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A note on strongly $\delta\theta$-$\mathcal{I}$-continuous functions

José Sanabria[1,*], Rafael Lozada-Yavina[2,3], José Tormet[4]

[1] *Departamento de Matemáticas, Facultad de Educación y Ciencias, Universidad de Sucre, Sincelejo, Colombia*
[2] *Facultad de Ciencias Básicas, Universidad Católica del Maule, Talca, Chile*
[3] *Facultad de Ciencias e Ingeniería, Universidad Tecnológica del Perú, Lima, Perú*
[4] *Departamento de Ciencias, Unidad de Estudios Básicos, Universidad de Oriente, Puerto La Cruz, Venezuela*

Abstract. In this article, we investigate some properties of strongly $\delta\theta$-$\mathcal{I}$-continuous functions and other types of related functions. Specially, we characterize strongly $\delta\theta$-$\mathcal{I}$-continuous, we investigate their relationship with other types of functions, and we study the behavior of certain topological notions under the action of these functions.

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1. Introduction

The concept of an ideal on a topological space (nowadays called a topological ideal) has played a fundamental role in several of the advances in general topology. In the last century, a large number of works have arisen that have enriched the literature related to the concept of topological ideal. Very recently, topological ideals have again received special attention for their versatility in tackling topology problems and in studying rough set models, as we can see in the references [19], [7], [3], [12], [16], [5], [9], [10].

In 2014, Hatir and Al-Omari [8] introduced the concept of δ-local function and studied some of its most relevant properties. The study carried out in [8] served as motivation to define the class of the $\delta\theta$-$\mathcal{I}$ -open sets in [11], which was later used in [14] to introduce new variants of continuous functions, called $\delta\theta$-$\mathcal{I}$-continuous, weakly δ-$\mathcal{J}$-continuous and strongly $\delta\theta$-$\mathcal{I}$-continuous functions. In this article, we study and characterize the strongly $\delta\theta$-$\mathcal{I}$-continuous functions, we investigate their relationship with other types of functions, and also, we explore the behavior of some topological notions under these classes of functions.

*Corresponding author.
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Email addresses: jesanabri@gmail.com, jose.sanabria@unisucre.edu.co (J. Sanabria), trobuyo@gmail.com (R. Lozada-Yavina), ajosetormet@gmail.com (J. Tormet)

2. Preliminaries

Throughout this paper, (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, we denote the closure of A and the interior of A by $Cl(A)$ and $Int(A)$, respectively. The definitions and results that we present below can be consulted in [8], [14] and [11]. A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) *point* of A if $Int(Cl(U)) \cap A \neq \emptyset$ (resp. $Cl(U) \cap A \neq \emptyset$) for each open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp. *θ-closure*) of A and is denoted by $\delta Cl(A)$ (resp. $Cl_\theta(A)$). A subset A of X is said to be *δ-closed* (resp. *θ-closed*) if $A = \delta Cl(A)$ (resp. $A = Cl_\theta(A)$). The complement of a δ-closed (resp. θ-closed) set is said to be a *δ-open* (resp. *θ-open*) set. Similarly, the θ-interior of a subset A of X, denoted by $Int_\theta(A)$, consists of all points x in X such that for some open set U containing x, $Cl(U) \subset A$. It is well known that, a subset A of X is θ-open if and only if $A = Int_\theta(A)$. It follows that the collection of all δ-open (resp. θ-open) sets in a topological space (X, τ) forms a topology on X which is denoted by τ_δ (resp. τ_θ). From the definitions it follows that $\tau_\theta \subset \tau_\delta \subset \tau$. The topology τ_δ is called the *semi-regularization* of τ. Observe that δCl is the closure operator with respect to τ_δ, but Cl_θ is not the closure operator with respect to τ_θ.

