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On the Weiner and Harary Index of Splitting Graphs

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Abstract. In this study we define a graph operation on a finite simple graph $G = (V, E)$ called the S-splitting graph of G where S is a non-empty subset of vertices of G. If $S = V$, then it is the splitting graph of G defined by E. Sampathkumar, and H.B. Walikar in the 1980's. This paper investigates the Wiener and Harary indices of the S-splitting graph of G for some families of graph.

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1. Introduction

Mathematical objects have been used to represent the structure of a chemical compound. One such representation is that each atom is described by vertices and the bond between atoms is described by an edge. With this, mathematical tools can now be used to analyze the properties of chemical compounds that may be related to its structure. A mathematical formula that represents chemical species which have made a variety of methods of chemical structure is called the *topological indices* [18].

Topological indices are helpful when interpreting chemical constitution into numerical values which can be used for correlation with physical properties in quantitative structure-property/activity relationship (QSPR/QSAR) studies. Quantitative structure-property relationship (QSPR) mathematical modeling method connects physical or chemical properties with a structure of a molecule [1]. Meanwhile, Quantitative structure-activity relationship (QSAR) is a mathematical modeling method that show relationships between biological activities and the structural properties of chemical compounds [13]. We have here some studies of toplogical indices in QSPR/QSAR. Shanmukha, et. al used 13 degree-based topological indices to study anticancer drugs in terms of QSPR [20]. Hosamani

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studied the QSPR of phytochemicals screened against SARS-CoV-2 $3CL^{pro}$ with the help of several topological indices [10].

In general, a topological index, also known as a graph-theoretic index, is a numerical invariant of a chemical graph. Harary index, Balaban index, molecular topological index, Wiener index, Hyper-Wiener index, and Zagreb indices are some of the well-studied topological indices.

Topological indices are used to represent each chemical structure with a numerical value. These values are used to model different physicochemical properties and biological activities of chemical compounds [15]. The first topological index was introduced by Harry Wiener in 1947. He computed the sum of the distances of the shortest path between all pairs of vertices of a graph called Wiener index [24]. The concept of the Wiener index was generalized by Milan Randic in 1993. It was the extension for all connected graphs and called it Hyper-Wiener index [14]. The sum of reciprocals of distances between all pairs of vertices in a graph G is called the Harary index, denoted by $H(G)$. It was introduced independently by Plavšić et al. [17] and by Ivanciuc et al. [11] in 1993.

A variety of topological indices have been studied. In particular, the Balaban index, also called J index was developed by Balaban [2]. De first derived explicit expression of reformulated first Zagreb index of generalized hierarchical product of two connected graphs [3]. Gao et. al developed some degree-based topological indices of networks derived from Honey comb networks [6]. Recently, Mondal et. al obtained some of the topological properties of some chemical structures used to inhibit the outbreak and transmission of COVID-19 in terms of some degree-based and some neighborhood degree sum-based indices [15].

While other researchers focused on the Wiener index of a graph obtained by some graph operations such as Yeh and Gutman in [25] studied the Wiener index of graphs obtained by means of certain binary operations on pairs of graphs. Stevanović generalized these results to determine the Wiener polynomial of the composite graphs [21]. The hyper-Wiener index of these operations determined by Khalifeh et. al in [12]. Eliasi et. al computed the values of the Wiener index of a graph obtained by some graph operations [5]. Moreover, a study on the Wiener index of a graph obtained from some graph operation is called sum of shadow graphs [7].

In this paper we only consider a finite simple undirected graph $G = (V, E)$ and $S \subseteq V = \{x_1, x_2, \ldots, x_n\}$ where $n \geq 1$. Define a graph $\Gamma = \Gamma(G, S)$ to be a graph obtained from G by replicating the vertices x in S as well as the edges adjacent to it, that is; $V(\Gamma) = V \cup \{x' | x \in S\}$ and $E(\Gamma) = E \cup \{x'u | u \in N_G(x)\}$ where $N(x)$ is the set of vertices in G adjacent to x. The graph $\Gamma(G, S)$ obtained is called an S-splitting of G. If $S = V$, then the graph $\Gamma(G, V)$ is the *splitting graph* of a graph G as defined by Sampathkumar et al. [19].

It is worthwhile to note that the definition of an S-splitting graph of G is similar to the double graph $\mathcal{D}[G]$ defined in [16]. In particular, $\Gamma(G) \subseteq \mathcal{D}[G]$. The case of double graphs is more simple than an S-splitting of G in terms of computing some topological indices.

The preliminary concepts are presented in Section 2. The S-splitting graph of G, denoted by $\Gamma(G, S)$ is formally defined in Section 3. The computations for the Weiner index of the S-splitting graph $\Gamma(G, S)$ are also included in this section. In chapter 4, we have the computations for the Harary index of $\Gamma(G, S)$.

2. Preliminary Concepts

In this study we only consider a simple undirected graph G with a non-empty finite set V of *vertices* and a finite set E of *edges*. We say $V(G)$ as the vertex set and $E(G)$ as the edge set. We denote any edge in $E(G)$ by $\{x_i, x_j\}$. We call *adjacent* these two distinct vertices x_i and x_j in $E(G)$. A set containing those vertices of G that are adjacent to some vertex a is called the *neighbor set of* a, denoted by $N(a)$. A graph G is said to be a *triangle free graph* if no three vertices form a triangle of edges.

The *adjacency matrix* denoted by $A(G)$ with vertex set $\{x_1, x_2, \ldots, x_n\}$ is the $n \times n$ binary matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if the vertices x_i, x_j are adjacent, and $a_{ij} = 0$ otherwise.

The distance $d(x_i, x_j)$ between two vertices x_i, and x_j is the length of the shortest path between the vertices x_i and x_j. Note that the sum of the degree of all vertices in a graph G is twice the number of edges in G. $D(G)$ is the matrix $[d_{ij}]$ where $d_{ij} = d(x_i, x_j)$ is called the *distance matrix* of a graph G.

