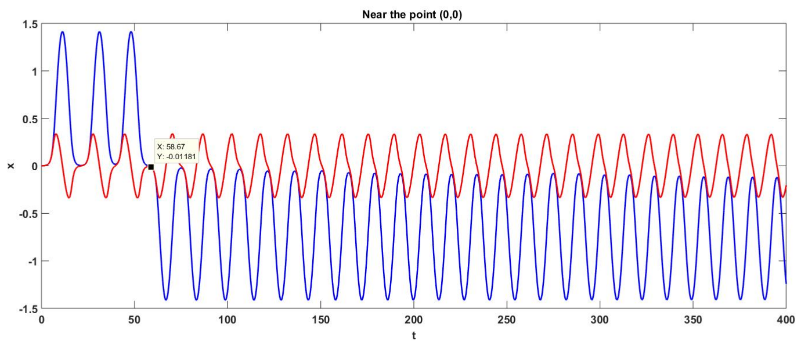


Figure 4: $(0.001, 0)$ $\alpha = 4.57$.

**CaseIV** Now for changing $\alpha$ to be non positive real number we see that the nonlinear oscillator intersect to each other when, $-0.5 < \alpha \leq 0$ like Figs. 5- 7, while when $\alpha \leq -0.5$ we see that the solution goes to infinity's shown in Fig. 8.

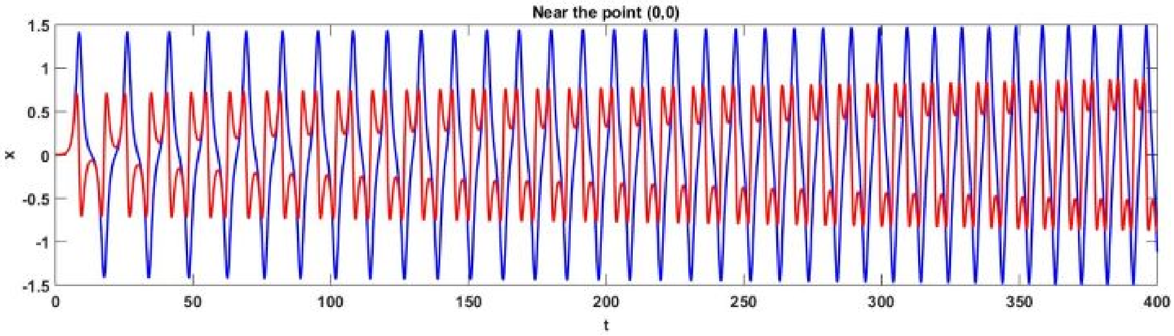


Figure 5: $(0.001, 0)$ $\alpha = 0$.

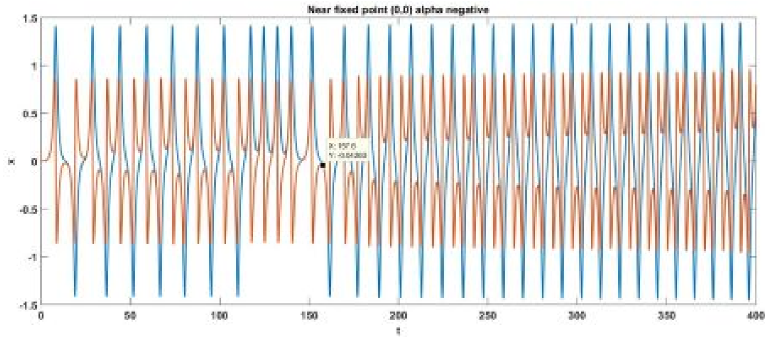


Figure 6: $(0.001, 0)$ $\alpha = -0.29$.