An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following two properties: (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An *ideal topological space* (or simply a space) is a topological space (X, τ) together with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. If $(X, \tau, \mathcal{I})$ is a space, then for any $A \subset X$, the *local function* (rep. δ-local function) of A with respect to $\mathcal{I}$ and τ, denoted by $A^\star(\mathcal{I}, \tau)$ (resp. $A^{\delta\star}(\mathcal{I}, \tau)$), is defined as $A^\star(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ (resp. $A^{\delta\star}(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every δ-open set U containing $x\}$). We simply write $A^\star$ (resp. $A^{\delta\star}$) in case there is no chance for confusion. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \{\emptyset\}$. We call the ideals which satisfy this condition *τ-boundary* ideals. A Kuratowski closure operator $Cl^\star(.)$ (resp. $\delta Cl^\star(.)$) for a topology $\tau^\star$ (res. $\tau^{\delta\star}$) finer than τ (resp. τ_δ), is defined by $Cl^\star(A) = A \cup A^\star$ (resp. $\delta Cl^\star(A) = A \cup A^{\delta\star}$). It is well known that $\tau_\delta \subset \tau \subset \tau^\star$ and $\tau_\delta \subset \tau_\delta^\star \subset \tau^\star$. A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-cluster point* of A if $\delta Cl^\star(U) \cap A \neq \emptyset$ for every open subset U of X containing x. The set of all $\delta\theta$-$\mathcal{I}$-cluster points of A is called the *$\delta\theta$-$\mathcal{I}$-closure* of A and is denoted by $\delta Cl_\theta^\star(A)$. A subset A of X is said to be *$\delta\theta$-$\mathcal{I}$-closed* if $\delta Cl_\theta^\star(A) = A$. The complement of a $\delta\theta$-$\mathcal{I}$-closed set is said to be *$\delta\theta$-$\mathcal{I}$-open* set. A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-interior point* of a subset A of X if there exists an open set U such that $x \in U \subset \delta\mathrm{Cl}^\star(U) \subset A$. The set of all $\delta\theta$-$\mathcal{I}$-interior points of A is called the *$\delta\theta$-$\mathcal{I}$-interior* of A and is denoted by $\delta Int_\theta^\star(A)$. In [11, Proposition 4.1] it was shown that a subset A of X is $\delta\theta$-$\mathcal{I}$-open if and only if $A = \delta Int_\theta^\star(A)$.

Since the main objective of this article is to study strongly $\delta\theta$-$\mathcal{I}$-continuous functions, now we will recall some variants of continuity related to the type of functions that we will study. In what follows, we consider that $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ are spaces.

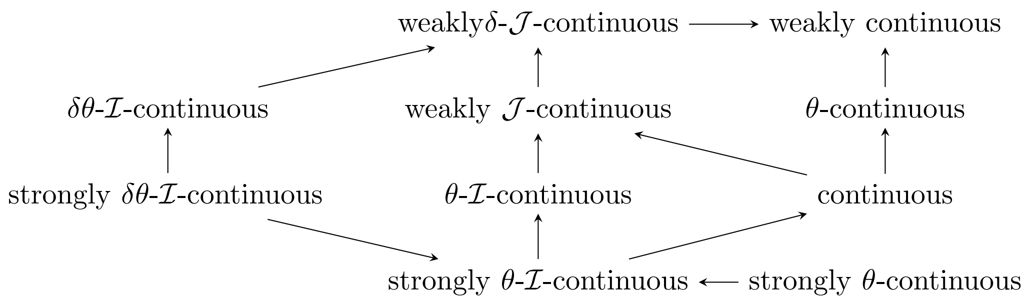
Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be* strongly θ-continuous *[18],* *(resp.* weakly continuous *[13],* θ-continuous *[15]) if for each* $x \in X$ *and each open set* V *in* Y *containing* $f(x)$*, there exists an open set* U *in* X *containing* x *such that* $f(Cl(U)) \subset V$ *(resp.* $f(U) \subset Cl(V)$*,* $f(Cl(U)) \subset Cl(V)$*).*

Definition 2. *A function* $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ *is said to be* weakly $\mathcal{J}$-continuous *[1] (resp.* weakly δ-$\mathcal{J}$-continuous *[14]), if for each* $x \in X$ *and each open set* V *in* Y *containing* $f(x)$*, there exists an open set* U *in* X *containing* x *such that* $f(U) \subset Cl^\star(V)$ *(resp.* $f(U) \subset \delta Cl^\star(V)$*).*

Definition 3. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is said to be* $\delta\theta$-$\mathcal{I}$-continuous *[14] (resp.* θ-$\mathcal{I}$-continuous *[17]), if for each* $x \in X$ *and each open set* V *in* Y *containing* $f(x)$*, there exists an open set* U *containing* x *such that* $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$ *(resp.* $f(Cl^\star(U)) \subset Cl^\star(V)$*).*

Definition 4. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is said to be* strongly θ-$\mathcal{I}$-continuous *[17] (resp.* strongly $\delta\theta$-$\mathcal{I}$-continuous *[14]), if for each* $x \in X$ *and each open set* V *in* Y *containing* $f(x)$*, there exists an open set* U *in* X *containing* x *such that* $f(Cl^\star(U)) \subset V$ *(resp.* $f(\delta Cl^\star(U)) \subset V$*).*

Remark 1. *The following diagram shows the relationship between the types of functions given in Definitions 1, 2, 3 and 4. In general none of the implications is reversible.*



3. Properties related to strongly $\delta\theta$-$\mathcal{I}$-continuous functions

In this section, we characterize strongly $\delta\theta$-$\mathcal{I}$-continuous. Also, we looking for some topological conditions in order to find the relationship between the strongly $\delta\theta$-$\mathcal{I}$-continuous functions, $\delta\theta$-$\mathcal{I}$-continuous functions and weakly δ-$\mathcal{J}$-continuous functions.