Readers are referred to [4, 9] for other elementary Graph Theoretic concepts. In this study, we focus on the Wiener index, and Harary index of a graph.

The Wiener index $W(G)$ of a graph G is defined as

$$W(G) = \sum_{\{v_i,v_j\} \subseteq V(G)} d(v_i, v_j) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(v_i, v_j).$$

The Harary index $H(G)$ of a graph G is defined as

$$H(G) = \sum_{\{v_i,v_j\} \subseteq V(G)} \frac{1}{d(v_i, v_j)} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{d(v_i, v_j)}.$$

The following are the exact values of Weiner and Harary indices for some families of graphs. We denote H_n be the n^{th} Harmonic number. That is, $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Theorem 1. *[17, 23] The Wiener and Harary index of the path graph on n vertices for $n \geq 1$ is given by $W(P_n) = \frac{1}{6} n(n^2 - 1)$ and $H(P_n) = nH_n - 1$.*

Theorem 2. *[17, 23] The Wiener and Harary index of the cycle graph on n vertices for $n \geq 3$ is given by*

$$W(C_n) = \begin{cases} \dfrac{n^3}{8} & n \text{ even} \\ \dfrac{(n-1)(n+1)n}{8} & n \text{ odd} \end{cases}$$

and

$$H(C_n) = \frac{1}{2}(1 + (-1)^n) + nH_{\lfloor (n-1)/2 \rfloor}.$$

Theorem 3. *[22, 23] The Wiener and Harary index of the complete graph on n vertices for $n \geq 1$ is given by $W(K_n) = H(K_n) = \dfrac{n(n-1)}{2}$.*

Theorem 4. *[22, 23] The Wiener and Harary index of the star graph on n vertices for $n \geq 1$ is given by $W(S_n) = (n-1)^2$ and $H(S_n) = \frac{1}{4}(n+2)(n-1)$.*

Theorem 5. *[22, 23] For $n \geq 4$, the Wiener and Harary index of the wheel graph on n vertices is given by $W(W_n) = (n-1)(n-2)$ and $H(W_n) = \frac{1}{4}(n+4)(n-1)$.*

Theorem 6. *[8, 23] For $m, n \geq 1$, the Wiener and Harary index of the complete bipartite graph $K_{m,n}$ is given $W(K_{m,n}) = m^2 + mn + n^2 - m - n$ and $H(K_{m,n}) = \frac{1}{4}(m^2 + n^2 - m - n) + mn$.*

3. S-splitting Graphs

Sampathkumar and Walikar introduced the Splitting graph in 1980 [19] and defined it as follows. For each vertex v of a graph G, we have a new vertex v', and connect v' to all the vertices of G adjacent to v. In this study, we look into a variation of the Splitting graph and define what we call an S-splitting graph of G where S is a non-empty subset of vertices in G.

3.1. The S-splitting Graph $\Gamma(G, S)$

Throughout, we consider a finite connected simple graph $G = (V, E)$ and $S \subseteq V = \{x_1, x_2, \ldots, x_n\}$ where $n \geq 2$ and $|E| = m \geq 1$. The graph $\Gamma(G, S)$ or simply Γ is an *S-splitting graph of G* is the graph obtained from G with the vertex set $V(\Gamma) = V \cup S'$ where $S' = \{x' | x \in S\}$ and the edge set $E(\Gamma) = E \cup \{\{x', u\} | u \in N_G(x)\}$ where $N_G(x)$ is the set of vertices in G adjacent to x. If $S = V$, then $\Gamma(G)$ or a V-splitting graph of G is *splitting graph* of a graph G as defined by Sampathkumar in [19]. From the definition of the S-splitting graph of G, the following statements can be easily shown.

Lemma 1. *Let $G = (V, E)$ is a simple graph on $n \geq 2$ vertices and $m \geq 1$ edges. Let S be a non-empty subset of V and $|S| = r$. Consider the S-splitting graph of G, $\Gamma(G, S)$ with $V(\Gamma) = V \cup S'$, where $S' = \{x' | x \in S\}$.*

(i) If $S = V$, then $|V(\Gamma)| = 2n$, $|E(\Gamma)| = 3m$.

(ii) If $S = V$ and $v \in S$, then $deg_\Gamma(v) = 2deg_G(v)$.

(iii) If $v \in S'$, then $deg_\Gamma(v') = deg_G(v)$.

(iv) If G is a connected graph, then $\Gamma(G, S)$ is also connected.

Proof. Let Γ be the S-splitting graph of G.

For (i), suppose $S = V$. Then $|S| = |V|$. Since $V(\Gamma) = V \cup V'$ and V' is the new set of the vertices obtained from $V(G)$, then $V \cap V' = \emptyset$. This implies that $|V(\Gamma)| = |V| + |V'| = 2|V(G)|$. Meanwhile, from the definition of $\Gamma(G, S)$, for every edge in G, 2 new edges are formed. So, $|E(\Gamma)| = |E(G)| + 2|E(G)| = 3|E(G)|$.

For (ii), Let $S = V$. From Γ, suppose $deg_G(v) = p$ and $v \in S$, then there are p new vertices connected to v. Thus, $deg_\Gamma(v) = 2deg_G(v)$.

For (iii), let $v \in V'$. Since v' is the new vertex that connects to all vertices of G adjacent to v, then $deg_\Gamma(v') = deg_G(v)$.