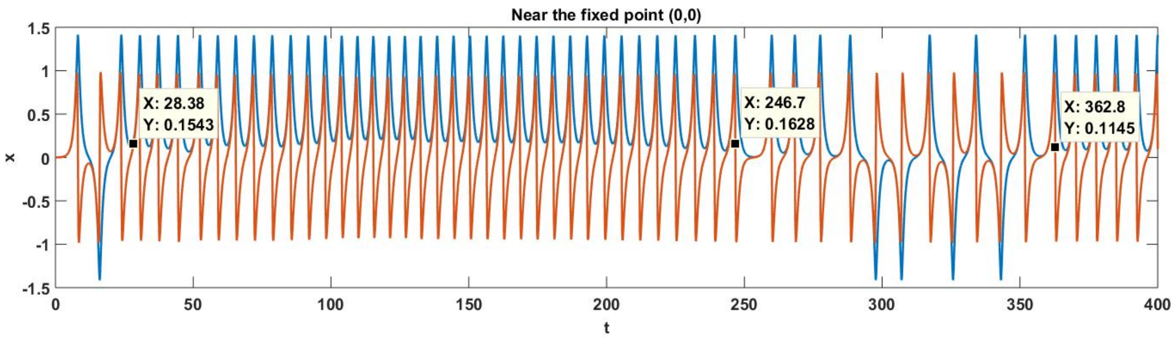


Figure 7: $(0.001, 0)$ $\alpha = -0.4$.

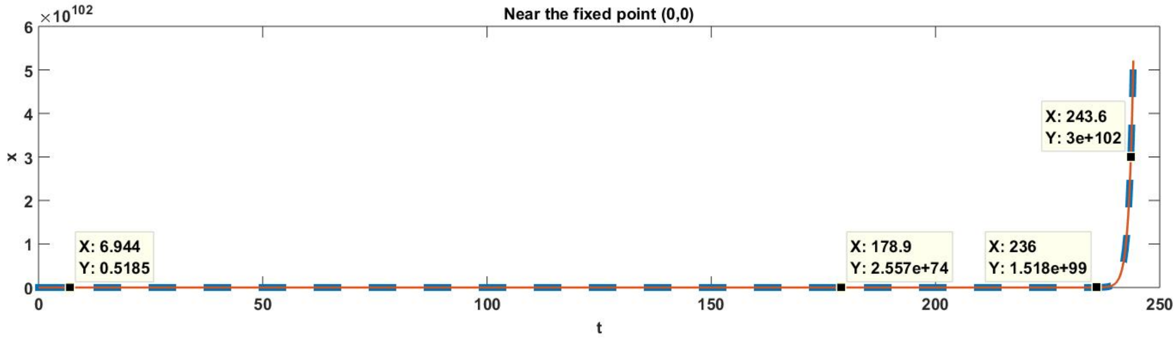


Figure 8: $(0.001, 0)$ $\alpha = -0.5$.

**CaseV**. We see that around the fixed point $(0, 0)$ ,we have an intersection between two oscillate system and they began to disjoint if we increase the vaues to be near the fixed point $(1, 0)$ or $(-1, 0)$ see Figs. 9- 10

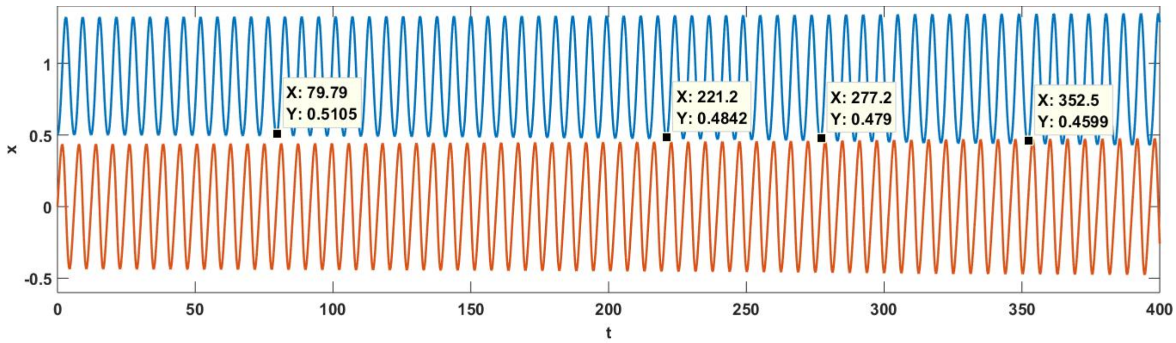


Figure 9: $(0.5, 0)$ $\alpha = -0.5$.