First, we characterize strongly $\delta\theta$-$\mathcal{I}$-continuous functions using $\delta\theta$-$\mathcal{I}$-open sets and $\delta\theta$-$\mathcal{I}$-closed sets.

Theorem 1. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$*. Then, the following properties are equivalent:*

(i) f *is strongly* $\delta\theta$-$\mathcal{I}$*-continuous.*

(ii) $f^{-1}(V)$ *is a* $\delta\theta$-$\mathcal{I}$*-open set in* X*, for each open set* $V \subset Y$*.*

(iii) $f^{-1}(F)$ *is a* $\delta\theta$-$\mathcal{I}$*-closed set in* X*, for each closed set* $F \subset Y$.

(iv) $f(\delta Cl^{\star}_{\theta}(A)) \subset Cl(f(A))$*, for each* $A \subset X$.

(v) $\delta Cl^{\star}_{\theta}(f^{-1}(B)) \subset f^{-1}(Cl(B))$*, for each* $B \subset Y$.

Proof. $(i) \Rightarrow (ii)$ Suppose that V is an open set in Y and $x \in f^{-1}(V)$. Since f is strongly $\delta\theta$-$\mathcal{I}$-continuous, there exists an open set U in X containing x such that $f(\delta Cl^{\star}(U)) \subset V$. Thus, $x \in U \subset \delta Cl^{\star}(U) \subset f^{-1}(V)$. Consequently, $f^{-1}(V)$ is a $\delta\theta$-$\mathcal{I}$-open set in X.

$(ii) \Rightarrow (iii)$ Let F be a closed set in Y. Then, $V = Y - F$ is an open set in Y and by (ii), $f^{-1}(V) = f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Therefore, $f^{-1}(F)$ is a $\delta\theta$-$\mathcal{I}$-closed set in X.

$(iii) \Rightarrow (iv)$ Let $A \subset X$. Since $Cl(f(A))$ is a closed set in Y, by hypothesis, it follows that $f^{-1}(Cl(f(A)))$ is a $\delta\theta$-$\mathcal{I}$-closed set. Then, we have $\delta Cl^{\star}_{\theta}(A) \subset \delta Cl^{\star}_{\theta}(f^{-1}(f(A))) \subset \delta Cl^{\star}_{\theta}(f^{-1}(Cl(f(A)))) = f^{-1}(Cl(f(A)))$, which implies that $f(\delta Cl^{\star}_{\theta}(A)) \subset Cl(f(A))$ for each $A \subset X$.

$(iv) \Rightarrow (v)$ Let $B \subset Y$. By hypothesis, we have $f(\delta Cl^{\star}_{\theta}(f^{-1}(B))) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$. Consequently, $\delta Cl^{\star}_{\theta}(f^{-1}(B)) \subset f^{-1}(Cl(B))$.

$(v) \Rightarrow (i)$ Let $x \in X$ and $V \in \sigma$ be such that $f(x) \in V$. Then, $Y - V$ is a closed set in Y and by hypothesis, $\delta Cl^{\star}_{\theta}(f^{-1}(Y - V)) \subset f^{-1}(Cl(Y - V)) = f^{-1}(Y - V)$, which tells us that $f^{-1}(Y - V)$ is a $\delta\theta$-$\mathcal{I}$-closed set in X. Since $f^{-1}(Y - V) = X - f^{-1}(V)$, we conclude that $f^{-1}(V)$ is a $\delta\theta$-$\mathcal{I}$-open set containing x. Thus, there exists $U \in \tau$ such that $x \in U \subset \delta Cl^{\star}(U) \subset f^{-1}(V)$. Therefore, $f(\delta Cl^{\star}(U)) \subset V$ and so, f is a strongly $\delta\theta$-$\mathcal{I}$-continuous function.

According to [14], we say that a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is $\delta\theta$-$\mathcal{I}$-irresolute, if $f^{-1}(V)$ is a $\delta\theta$-$\mathcal{I}$-open set in X, for each $\delta\theta$-$\mathcal{J}$-open set $V \subset Y$. This type of functions is used in the following theorem, where we analyze when the composition of functions is strongly $\delta\theta$-$\mathcal{I}$-continuous.