For (iv), to show that Γ is connected, we need to show that for any pair of vertices x, y in Γ, there exists a path from vertex x to vertex y. We consider three cases: $x, y \in V$; $x \in V, y \in V'$; and $x, y \in V'$. Since G is a connected graph, then there is a path between any pair of vertices x, y in V. Let $x \in V, y \in V'$, if $y = x'$ then $d(x, y) \geq 2$ from the definition of an S-splitting of G. Moreover, y must be adjacent to a vertex in $N_G(x)$ and thus there exist a path of length 2 from x to y, which shows that $d(x, y) = 2$. We now consider vertex $u \in V$ associated to y from the definition of Γ where $y \neq x'$. Since G is connected, then there must be a path from x to u. Suppose the sequence of vertices from this path is $x = a_1, \ldots, a_k = u$, then there is a path from x to y using the sequence of vertices $x = a_1, \ldots, a_{k-1} = y$. For the last case, suppose both x, y are in V'. Let u, v be the vertices in V associated with x, y respectively from the definition of Γ. Since G is connected, then there must be a path from u to v in G. Suppose $a_1, a_2, \ldots, a_{k-1}, a_k$ is the sequence of vertices from this path. From the definition of Γ, x is adjacent to a_2 and y is adjacent to a_{k-1}, thus there is a path from x to y in Γ. Therefore, $\Gamma(G, S)$ is a connected graph.

Lemma 2. *Let G be a connected graph triangle free graph with at least two vertices. Consider $\Gamma(G)$, then we have,*

(i) $d(x_i, x_i') = 2, \quad$ *for $i = 1, \ldots, n$;*

(ii) $d(x_i, x_j) = d(x_i', x_j)$, *for $1 \leq i, j \leq n, i \neq j$;*

(iii) $d(x_i', x_j') = 3$ *for adjacent vertices x_i and x_j;*

(iv) $d(x_i, x_j) = d(x_i', x_j')$ *where x_i and x_j are non-adjacent vertices*

where $V(\Gamma) = \{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$.

Lemma 3. *Let G be a connected graph with at least two vertices such that any pair of adjacent vertices has a common neighbor. Then we have the following:*

(i) $d(x_i, x_i') = 2$ *for $i = 1, \ldots n$;*

(ii) $d(x_i, x_j) = d(x_i', x_j)$ *for $1 \leq i, j \leq n, i \neq j$;*

(iii) $d(x_i', x_j') = 2$ *for adjacent vertices x_i and x_j;*

(iv) $d(x_i, x_j) = d(x'_i, x'_j)$ *where* x_i *and* x_j *are non-adjacent vertices in* G

where $V(\Gamma) = \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}$.

If G is a triangle free graph, then by Lemma 2 the distance matrix of the V-splitting of G , $\Gamma(G) = \Gamma$ can view as a block matrix given by

$$D(\Gamma) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ (D(G) + 2I_n) & D(G) + 2A(G) \end{bmatrix}$$

While, in view of Lemma 3, if every pair of adjacent vertices of a graph G has a common neighbor, then the distance matrix of Γ can be written as the block matrix given by

$$D(\Gamma) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) + A(G) \end{bmatrix}$$

where $D(G)$ is the distance matrix of a graph G, $A(G)$ is the adjacency matrix, and I_n is the identity matrix with size n.

The following theorem now gives us a general description of the distance matrix of a V-splitting graph of G.

Theorem 7. *The distance matrix of the* V*-splitting graph of* G *can be viewed as a* 2×2 *block matrix given by*

$$D(\Gamma) = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

$X = D(G), Y = Z = D(G) + 2I_n$ *and* W *where* $W = [w_{ij}]$ *is the matrix given by:*

$$w_{ij} = \begin{cases} 2 & \textit{if } \{x_i, x_j\} \in E(G) \textit{ and } N_G(i) \cap N_G(j) \neq \emptyset \\ 3 & \textit{if } \{x_i, x_j\} \in E(G) \textit{ and } N_G(i) \cap N_G(j) = \emptyset \\ 0 & \textit{if } i = j \\ d(x_i, x_j) & \textit{otherwise.} \end{cases}$$

Proof. Consider the distance matrix of the V-splitting graph of G where first n rows and columns are indexed by the vertices in G say $x_1, \ldots, x_n$ and the last n and columns rows by the new vertices say $x'_1, \ldots, x'_n$. From Lemma 2 and Lemma 3, it is easy to see that $X = D(G), Y = Z = D(G) + 2I_n$. Let us now consider the entries in $W = [w_{ij}]$. First note that in the distance matrix of G, $D(G) = [d_{ij}]$, $d_{ij} = 1$ whenever vertex x_i is adjacent to vertex x_j in G. Thus, the non-zero entries in $D(G) - A(G)$ denotes the distances of two distinct non-adjacent vertices in G. By Lemma 2 and Lemma 3, $w_{ij} = d(x_i, x_j)$ for any two distinct non-adjacent vertices in G. Now, we partition the set of all pairs of adjacent vertices in G say V_1, V_2 where V_1 contains all pairs having a common neighbor and V_2 contains pairs of adjacent vertices having no common neighbor. For the pair of vertices

x_i, x_j in V_1, by Lemma 3 (iii), $w_{ij} = d(x'_i, x'_j) = 2$ and for the pairs of vertices x_i, x_j in V_2, by Lemma 3 (iii), $w_{ij} = d(x'_i, x'_j) = 3$.

Note that the matrix W in Theorem 3.1.7 can be expressed as

$$W = D(G) - A(G) + 2A'(G) + 3A''(G)$$

where $D(G), A(G)$ is the distance matrix and adjacency matrix of G respectively, and $A'(G)$ is an $n \times n$ matrix whose ij-entry is 1 if vertex i is adjacent to vertex j in G such that $N_G(i) \cap N_G(j) \neq \emptyset$ and 0 otherwise; and $A''(G)$ is an $n \times n$ matrix whose ij-entry is 1 if vertex i is adjacent to vertex j in G such that $N_G(i) \cap N_G(j) = \emptyset$ and 0 otherwise.

3.2. Wiener Index of $\Gamma(G, V)$

To simplify our computations for the Wiener index of a graph we use the following notation, for any matrix A we denote the sum of all entries in A by $\sum A$. Recall that for a constant c, then $\sum cA = c\sum A$.

Theorem 8. *Let G be a connected triangle free graph on n vertices and m edges. Suppose $V = S$, the splitting graph of G $\Gamma = \Gamma(G)$ has Wiener index given by*

$$W(\Gamma) = 4W(G) + 2n + 2m.$$

Proof.