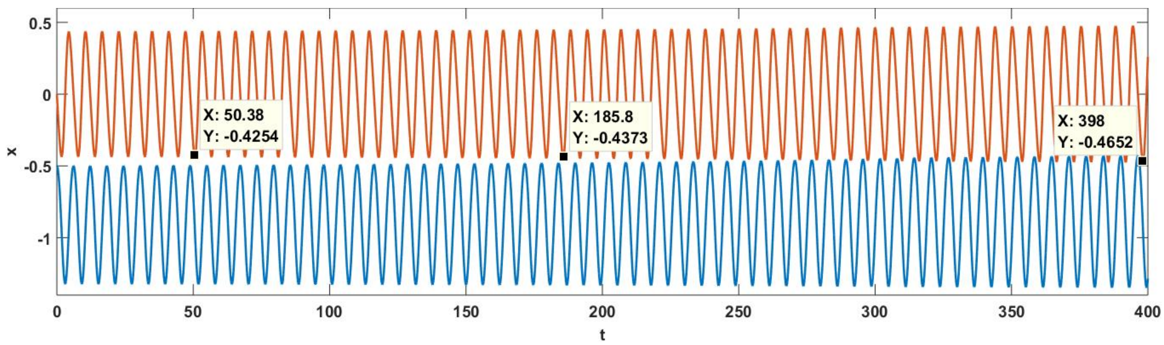


Figure 10: $(-0.5, 0)$ $\alpha = -0.5$.

**CaseVI**. We see if we increase the value to reach the fixed point $(\pm 1, 0)$ which they are unstable for any real number $\alpha$ see Fig. 11

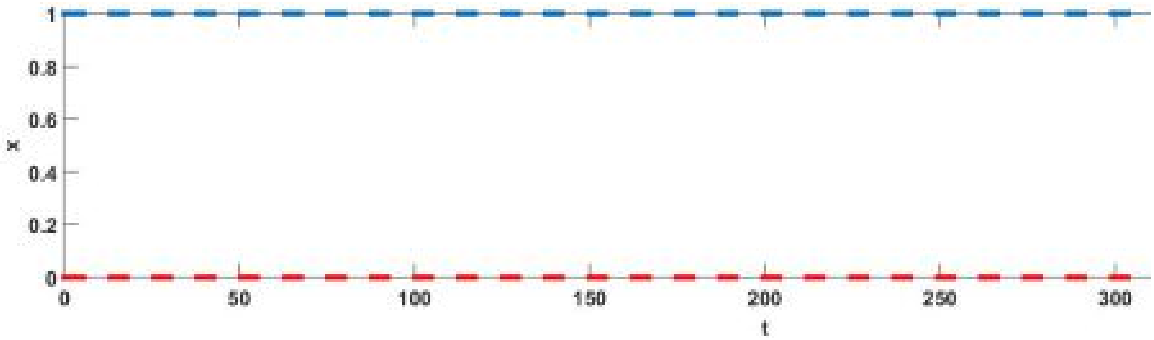


Figure 11: Fixed points $(0, 0)$ and $(1, 0)$ where $\alpha \in R$.