Theorem 2. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *and* $g : (Y, \sigma, \mathcal{J}) \to (Z, \varphi)$ *be two functions. Then, the following properties hold:*

(i) If f *is strongly* $\delta\theta$-$\mathcal{I}$*-continuous and* g *is continuous, then* $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \varphi)$ *is strongly* $\delta\theta$-$\mathcal{I}$*-continuous.*

(ii) If f *is* $\delta\theta$-$\mathcal{I}$*-irresolute and* g *is strongly* $\delta\theta$-$\mathcal{J}$*-continuous, then* $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \varphi)$ *is strongly* $\delta\theta$-$\mathcal{I}$*-continuous.*

Proof. (i) Let $V \in \varphi$. Since g is continuous, $g^{-1}(V)$ is an open set in Y, and as f is strongly $\delta\theta$-$\mathcal{I}$-continuous, by Theorem 1, it follows that $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Again, by Theorem 1, we get that $g \circ f$ is a strongly $\delta\theta$-$\mathcal{I}$-continuous function.

The proof of (ii) is similar to that of (i).

Theorem 3. *Let $\{(Y_\lambda, \sigma_\lambda) : \lambda \in \Lambda\}$ be a collection of topological spaces and let $Y = \prod\{Y_\lambda : \lambda \in \Lambda\}$ with product topology $\sigma = \prod\{\sigma_\lambda : \lambda \in \Lambda\}$ induced by the projections $p_\lambda : Y \to Y_\lambda$, $\lambda \in \Lambda$. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly $\delta\theta$-$\mathcal{I}$-continuous if and only if every composition $p_\lambda \circ f$ is strongly $\delta\theta$-$\mathcal{I}$-continuous.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a strongly $\delta\theta$-$\mathcal{I}$-continuous function and let $\lambda \in \Lambda$. Since the projection $p_\lambda : Y \to Y_\lambda$ is continuous, by Theorem 2, we get that $p_\lambda \circ f : (X, \tau, \mathcal{I}) \to (Y_\lambda, \sigma_\lambda)$ is strongly $\delta\theta$-$\mathcal{I}$-continuous for every $\lambda \in \Lambda$. Conversely, suppose that every composition $p_\lambda \circ f : (X, \tau, \mathcal{I}) \to (Y_\lambda, \sigma_\lambda)$ is strongly $\delta\theta$-$\mathcal{I}$-continuous. Since the collection $\tau_{\delta\theta-\mathcal{I}}$ of all $\delta\theta$-$\mathcal{I}$-open sets in X is a topology, it suffices to show that the inverse image under f of each subbasic open set of $Y = \prod\{Y_\lambda : \lambda \in \Lambda\}$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Let $p_\lambda^{-1}(V_\lambda)$ be a subbasic open set in Y. Then, $V_\lambda \in \sigma_\lambda$ and so, $f^{-1}\left(p_\lambda^{-1}(V_\lambda)\right) = (p_\lambda \circ f)^{-1}(V_\lambda)$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Therefore, f is strongly $\delta\theta$-$\mathcal{I}$-continuous.

Corollary 1. *Let $(X, \tau, \mathcal{I})$ be a space, $\{(Y_\lambda, \sigma_\lambda) : \lambda \in \Lambda\}$ be a collection of topological spaces and $f_\lambda : (X, \tau, \mathcal{I}) \to (Y_\lambda, \sigma_\lambda)$ be a function for every $\lambda \in \Lambda$. Let $\sigma = \prod\{\sigma_\lambda : \lambda \in \Lambda\}$ be the product topology on $Y = \prod\{Y_\lambda : \lambda \in \Lambda\}$ and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be the function defined by $f(x) = (f_\lambda(x))_{\lambda \in \Lambda}$ for each $x \in X$. Then, f is strongly $\delta\theta$-$\mathcal{I}$-continuous if and only if f_λ is strongly $\delta\theta$-$\mathcal{I}$-continuous for every $\lambda \in \Lambda$.*

Next, we present some topological notions to establish some properties related to strongly $\delta\theta$-$\mathcal{I}$-continuous functions.

Definition 5. *A space $(X, \tau, \mathcal{I})$ is said to be $\delta\star$-regular [14] (resp. $\star$-regular [2]), if for each pair consisting of a closed set F and a point $x \notin F$, there exist $V \in \tau$ and $U \in \tau^{\delta\star}$ (resp. $U \in \tau^\star$) such that $x \in V$, $F \subset U$ and $U \cap V = \emptyset$.*

The concepts of $\delta\star$-regular space and $\star$-regular space are related as follows.

Lemma 1. *Every $\delta\star$-regular space is a $\star$-regular.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is a $\delta\star$-regular space. Let F be a closed set and $x \notin F$. Then, there exist $V \in \tau$ and $U \in \tau^{\delta\star}$ such that $x \in V$, $F \subset U$ and $U \cap V = \emptyset$. Since $\tau^{\delta\star} \subset \tau^\star$, we have $U \in \tau^\star$ and hence, $(X, \tau, \mathcal{I})$ is a $\star$-regular space.

The following example shows that, in general, the converse of Lemma 1 is not true.