Let G be any connected graph of order n such that any pair of adjacent vertices has no common neighbor, that is G is a triangle free graph. Denote the splitting graph of G by Γ. Then we can describe the distance matrix of Γ as a 2x2 block matrix entries that depends on the distance matrix, adjacency matrix and identity matrix of the graph G by

$$D(\Gamma) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) + 2A(G) \end{bmatrix}$$

Since the Wiener index of Γ is half of the distance matrix of Γ then we have the following:

$$\begin{aligned} W(\Gamma) &= \frac{1}{2}\sum D(\Gamma) = \frac{1}{2}\Big(\sum 4D(G) + 4\sum I_n + \sum 2A(G)\Big) \\ &= 4\Big(\frac{1}{2}\sum D(G)\Big) + 2\sum I_n + \sum A(G) = 4W(G) + 2n + \sum A(G) \end{aligned}$$

Observe that the sum of the entries in an adjacency matrix is twice the number of edges m. That is, $W(\Gamma) = 4W(G) + 2n + 2m$.

We note that the path P_n , star graph S_n, and cycle $C_n, n > 3$, are triangle free graph. The following statements follows easily.

Corollary 1. *For $n > 3$, the Wiener index of the V-splitting graph of G, Γ is given by*

(i) $W(\Gamma(P_n, V)) = \frac{2n^3+10n-6}{3}$;

(ii) $W(\Gamma(S_n, V)) = 4n^2 - 4n + 2$;

(iii) $W(\Gamma(C_n, V)) = \begin{cases} \frac{n^3+8n}{2} & n \text{ even} \\ \frac{n^3+7n}{2} & n \text{ odd.} \end{cases}$

Proof. For (i) The result follows from Theorem 1 and Theorem 8, and for (ii), the result follows from Theorem 4 and Theorem 8 while the result for (iii) follows from Theorem 2 and Theorem 8.

Theorem 9. *For a connected graph $G = (V, E)$ on $n \geq 2$ vertices and $m \geq 1$ edges such that any pair of adjacent vertices have at least one common neighbor and $V = S$, the splitting graph of G, say $\Gamma = \Gamma(G)$ has Wiener index given by*

$$W(\Gamma) = 4W(G) + 2n + m.$$

Proof. Then distance matrix of Γ as a 2×2 block matrix entries that depends on the distance, adjacency and identity matrices of the graph G by

$$D(\Gamma) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) + A(G) \end{bmatrix}$$

where $D(G)$ and $A(G)$ are the distance, adjacency matrices of the graph G and I_n is the identity matrix of size n. Notice that we can express the Wiener index of Γ using the distance matrix of Γ then have the notation:

$$\begin{aligned} W(\Gamma) &= \frac{1}{2}\sum D(\Gamma) = \frac{1}{2}\Big(\sum 4D(G) + 4\sum I_n + \sum A(G)\Big) \\ &= 4\Big(\frac{1}{2}\sum D(G)\Big) + 2\sum I_n + \frac{1}{2}\sum A(G) = 4W(G) + 2n + \frac{1}{2}\sum A(G) \end{aligned}$$

Note that the sum of the entries in an adjacency matrix is twice the number of edges m. Thus we have $W(\Gamma) = 4W(G) + 2n + \frac{1}{2}(2m) = 4W(G) + 2n + m$.

Corollary 2. *For $n > 3$, the Wiener index of the V-splitting graph of G, Γ is given by*

(i) $W(\Gamma(K_n, V)) = \frac{5n^2-n}{2}$;

(ii) $W(\Gamma(W_n, V)) = 4n^2 - 8n + 6$.

Proof. For (i), the result follows from Theorem 3 and Theorem 9, while for (ii), the results are immediate from Theorem 5 and Theorem 9.

The following observations can be easily verified from the definition of the S-splitting graph of G.

- If G is either a complete graph K_n or a cycle graph C_n, then for any $x, y \in V(G)$ we have $\Gamma(G, \{x\}) \cong \Gamma(G, \{y\})$.

- Let G be a complete graph K_n and $n \geq 3$. Then the graphs $\Gamma(G, S_1)$ and $\Gamma(G, S_2)$ are isomorphic for any $S_1, S_2 \subset V(G)$ such that $|S_1| = |S_2|$.

We now determine the Wiener index of $\Gamma(G, S)$ where $S \neq V$ for some families of graphs. The graphs included are complete graphs, complete bipartite graphs, cycle graphs, path graphs, star graphs and wheel graphs.

The following lemma follows directly from the definition of an S-splitting of G.

Lemma 4. *Let $G = K_{m,n}$ with vertex partition $V = V_1 \cup V_2$. Suppose $S \subseteq V_1$, where $|S| = p$ then $\Gamma(G, S) \cong K_{m+p,n}$.*

Theorem 10. *Let $G = K_n$, for $n \geq 3$ and let $S \subset V(K_n)$ such that $|S| = r$ where $1 \leq r \leq n-1$. Then the Wiener index of the splitting graph of K_n is given by $W(\Gamma(K_n, S)) = \frac{n^2-n+2r^2+2nr}{2}$.*

Proof. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, suppose $S = \{x_1, x_2, \ldots, x_r\}$. We denote the ordering of the vertices of the graph $\Gamma = \Gamma(K_n, S)$ by $x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_r$. Then we can write the distance matrix of Γ as a block matrix as follows.