**Case VII**. Now if we change the initial value we see the oscillate movement has to periodic in two different time see Fig. 12

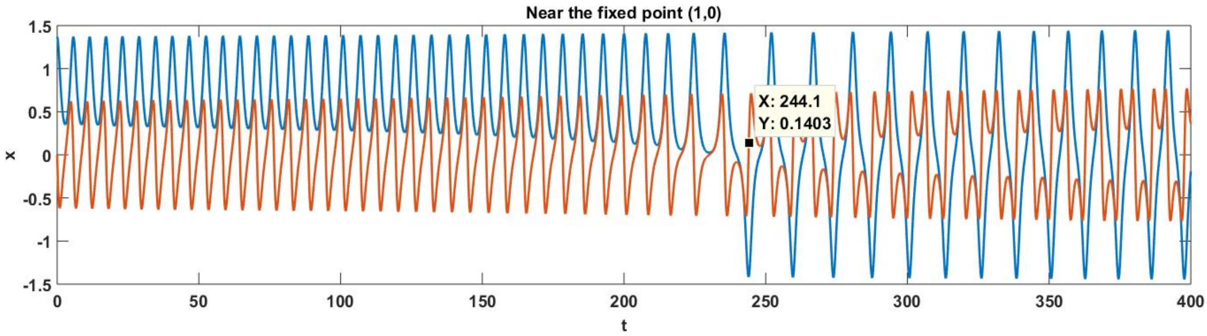


Figure 12: Initial value $(1.37, 0)$ $\alpha = 0$.

**Finally** we see an socialite movement for this equation with periodic motion just when we begin a little bet a way from the fixed point when $\alpha \geq 2$ there is two disjoint oscillate like Figs 13- 14, but when $-0.53 \leq \alpha < 2$ the oscillate movement will intersect see Fig.15, while the solution goes to infinty when $\alpha > -0.53$ as shown in Fig.16.

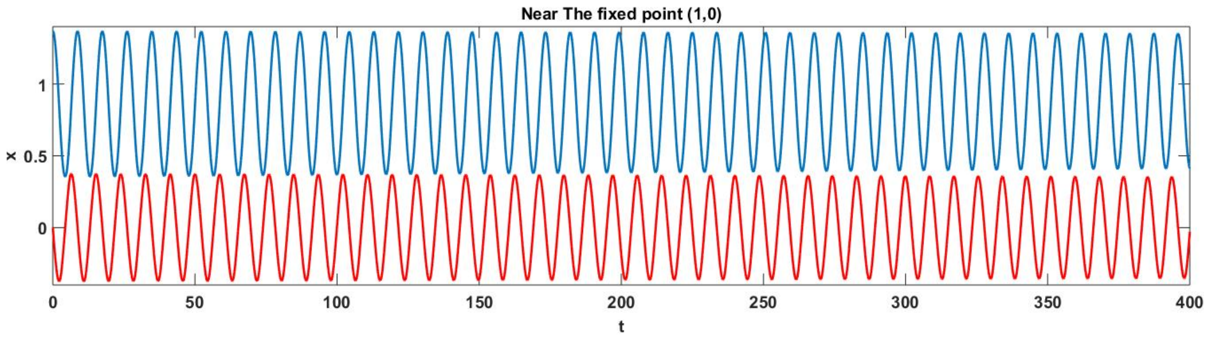


Figure 13: Initial value $(1.37, 0)$ $\alpha = 0$.

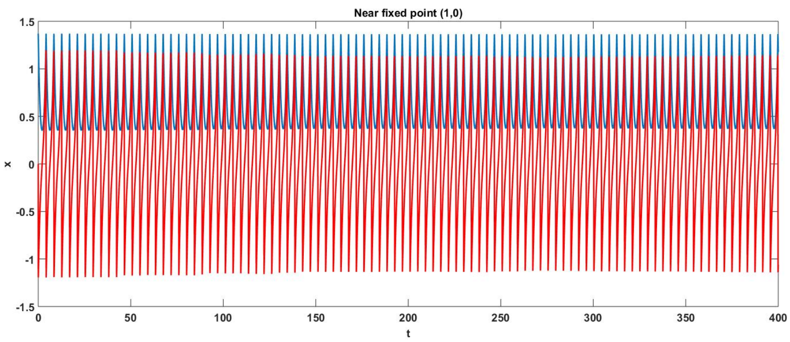


Figure 14: Initial value $(1.37, 0)$ $\alpha = 5$.