Example 1. ***A $\star$-regular space need not be $\delta\star$-regular space.*** *Consider the space $(X, \tau, \mathcal{I})$, where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{a, b, d\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Observe that the collection of all closed sets is $\{\emptyset, X, \{c\}, \{d\}, \{c, d\}\}$. Also, $\tau^\star = \mathcal{P}(X)$, $\tau_\delta = \{\emptyset, X\}$ and $\tau^{\delta\star} = \{\emptyset, X, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$. Then, we have:*

(i) For $F_1 = \{d\}$ and $a \notin F_1$, there exist $V_1 = \{a, b\} \in \tau$ and $U_1 = \{c, d\} \in \tau^\star$ such that $a \in V_1$, $F_1 \subset U_1$ and $V_1 \cap U_1 = \emptyset$.

(ii) For $F_1 = \{d\}$ and $b \notin F_1$, there exist $V_1 = \{a, b\} \in \tau$ and $U_1 = \{c, d\} \in \tau^\star$ such that $b \in V_1$, $F_1 \subset U_1$ and $V_1 \cap U_1 = \emptyset$.

(iii) *For* $F_1 = \{d\}$ *and* $c \notin F_1$*, there exist* $V_2 = \{a, b, c\} \in \tau$ *and* $U_2 = \{d\} \in \tau^\star$ *such that* $c \in V_2$*,* $F_1 \subset U_2$ *and* $V_2 \cap U_2 = \emptyset$*.*

(iv) *For* $F_2 = \{c\}$ *and* $a \notin F_2$*, there exist* $V_1 = \{a, b\} \in \tau$ *and* $U_1 = \{c, d\} \in \tau^\star$ *such that* $a \in V_1$*,* $F_2 \subset U_1$ *and* $V_1 \cap U_1 = \emptyset$*.*

(v) *For* $F_2 = \{c\}$ *and* $b \notin F_2$*, there exist* $V_1 = \{a, b\} \in \tau$ *and* $U_1 = \{c, d\} \in \tau^\star$ *such that* $b \in V_1$*,* $F_2 \subset U_1$ *and* $V_1 \cap U_1 = \emptyset$*.*

(vi) *For* $F_2 = \{c\}$ *and* $d \notin F_2$*, there exist* $V_3 = \{a, b, d\} \in \tau$ *and* $U_3 = \{c\} \in \tau^\star$ *such that* $d \in V_3$*,* $F_2 \subset U_3$ *and* $V_3 \cap U_3 = \emptyset$*.*

(vii) *For* $F_3 = \{c, d\}$ *and* $a \notin F_3$*, there exist* $V_1 = \{a, b\} \in \tau$ *and* $U_1 = \{c, d\} \in \tau^\star$ *such that* $a \in V_1$*,* $F_3 \subset U_1$ *and* $V_1 \cap U_1 = \emptyset$*.*

(viii) *For* $F_3 = \{c, d\}$ *and* $b \notin F_3$*, there exist* $V_1 = \{a, b\} \in \tau$ *and* $U_1 = \{c, d\} \in \tau^\star$ *such that* $b \in V_1$*,* $F_3 \subset U_1$ *and* $V_1 \cap U_1 = \emptyset$*.*

By (i)*-*$(viii)$*, we deduce that* $(X, \tau, \mathcal{I})$ *is a* $\star$*-regular space. Now, we will show that* $(X, \tau, \mathcal{I})$ *is not a* $\delta\star$*-regular space. Indeed, let* $F = \{d\}$*. Then,* F *is a closed set and* $c \notin F$*. Observe that the only open sets containing* c *are* $V_2 = \{a, b, c\}$ *and* $V_4 = X$*, and the only* $\tau^{\delta\star}$*-open sets containing* F *are* $W_1 = \{c, d\}$*,* $W_2 = \{b, c, d\}$ *and* $W_3 = \{a, c, d\}$*. In addition,* $V_2 \cap W_1 = \{c\} \neq \emptyset$*,* $V_2 \cap W_2 = \{b, c\} \neq \emptyset$*,* $V_2 \cap W_3 = \{a, c\} \neq \emptyset$*,* $V_4 \cap W_1 = W_1 \neq \emptyset$*,* $V_4 \cap W_2 = W_2 \neq \emptyset$*,* $V_4 \cap W_3 = W_3 \neq \emptyset$*. Therefore,* $(X, \tau, \mathcal{I})$ *is not a* $\delta\star$*-regular space.*

Lemma 2. *[14] A space* $(X, \tau, \mathcal{I})$ *is* $\delta\star$*-regular if and only if for each* $x \in X$ *and each open set* V *containing* x*, there exists an open set* U *such that* $x \in U \subset \delta Cl^\star(U) \subset V$*.*

The following result shows that the strongly $\delta\theta$-$\mathcal{I}$-continuity and the continuity are equivalent under the assumption that the domain is a $\delta\star$-regular space.