$$D(\Gamma) = \left[\begin{array}{c|c} D(K_n) & \begin{array}{c} A(K_r) + 2I_r \\ J_{n-r\times r} \end{array} \\ \hline A(K_r) + 2I_r \quad J_{r\times n-r} & 2A(K_r) \end{array}\right]$$

where $D(K_n), A(K_r), I_r, J_{r\times n-r}$ are the distance matrix of K_n, adjacency matrix of K_r, identity matrix of size r, and the all one's matrix of size $r \times n-r$ respectively. Hence, we can compute the Wiener index by

$$\begin{aligned} W(\Gamma) &= \frac{1}{2}\sum D(\Gamma) \\ &= \frac{1}{2}\Big(\sum D(K_n) + 4\sum A(K_r) + 4\sum I_r + \sum J_{n-r\times r} + \sum J_{r\times n-r}\Big) \\ &= \frac{1}{2}D(K_n) + 2\sum A(K_r) + 2\sum I_r + \frac{1}{2}\sum J_{n-r\times r} + \frac{1}{2}\sum J_{r\times n-r} \\ &= W(K_n) + 2\big(2|E(K_r)|\big) + 2\sum I_r + \frac{1}{2}\sum J_{n-r\times r} + \frac{1}{2}\sum J_{r\times n-r} \\ &= \frac{n(n-1)}{2} + 2r(r-1) + 2r + \frac{1}{2}(n-r)r + \frac{1}{2}(n-r)r \\ &= \frac{n(n-1)}{2} + 2r^2 - 2r + 2r + r(n-r) \\ &= \frac{n^2 - n + 2r^2 + 2nr}{2} \end{aligned}$$

.

Theorem 11. *Let $n > 3$, and $S = \{x\}$ such that $x \in V(C_n)$. Then the Wiener index of the S-splitting graph of C_n is given by* *(i)* $W(\Gamma(C_n,S)) = \frac{n(n-1)(n+1)}{8} + k^2 + k + 2$ *if* n *is odd;*

(ii) $W(\Gamma(C_n,S)) = \frac{n^3}{8} + k^2 + 2$ *if* n *is even.*

Proof. Suppose $n = 2k+1$. Let $V(C_n) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, suppose $S = \{x_1\}$. For some ordering of the vertices of Γ say, $x_1, x_2, x_n, x_3, x_{n-1}, \ldots, x_{k+1}, x_{k+2}, x_1'$, the distance matrix of $D(\Gamma)$ can describe as a bloc matrix given by

$$D(\Gamma) = \begin{bmatrix} D(C_n) & A \\ A^t & 0 \end{bmatrix}$$

where $D(C_n)$ is the distance matrix of C_n and A^t is the $1\times n$ matrix $[2,1,1,2,2,3,3,\ldots,k,k]$. Thus we have,

$$\begin{aligned} W(\Gamma) &= W(C_n) + 2 + 2\sum_{i=1}^{k} i = \frac{n(n-1)(n+1)}{8} + 2 + 2\Big(\frac{k(k+1)}{2}\Big) \\ &= \frac{n(n-1)(n+1)}{8} + 2 + k(k+1) = \frac{n(n-1)(n+1)}{8} + k^2 + k + 2. \end{aligned}$$

Now suppose $n = 2k$, given the ordering of the vertices of Γ by $x_1, x_2, x_n, x_3, x_{n-1}, \ldots, x_{k-1}, x_{k+1}, x_k, x_1'$ where $n = 2k$. Then we can write the distance matrix of Γ by

$$D(\Gamma) = \begin{bmatrix} D(C_n) & A \\ A^t & 0 \end{bmatrix}$$

where $D(C_n)$ is the distance matrix of C_n and A^t is the $1\times n$ matrix $[2,1,1,2,2,\ldots,k-1,k-1,k]$. Computing for the Wiener index of the graph $\Gamma(C_n,S)$, we have:

$$\begin{aligned} W(\Gamma) &= W(C_n) + 2 + 2\sum_{i=1}^{k-1} i + k = \frac{n^3}{8} + 2 + 2\Big(\frac{k(k-1)}{2}\Big) + k \\ &= \frac{n^3}{8} + 2 + 2\Big(\frac{k(k-1)}{2}\Big) + k = \frac{n^3}{8} + k^2 + 2. \end{aligned}$$

Theorem 12. *Let G be a path graph and $S \subset V(P_n) = \{x_1, x_2, \ldots, x_n\}$ where $n \geq 2$. Then*

(i) If $S = \{x_1\}$ or $S = \{x_n\}$, then $W(\Gamma(P_n,S)) = \frac{n^3+3n^2-4n+12}{6}$;

(ii) If $S = \{x_i\}$ such that $2 \leq i \leq n-1$, then $W(\Gamma(P_n,S)) = \frac{n^3+3n^2+2n+6i^2-6in-6i+12}{6}$.

Proof. Let $S = \{x_1\}$. Given the ordering of the vertices $x_1, x_2, \ldots, x_n, x_1'$. We have the distance matrix for the graph $\Gamma(S, P_n)$ as follows.

$$D(\Gamma) = \begin{bmatrix} D(P_n) & A \\ A^t & 0 \end{bmatrix}$$

where $D(P_n)$ is the distance matrix of P_n and A^t is the $1 \times n$ matrix $[2, 1, 2, \ldots, n-1]$. From this, we can now compute the Wiener index of $\Gamma(P_n, S)$ as follows

$$\begin{aligned} W(\Gamma) &= W(P_n) + 2 + \sum_{i=1}^{n-1} i \\ &= \frac{n(n^2-1)}{6} + 2 + \frac{n(n-1)}{2} \\ &= \frac{n^3 + 3n^2 - 4n + 12}{6}. \end{aligned}$$

Moreover, suppose $S = \{x_i\}$ where $2 \leq i \leq n-1$. Now, we consider the ordering of the vertices $x_1, x_2, \ldots, x_n, x_i'$ of Γ. So, the distance matrix of the graph $\Gamma(P_n, S)$ is given by

$$D(\Gamma) = \begin{bmatrix} D(P_n) & A \\ A^t & 0 \end{bmatrix}$$

where $D(P_n)$ is the distance matrix of P_n and A^t is the $1 \times n$ matrix $[i-1, i-2, \ldots, 2, 1, 2, \ldots, n-i]$. From this, we now have

$$\begin{aligned} W(\Gamma) &= W(P_n) + 2 + \sum_{j=1}^{i-1} j + \sum_{j=1}^{n-1} j \\ &= W(P_n) + 2 + \frac{i^2-1}{2} + \frac{n^2+i^2-2in+n-i}{2} \\ &= \frac{n(n^2-1)}{6} + 2 + \frac{i^2-1}{2} + \frac{n^2+i^2-2in+n-i}{2} \\ &= \frac{n^3+3n^2+2n+6i^2-6in-6i+12}{6}. \end{aligned}$$