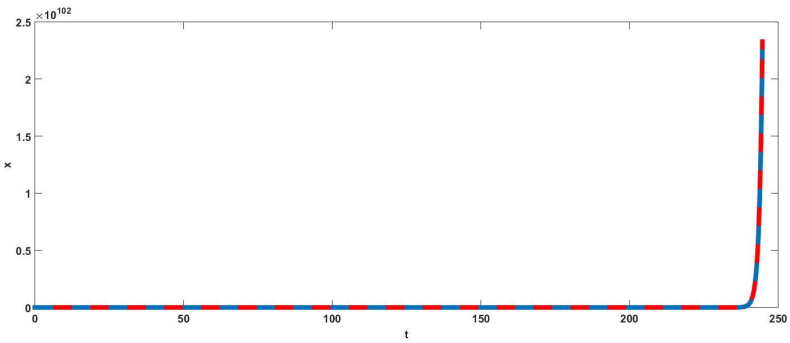


Figure 15: Initial value $(1.37, 0)$ $\alpha = -0.53$.

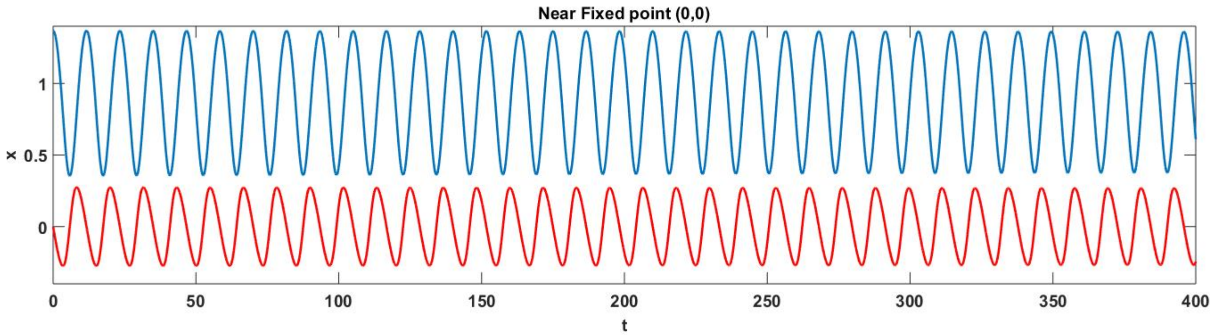


Figure 16: Initial value $(1.37, 0)$ $\alpha = -0.54$.

Studying the direction field around the fixed points of the system, we see first how the solution goes a way from (0,0), then the fixed points $(\pm 1, 0)$. Since every fixed point change it direction when it go away from left or right side as an opposite position from each other,now the graph of the direction field contain the three fixed points that shows their stability see Fig. 17.

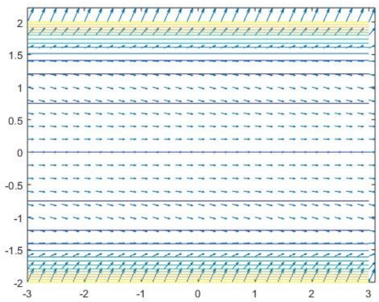

Figure 17: Direction field about the points $(0, 0)$, $(1, 0)$, $(-1, 0)$.

## 5. Conclusion

An exact analytical solution for a nonlinear oscillator with coordinate dependent- mass was obtained. From the obtained figures we conclude that the oscillator have a periodic oscillation at all time and in all position except at the equilibrium points, that change the direction and make intersection of oscillator or disconnect each other in different time and position.

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