Theorem 4. *Let* $(X, \tau, \mathcal{I})$ *be a* $\delta\star$*-regular space. A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is strongly* $\delta\theta$*-*$\mathcal{I}$*-continuous if and only if it is continuous.*

Proof. By Remark 1, if f is strongly $\delta\theta$-$\mathcal{I}$-continuous then it is continuous. Conversely, suppose that f is continuous. Let $x \in X$ and V be any open set in Y containing $f(x)$. Then, there exists an open set U containing x such that $f(U) \subset V$. Since $(X, \tau, \mathcal{I})$ is a $\delta\star$-regular space, by Lemma 2, there exists an open set W such that $x \in W \subset \delta Cl^\star(W) \subset U$. Thus, $f(x) \in f(\delta Cl^\star(W)) \subset f(U) \subset V$, which implies that $f(\delta Cl^\star(W)) \subset V$. Therefore, f is strongly $\delta\theta$-$\mathcal{I}$-continuous.

Theorem 5. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be a function and* $(X, \tau, \mathcal{I})$ *be a* $\delta\star$*-regular space. If* f *is strongly* θ*-continuous, then it is strongly* $\delta\theta$*-*$\mathcal{I}$*-continuous.*

Proof. The proof is similar to the part of the proof of Theorem 4, where it was proved that if $(X, \tau, \mathcal{I})$ is $\delta\star$-regular and f is continuous, then f is strongly $\delta\theta$-$\mathcal{I}$-continuous.

The following example shows that a strongly $\delta\theta$-$\mathcal{I}$-continuous function is not necessarily strongly θ-continuous, even though the domain is a $\delta\star$-regular space.

Example 2. *Consider* $X = \{a, b, c, d\}$ *with the topology* $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ *and the ideal* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. *Let us define a function* $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ *as follows:* $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = b$. *Note that:*

(i) The only open sets in X *containing* $f(a) = a$ *are* $V_1 = \{a, b\}$, $V_2 = \{a, b, c\}$, $V_3 = \{a, b, d\}$ *and* $V_4 = X$. *In addition,* V_1 *satisfies that* $f(\delta Cl^\star(V_1)) = f(\delta Cl^\star(\{a, b\})) = f(\{a, b\}) = \{a, b\} = V_1$, $f(\delta Cl^\star(V_1)) = V_1 \subset V_2$, $f(\delta Cl^\star(V_1)) = V_1 \subset V_3$ *and* $f(\delta Cl^\star(V_1)) = V_1 \subset V_4$.

(ii) Using the same argument from part (1), we get the result for $f(b) = b$.

(iii) The only open sets in X *containing* $f(c) = c$ *are* $V_2 = \{a, b, c\}$ *and* $V_4 = X$. *In addition,* V_2 *satisfies that* $f(\delta Cl^\star(V_2)) = f(\delta Cl^\star(\{a, b, c\})) = f(\{a, b, c\}) = \{a, b, c\} = V_2$ *and* $f(\delta Cl^\star(V_2)) = V_2 \subset V_4$.

(iv) The only open sets in X *containing* $f(d) = b$ *are* $V_1 = \{a, b\}$, $V_2 = \{a, b, c\}$, $V_3 = \{a, b, d\}$ *and* $V_4 = X$. *In addition,* $V_3 = \{a, b, d\}$ *is an open set in* X *containing* d *such that* $f(\delta Cl^\star(V_3)) = f(\delta Cl^\star(\{a, b, d\})) = f(\{a, b, d\}) = \{a, b\} = V_1$ *which is contained in* V_1, V_2, V_3 *and* V_4.

By (*i*)*-*(*iv*), *we conclude that* f *is strongly* $\delta\theta$*-*$\mathcal{I}$*-continuous. On the other hand, since* $Cl(V_1) = Cl(V_2) = Cl(V_3) = Cl(V_4) = X$ *and* $f(X) = V_2 \not\subset V_1$, *we conclude that* f *is not strongly* θ*-continuous. Observe that* $(X, \tau, \mathcal{I})$ *is a* $\delta\star$*-regular space that is not regular.*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is super-continuous [6] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Int(Cl(U))) \subset V$. This type of function is characterized by the property that the inverse image of each open set in Y is a δ-open set in X. Clearly, every super-continuous function is continuous, but the converse, in general, is not true.