Theorem 13. *Let G be the complete bipartite graph $K_{m,n}$ with vertex partition $V = V_1 \cup V_2$ such that $|V_1| = m$ and $|V_2| = n$ and $S \subset V_k$. Then the Wiener index of $\Gamma = \Gamma(K_{m,n}, S)$ is given by*

$$W(\Gamma) = m^2 + n^2 + i^2 + j^2 - m - n - i - j + mn + ni + mj + 2mi + 2nj + 3ij$$

where $|S \cap V_1| = i$ and $|S \cap V_2| = j$.

Proof. Let $V = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$ be the vertex set of $K_{m,n}$. Without loss of generality, suppose $S = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j\}$. Now, consider the vertex set of $\Gamma = \Gamma(K_{m,n}, S)$ say, $\{x_1, \ldots, x_m, y_1, \ldots, y_n, x'_1, \ldots, x'_i, y'_1, \ldots, y'_j\}$. The distance matrix for the S-splitting of $K_{m,n}$ can be written as

$$D(\Gamma) = \begin{bmatrix} 2J_m - 2I_m & J_{m\times n} & 2J_{m\times i} & J_{m\times j} \\ J_{n\times m} & 2J_n - 2I_n & J_{n\times i} & 2J_{n\times j} \\ 2J_{i\times m} & J_{i\times n} & 2J_i - 2I_i & 3J_{i\times j} \\ J_{j\times m} & 2J_{j\times n} & 3J_{j\times i} & 2J_j - 2I_j \end{bmatrix}$$

Notice that from the distance matrix, the Wiener index of Γ is the sum of $\sum J_m - I_m$, $\sum J_n - I_n$, $\sum J_i - I_i$, $\sum J_j - I_j$, $\sum J_{m\times n}$, $\sum J_{n\times i}$, $\sum J_{m\times j}$, $3\sum J_{i\times j}$, $2\sum J_{m\times i}$, and $2\sum J_{n\times j}$. Thus, the result follows.

3.3. Harary Index of $\Gamma(G, V)$

Another well-known topological index of a graph studied by Plavšić et al. [17] and by Ivanciuc et al. [11] is Hararay index. We recall the Harary Index of the graph G and is defined as follows.

$$H(G) = \sum_{\{v_i,v_j\}\subseteq V(G)} \frac{1}{d_G(v_i, v_j)} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{d_G(v_i, v_j)}.$$

The Harary index of $\Gamma = \Gamma(G, S)$ can be computed using the distance matrix of Γ viewed as a block matrix similar to the computation of the Wiener index of Γ.

In order to compute the Harary index of the S-splitting graph of G, we consider a matrix whose entries are the reciprocals of the nonzero entries of its distance matrix. For any matrix A, we let $\overline{A}$ be the matrix whose entries are the reciprocals of the non zero entries in A and $\sum \overline{A}$ be the sum of the reciprocals of the nonzero entries in A, that is, if $A = [a_{ij}]$, then $\overline{A} = [\overline{a_{ij}}]$ where $\overline{a_{ij}} = \frac{1}{a_{ij}}$ if $a_{ij} \neq 0$ and 0 otherwise. If the entries in A are 0 or 1, then $\sum \overline{A} = \sum A$. Additionally, for any nonzero c, we have $\sum c\overline{A} = c \sum \overline{A}$.

We will use the following lemma in order to compute for the Harary index of $\Gamma(G, S)$.

Lemma 5. *Let $A = [a_{ij}], B = [b_{ij}]$ be $n \times n$ square matrices with real entries such that at least one of a_{ij} or b_{ij} is zero for all $0 \leq i, j \leq n$, then*

$$\sum \overline{A + B} = \sum \overline{A} + \sum \overline{B}.$$

Proposition 1. *Let D, A be the distance matrix and adjacency matrix of a graph G on n vertices and I be the identity matrix of size n. Let c any nonzero real number.*

(i) $\sum \overline{cA} = \frac{1}{c} \sum A$

(ii) $\sum \overline{D + cA} = \sum \overline{D - A} + \sum \overline{(c+1)A}$

(iii) $\sum \overline{D + cI} = \sum \overline{D} + \sum \overline{cI}$

Proof. Let $D = [d_{ij}], A = [a_{ij}]$.

For (i) : Note that since a_{ij} is either 0 or 1, for $0 \leq i, j \leq n$, then we have

$$\sum \overline{cA} = \sum \frac{1}{ca_{ij}} = \frac{1}{c} \sum \frac{1}{a_{ij}} = \frac{1}{c} \sum a_{ij} = \frac{1}{c} \sum A$$

where the summation runs over all non-zero a_{ij}.

For (ii) : We note that $d_{ij} = a_{ij} = 1$ whenever vertex i is adjacent to vertex j. Note that the diagonal entries of $D - A$ and and $(c + 1)A$ are all zero. Moreover, the ij-entry of $D - A$ is zero whenever vertex i is adjacent to vertex j and greater than 0 whenever vertex i is not adjacent to vertex j. Furthermore, the ij-entry in the matrix $(c + 1)A$ is zero whenever vertex i is not adjacent to vertex j and $(c + 1)$ whenever vertex i is adjacent to vertex j. Since $D + cA = (D - A) + (c + 1)A$ and by Lemma 5, we have $\sum \overline{D + cA} = \sum \overline{D - A} + \sum \overline{(c + 1)A}$.

For (iii): From the definition of D the entries in the main diagonal are all zero and that the entries outside the main diagonal of $(c+1)I$ are all zero. Thus, by Lemma 5, the statement follows.

Theorem 14. *Suppose $G = (V, E)$ is a connected triangle free graph on n vertices and m edges, then the Harary index of $\Gamma(G, V)$ is given by*

$$H(\Gamma) = 4H(G) + \frac{n}{2} - \frac{2}{3}m.$$

Proof. Let G be any connected graph of order n and m edges such that any pair of adjacent vertices has no common neighbor, that is G is a triangle free graph. Denote the splitting graph of G by Γ. The distance matrix of Γ can be written as a 2×2 block matrix given by

$$D(\Gamma) = \begin{bmatrix} D(G) & D(G) + 2I_n \\ D(G) + 2I_n & D(G) + 2A(G) \end{bmatrix}$$

where $D(G)$ and $A(G)$ are the distance matrix, adjacency matrix of the graph G and I_n is the identity matrix of size n. Since the Harary index of Γ is half the sum of the entries in $\overline{D(\Gamma)}$, then we have the following computations:

$$\begin{aligned}
H(\Gamma) &= \frac{1}{2} \sum \overline{D(\Gamma)} \\
&= \frac{1}{2}\Big(\sum \overline{D(G)} + \sum \overline{D(G) + 2I_n} + \sum \overline{D(G) + 2I_n} + \sum \overline{D(G) + 2A(G)} \Big) \\
&= \frac{1}{2}\Big(\sum \overline{D(G)} + \sum \overline{D(G)} + \sum \overline{2I_n} + \sum \overline{D(G)} + \sum \overline{2I_n} \\
&\quad + \sum \overline{D(G) - A(G)} + \sum \overline{3A(G)} \Big) \\
&= \frac{1}{2}\Big(4 \sum \overline{D(G)} + 2 \sum \overline{2I_n} + \sum \overline{3A(G)} - \sum \overline{A(G)} \Big)
\end{aligned}$$

$$\begin{aligned}
&= 4\Big(\frac{1}{2}\sum\overline{D(G)}\Big)+\sum\overline{2I_n}+\frac{1}{2}\sum\overline{3A(G)}-\frac{1}{2}\sum\overline{A(G)}\\
&= 4H(G)+\frac{n}{2}+\frac{1}{2}\sum\overline{3A(G)}-\frac{1}{2}\sum\overline{A(G)}
\end{aligned}$$

Observe that the sum of the entries in an adjacency matrix is twice the number of edges m. That is,

$$\begin{aligned}
H(\Gamma) &= 4H(G)+\frac{n}{2}+\frac{1}{2}\Big(\frac{2m}{3}\Big)-\frac{1}{2}\Big(2m\Big)\\
&= 4H(G)+\frac{n}{2}+\frac{m}{3}-m\\
&= 4H(G)+\frac{n}{2}-\frac{2m}{3}
\end{aligned}$$

Corollary 3. *Let G be a cycle graph C_n. Then the V-splitting graph of C_n has a Harary index given by*

$$H(\Gamma)=2\Big(1+\big(-1\big)^n\Big)+n\Big(4H_{\lfloor\frac{n-1}{2}\rfloor}-\frac{1}{6}\Big).$$

Corollary 4. *Let G be a path graph P_n. Then the Harary index of V-splitting graph of P_n is given by*

$$H(\Gamma)=n\Big(4H_n-\frac{25}{6}\Big)+\frac{2}{3}.$$

Corollary 5. *Let G be a star graph S_n. Then the V-splitting graph of S_n has a Harary index given by*

$$H(\Gamma)=\frac{1}{6}\Big(6n^2+5n-8\Big).$$

Theorem 15. *Suppose $G=(V,E)$ is a connected graph on $n\geq 2$ vertices and $m\geq 1$ edges such that every pair of adjacent vertices have a common neighbor, then the Harary index of $\Gamma(G,V)$ is given by*

$$H(\Gamma)=4H(G)+\frac{n-m}{2}.$$

Proof. Let G be a connected graph of order n and m edges such that any pair of adjacent vertices has at least common neighbor and let Γ be the splitting graph of G. This implies that the distance matrix of Γ can be written as a 2×2 block matrix given by

$$D(\Gamma)=\begin{bmatrix} D(G) & D(G)+2I_n\\ D(G)+2I_n & D(G)+A(G)\end{bmatrix}$$

where $D(G)$ and $A(G)$ are the distance matrix, adjacency matrix of the graph G and I_n is the identity matrix of size n. We can determine the Harary index of Γ using the distance matrix of Γ given by the following:

$$
\begin{aligned}
H(\Gamma) &= \frac{1}{2}\sum\overline{D(\Gamma)} \\
&= \frac{1}{2}\Big(\sum\overline{D(G)}+\sum\overline{D(G)+2I_n}+\sum\overline{D(G)+2I_n}+\sum\overline{D(G)+A(G)}\Big) \\
&= \frac{1}{2}\Big(\sum\overline{D(G)}+\sum\overline{D(G)}+\sum\overline{2I_n}+\sum\overline{D(G)}+\sum\overline{2I_n} \\
&\quad +\sum\overline{D(G)-A(G)}+\sum\overline{2A(G)}\Big) \\
&= \frac{1}{2}\Big(4\sum\overline{D(G)}+2\sum\overline{2I_n}+\sum\overline{2A(G)}-\sum\overline{A(G)}\Big) \\
&= 4\Big(\frac{1}{2}\sum\overline{D(G)}\Big)+\sum\overline{2I_n}+\frac{1}{2}\sum\overline{2A(G)}-\frac{1}{2}\sum\overline{A(G)} \\
&= 4H(G)+\frac{n}{2}+\frac{1}{2}\sum\overline{2A(G)}-\frac{1}{2}\sum\overline{A(G)}
\end{aligned}
$$