Theorem 6. *Let* (Y, σ) *be a regular space and* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be a function. If* f *is super-continuous, then it is strongly* $\delta\theta$*-*$\mathcal{I}$*-continuous.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Since Y is regular, there exists an open set U such that $f(x) \in U \subset Cl(U) \subset V$. On the other hand, as f is super-continuous, there exists a δ-open set W containing x such that $f(W) \subset U$. We will show that $f(\delta Cl^\star(W)) \subset Cl(U)$. Indeed, suppose that $y \notin Cl(U)$. Then, we choose an open set O such that $y \in O$ and $O \cap U = \emptyset$. By the super-continuity of f, we have $f^{-1}(O)$ is a δ-open set such that $f^{-1}(O) \cap f^{-1}(U) = \emptyset$, which implies that $f^{-1}(O) \cap W = \emptyset$. We affirm that $f^{-1}(O) \cap \delta Cl^\star(W) = \emptyset$. Otherwise, there exists a point $z \in f^{-1}(O) \cap \delta Cl^\star(W)$ and so, $z \in f^{-1}(O)$ and $z \in \delta Cl^\star(W)$. It follows that $z \in W$ or $z \in W^{\delta\star}$. If $z \in W$ then $z \in f^{-1}(O) \cap W$, which is a contradiction. If $z \in W^{\delta\star}$ then $G \cap W \notin \mathcal{I}$ for each δ-open set G containing z; in particular, $f^{-1}(O) \cap W \notin \mathcal{I}$, which implies that $f^{-1}(O) \cap W \neq \emptyset$, so again we get a contradiction. Therefore, we deduce that $f^{-1}(O) \cap \delta Cl^\star(W) = \emptyset$. Thus, $O \cap f(\delta Cl^\star(W)) = \emptyset$ and hence, $y \notin f(\delta Cl^\star(W))$. This shows that $f(\delta Cl^\star(W)) \subset Cl(U) \subset V$ and so, f is strongly $\delta\theta$-$\mathcal{I}$-continuous.

In the following example, we show that there exists a strongly $\delta\theta$-$\mathcal{I}$-continuous function that is not super-continuous.

Example 3. *The function f given in Example 2 is strongly continuous, but it is not super-continuous, because in this case $\tau_\delta = \{\emptyset, X\}$ and $\{a,b\}$ is an open set such that $f^{-1}(\{a,b\}) = \{a,b\} \notin \tau_\delta$.*

Theorem 7. *Let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be any function and $g : (X,\tau,\mathcal{I}) \to (X \times Y, \tau \times \sigma)$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$, where $\tau \times \sigma$ is the product topology on $X \times Y$. Then, g is strongly $\delta\theta$-$\mathcal{I}$-continuous if and only if f is strongly $\delta\theta$-$\mathcal{I}$-continuous and $(X,\tau,\mathcal{I})$ is $\delta\star$-regular.*

Proof. Clearly, $g(x) = (1_X(x), f(x))$ for every $x \in X$, where $1_X : (X,\tau,\mathcal{I}) \to (X,\tau)$ is the identity function on X. Then, by Corollary 1, g is strongly $\delta\theta$-$\mathcal{I}$-continuous if and only if 1_X and f are strongly $\delta\theta$-$\mathcal{I}$-continuous. In addition, 1_X is strongly $\delta\theta$-$\mathcal{I}$-continuous if and only if for each $x \in X$ and each open set V in X containing x, there exists an open set U in X containing x such that $\delta Cl^\star(U) \subset V$, but by Lemma 2, the latter is equivalent to that $(X,\tau,\mathcal{I})$ is a $\delta\star$-regular space.

Definition 6. *[14] A space $(X,\tau,\mathcal{I})$ is said to be $\delta\star$-Urysohn, if for each pair of distinct points x and y in X, there exist two open subsets U and V of X containing x and y respectively, such that $\delta Cl^\star(U) \cap \delta Cl^\star(V) = \emptyset$.*

Theorem 8. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is a strongly $\delta\theta$-$\mathcal{I}$-continuous injective function and (Y,σ) is T_2, then $(X,\tau,\mathcal{I})$ is $\delta\star$-Urysohn.*

Proof. Let x and y be distinct points of X. Then, $f(x) \neq f(y)$ and as (Y,σ) is T_2, there exist disjoint open sets V and W in Y containing $f(x)$ and $f(y)$, respectively. Since f is strongly $\delta\theta$-$\mathcal{I}$-continuous, there exist two open sets G and H in X containing x and y, respectively, such that $f(\delta Cl^\star(G)) \subset V$ and $f(\delta Cl^\star(H)) \subset W$. It follows that $\delta Cl^\star(G) \cap \delta Cl^\star(H) \subset f^{-1}(f(\delta Cl^\star(G))) \cap f^{-1}(f(\delta Cl^\star(H))) \subset f^{-1}(V) \cap f^{-1}(W) = \emptyset$, and hence, $\delta Cl^\star(G) \cap \delta Cl^\star(H) = \emptyset$. This shows that $(X,\tau,\mathcal{I})$ is $\delta\star$-Urysohn.

Theorem 9. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is a strongly $\delta\theta$-$\mathcal{I}$-continuous injective function and (Y,σ) is T_0, then $(X,\tau^\star)$ is T_2.*

Proof. Let x and y be distinct points of X. Then, $f(x) \neq f(y)$ and as (Y,σ) is T_0, there exists an open set V in Y containing one the points $f(x)$ and $f(y)$ but not both. Without loss of generality, we assume that $f(x) \in V$ and $f(y) \notin V$. Since f is a strongly $\delta\theta$-$\mathcal{I}$-continuous function, there exists an open set U in X containing x such that $f(\delta Cl^\star(U)) \subset V$. Thus, we obtain that $x \in U \subset \delta Cl^\star(U) \subset f^{-1}(f(\delta Cl^\star(U))) \subset f^{-1}(V)$ and $y \notin \delta Cl^\star(U)$, which implies that U and $X \setminus \delta Cl^\star(U)$ are two disjoint $\tau^\star$-open sets in X containing x and y, respectively. Therefore, $(X,\tau^\star)$ is a T_2-space.

Definition 7. *A* strongly $\delta\theta$-$\mathcal{I}$-continuous retraction *is a strongly $\delta\theta$-$\mathcal{I}$-continuous function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$, where $Y \subset X$ and $f|_Y = 1_Y$ the identity function on Y.*

Theorem 10. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a strongly $\delta\theta$-$\mathcal{I}$-continuous retraction and (Y, σ) is T_2, then Y is a $\delta\theta$-$\mathcal{I}$-closed set in X.*

Proof. We will show that $X \setminus A$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Let $x \in X \setminus A$. Then, $f(x) \in A$ and $x \notin A$ because f is a strongly $\delta\theta$-$\mathcal{I}$-continuous retraction. Since (Y, σ) is T_2, there exist two disjoint open sets V and W in Y containing x and $f(x)$, respectively. By the strongly $\delta\theta$-$\mathcal{I}$-continuity of f, we have $f^{-1}(W)$ is a $\delta\theta$-$\mathcal{I}$-open set in X containing x. Thus, there exists an open set G in X such that $x \in G \subset \delta Cl^\star(G) \subset f^{-1}(W)$. Let us observe that $U = G \cap V$ is an open set in X containing x. We affirm that $\delta Cl^\star(U) \subset X \setminus A$ and so, $X \setminus A$ is a $\delta\theta$-$\mathcal{I}$-open set in X. Indeed, if $y \in \delta Cl^\star(U)$ then $y \in \delta Cl^\star(G) \subset f^{-1}(W)$, which implies that $f(y) \in W$, and as V and W are disjoint sets, it follows that $f(y) \notin V$, but since $y \in V$, we get that $y \neq f(y)$ and hence, $y \notin A$.

Definition 8. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be* strongly $\delta\theta$-$\mathcal{I}$-closed with respect to X, *if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist two open sets U and V containing x and y, respectively, such that $(\delta Cl^\star(U) \times V) \cap G(f) = \emptyset$.*

Lemma 3. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly $\delta\theta$-$\mathcal{I}$-closed with respect to X if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist two open sets U and V containing x and y, respectively, such that $f(\delta Cl^\star(U)) \cap V = \emptyset$.*

Theorem 11. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly $\delta\theta$-$\mathcal{I}$-continuous and (Y, σ) is T_2, then $G(f)$ is strongly $\delta\theta$-$\mathcal{I}$-closed with respect to X.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then, $f(x) \neq y$ and as (Y, σ) is T_2, there exist open sets V and W in Y containing $f(x)$ and y, respectively, such that $V \cap W = \emptyset$. Since f is strongly $\delta\theta$-$\mathcal{I}$-continuous, there exists an open sets U in X containing x such that $f(\delta Cl^\star(U)) \subset V$, which implies that $f(\delta Cl^\star(U)) \cap W = \emptyset$. By Lemma 3, we conclude that $G(f)$ is strongly $\delta\theta$-$\mathcal{I}$-closed with respect to X.

4. Conclusion

The notion of a continuous function and its generalizations have important applications in various areas of mathematics and related sciences; for example, this notion is widely used in physics and information systems, as described in [4]. In this article we have studied a generalization of continuous functions using concepts recently derived from the theory of topological ideals, such as $\delta\theta$-$\mathcal{I}$-open set and $\delta\theta$-$\mathcal{I}$-closure operator. This class of functions can have applications in computation and image design, especially in digital topology, as well as in information systems and quantum physics. On the other hand, the notions discussed here could be extended to contexts such as a topological space endowed with a hereditary class, fuzzy ideal topological spaces and soft ideal topological spaces, where the results could be used in problems dealing with uncertainty and vagueness.

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