Note that the sum of the entries in an adjacency matrix is twice the number of edges m. Then, we have

$$
\begin{aligned}
H(\Gamma) &= 4H(G)+\frac{n}{2}+\frac{1}{2}\Big(\frac{2m}{2}\Big)-\frac{1}{2}(2m) \\
&= 4H(G)+\frac{n}{2}+\frac{m}{2}-m \\
&= 4H(G)+\frac{n}{2}-\frac{m}{2} \\
&= 4H(G)+\frac{n-m}{2}.
\end{aligned}
$$

Corollary 6. *Let G be a complete graph K_n. Then the V-splitting graph of K_n has a Harary index given by $H(\Gamma)=\frac{1}{4}\big(7n^2-5n\big)$.*

Corollary 7. *Let G be the wheel graph W_n. Then the splitting graph $\Gamma(W_n,S)$ has a Harary index given by $W(\Gamma)=\frac{1}{2}(2n^2+5n-6)$.*

Theorem 16. *Let G be a complete graph for $n\geq 3$ and $S\subset V(K_n)$ such that $|S|=r$ where $1\leq r\leq n-1$. Then the Harary index of the splitting graph of K_n is given by $H(\Gamma(K_n,S))=\frac{2n^2-2n+r^2-3r+4nr}{4}$.*

Proof. Let $V(K_n)=\{x_1,x_2,\ldots,x_n\}$. Without loss of generality, suppose $S=\{x_1,x_2,\ldots,x_r\}$. Suppose the ordering of the vertices of the graph $\Gamma=\Gamma(K_n,S)$ is $x_1,x_2,\ldots,x_n,x_1',x_2',\ldots,x_r'$. Then we can write the distance matrix of Γ as a block matrix as follows.

$$
D(\Gamma)=\left[\begin{array}{cc|c}
\multicolumn{2}{c|}{D(K_n)} & \begin{matrix} A(K_r)+2I_r \\ J_{n-r\times r}\end{matrix} \\
\hline
A(K_r)+2I_r & J_{r\times n-r} & 2A(K_r)
\end{array}\right]
$$

where $D(K_n), A(K_r), I_r, J_{r\times n-r}$ are the distance matrix of K_n, adjacency matrix of K_r, identity matrix of size r, and the all one's matrix of size $r \times n - r$ respectively. Hence, we can compute the Harary index by

$$\begin{aligned}
H(\Gamma) &= \frac{1}{2}\sum \overline{D(\Gamma)} \\
&= \frac{1}{2}\Big(\sum D(K_n) + 2\sum \overline{A(K_r) + 2I_r} + \sum \overline{2A(K_r)} + \sum J_{n-r\times r} + \sum J_{r\times n-r}\Big) \\
&= \frac{1}{2}D(K_n) + \sum \overline{A(K_r) + 2I_r} + \frac{1}{2}\sum \overline{2A(K_r)} + \frac{1}{2}\sum J_{n-r\times r} + \frac{1}{2}\sum J_{r\times n-r} \\
&= H(K_n) + \sum \overline{A(K_r)} + \sum \overline{2I_r} + \frac{1}{2}\sum \overline{2A(K_r)} + \frac{1}{2}\sum J_{n-r\times r} + \frac{1}{2}\sum J_{r\times n-r} \\
&= \frac{n(n-1)}{2} + 2|E(K_r)| + \frac{r}{2} + \frac{1}{2}\Big(\frac{2|E(K_r)|}{2}\Big) + \frac{1}{2}(n-r)r + \frac{1}{2}(n-r)r \\
&= \frac{n(n-1)}{2} + 2|E(K_r)| + \frac{r}{2} + \frac{1}{2}|E(K_r)| + r(n-r) \\
&= \frac{n(n-1)}{2} + \frac{5}{2}|E(K_r)| + \frac{r}{2} + nr - r^2 \\
&= \frac{n(n-1)}{2} + \frac{5}{2}\Big(\frac{r(r-1)}{2}\Big) + \frac{r}{2} + nr - r^2 \\
&= \frac{n(n-1)}{2} + \frac{5r^2 - 5r}{4} + \frac{r}{2} + nr - r^2 \\
&= \frac{2n^2 - 2n + r^2 - 3r + 4nr}{4}
\end{aligned}$$

Using the distance matrices of the corresponding S-splitting graph in Theorems 11,12,and 13 respectively and the definition of the Harary index of a graph, the following results hold.

Theorem 17. *Let G be a cycle graph, and $S = \{x\}$ such that $x \in V(G)$. Then*

(i) If $n = 2k+1$, then $H(\Gamma(G,S)) = \frac{1}{2}\big(2 + (-1)^n\big) + nH_{\lfloor (n-1)/2\rfloor} + 2H_k$; and

(ii) If $n = 2k$, then $H(\Gamma(G,S)) = \frac{1}{2}\big(2 + (-1)^n\big) + nH_{\lfloor (n-1)/2\rfloor} + 2H_{k-1} + \frac{1}{k}$.

Theorem 18. *Let G be a path graph and $S \subset V(P_n) = \{x_1, \dots, x_n\}$ where $n \geq 2$. Then*

(i) If $S = \{x_1\}$ or $S = \{x_n\}$, then $H(\Gamma(P_n, S)) = (n+1)H_{n-1} - n + \frac{3}{2}$;

(ii) If $S = \{x_i\}$ such that $2 \leq i \leq n-1$, then $H(\Gamma(P_n,S)) = n\big(H_n - 1\big) + H_{n-i} + H_{i-1} + \frac{1}{2}$.

Theorem 19. *Let G be the complete bipartite graph $K_{m,n}$ with vertex partition $V = V_1 \cup V_2$ such that $|V_1| = m$ and $|V_2| = n$. Then the Harary index of $\Gamma = \Gamma(K_{m,n}, S)$ is given by*

$$H(\Gamma) = \frac{1}{4}\Big(m^2 + n^2 + i^2 + j^2 - m - n - i - j\Big) + \frac{ij}{3} + \frac{mi}{2} + \frac{nj}{2} + mn + mj + ni$$

where $|S \cap V_1| = i$ and $|S \cap V_2| = j$.

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