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Atomic Solution of Poisson Type Equation

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Abstract. In this paper we find a certain solution of Poisson type fractional differential equation using theory of tensor product of Banach spaces.

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1. Introduction

In [5], a new definition, which is called conformable fractional derivative was introduced.

For a given a function $f : [0, \infty) \to R$. The **conformable fractional derivative** of f of order α for $\alpha \in (0, 1]$, is defined by:

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

If the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

For $\alpha \in (0, 1]$ and for f, g which $\alpha-$differentiable functions at a point t, the conformable derivative satisfies:

1. $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$, for all a, $b \in \mathbb{R}$.
2. $D^\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
3. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.
4. $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.

We list here the fractional derivatives of certain functions:

1. $D^\alpha(c) = 0$, where c is constant.
2. $D^\alpha(e^{ct}) = ct^{1-\alpha}e^{ct}, c \in \mathbb{R}$
3. $D^\alpha(\cos bt) = -bt^{1-\alpha} \sin bt, b \in \mathbb{R}$.
4. $D^\alpha(\sin bt) = bt^{1-\alpha} \cos bt, b \in \mathbb{R}$.

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5. $D^\alpha(\frac{1}{\alpha}t^\alpha) = \alpha(\frac{1}{\alpha}t^{\alpha-\alpha}) = 1$.
6. $D^\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$.
7. $D^\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$.
8. $D^\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

2. Atomic solution

Let X and Y be Banach spaces and X^* denotes the dual of X. For $x \in X$ and $y \in Y$ define

$T_{(x,y)} : X^* \to Y$ as:

$T_{(x,y)}(x^*) = \langle x^*, x\rangle\, y$.

We denote $T_{(x,y)}$ by $x \otimes y$ and we call $x \otimes y$ an atom [6].

Atoms are used in theory of best approximation in Banach spaces, see [3].

One of the known results, see [4], that we need in our paper is that: If the sum of two atoms is an atom, then either

the first components are dependent or the second ones are dependent.

Let us write $D^{2\alpha}f$ to mean $D^\alpha D^\alpha f$. Further we write $f^{(\alpha)}, f^{(2\alpha)}$ to denote $D^\alpha f$, $D^{2\alpha}f$ respectively.

If u is a function of two variables, say x, y , we write $D_x^\alpha u$ for the partial $\alpha-$derivative of u with respect to x, and by $D_x^{2\alpha}u$ we mean $D_x^\alpha D_x^\alpha u$. Similarly for derivatives with respect to y.

Our main object in this paper is to find an atomic solution of the Poisson type fractional differential equation:

$$D_x^{2\alpha}u(x,y) + D_x^\alpha D_y^\beta u(x,y) = f(x,y), 0 < \alpha, \beta < 1, \tag{1}$$

where $f(x,y)$ is a given function.

3. Procedure

In order to find a certain atomic solution, assume $f(x,y) = A(x)B(y)$, is given.

Now, put

$$u(x,y) = P(x)Q(y) \tag{2}$$

and assume that

$$P(0) = 1, P^{(\alpha)}(0) = 1, \text{and } Q(0) = 1.$$

Substitute $u(x,y) = P(x)Q(y)$ in (1) to get:

$$P^{(2\alpha)}(x)Q(y) + P^{(\alpha)}(x)Q^{(\beta)}(y) = A(x)B(y) \tag{3}$$

This can be written in tensor product form as:

$$P^{(2\alpha)}(x) \otimes Q(y) + P^{(\alpha)}(x) \otimes Q^{(\beta)}(y) = A(x) \otimes B(y) \tag{4}$$

Now, we have a situation where the sum of two atoms is an atom. Hence, we have two cases:

Case (i) $P^{(2\alpha)}(x) = P^{(\alpha)}(x) = A(x)$.
Consider $\quad P^{(2\alpha)}(x) = P^{(\alpha)}(x)$
this is a $2\alpha-$ order linear differential equation [7], so the corresponding auxiliary equation is

$$r^2 - r = 0$$

Hence $r = 0, 1$,and

$$P(x) = c_1 + c_2 e^{x^\alpha/\alpha}$$

Since $P^{(\alpha)}(0) = 1$,we have $c_2 = 1$
and since $P(0) = 1$, then $c_1 = 0$
So

$$P^{(\alpha)}(x) = c_2 e^{x^\alpha/\alpha}$$

Hence

$$P(x) = e^{x^\alpha/\alpha}. \tag{5}$$

Thus $A(x)$ must equal to $e^{x^\alpha/\alpha}$ in order to an atomic solution to be exist.
Substitute in (3) to get

$$e^{x^\alpha/\alpha}Q(y) + e^{x^\alpha/\alpha}(x)Q^{(\beta)}(y) = e^{x^\alpha/\alpha}B(y).$$

Hence

$$Q(y) + Q^{(\beta)}(y) = B(y). \tag{6}$$

This is a linear fractional differential equation of order β. Hence, using result in [1], we multiply equation (6) by the integrating factor

$$\mu(y) = e^{I_\beta(1)} = e^{\int\limits_a^y \frac{1}{t^{1-\beta}} dt} = e^{y^\beta/\beta}.$$

to get

$$e^{y^\beta/\beta} Q(y) + e^{y^\beta/\beta} Q^{(\beta)}(y) = e^{y^\beta/\beta} B(y),$$

$$D^\beta[e^{y^\beta/\beta} Q(y)] = e^{y^\beta/\beta} B(y).$$

Hence

$$Q(y) = e^{-y^\beta/\beta} I_\beta[e^{y^\beta/\beta} B(y)].$$

So

$$Q(y) = e^{-y^\beta/\beta} \int^y \frac{e^{t^\beta/\beta} B(t)}{t^{1-\beta}} dt. \tag{7}$$

Equations (5) and (7) give that:

$$u(x, y) = e^{x^\alpha/\alpha - y^\beta/\beta} \int\limits_0^y \frac{e^{y^\beta/\beta} B(y)}{t^{1-\beta}} dt. \tag{8}$$

This is the atomic solution for case (i).

Case (ii) : $Q^{(\beta)}(y) = Q(y) = B(y)$.
Consider

$$Q^{(\beta)}(y) = Q(y).$$

Using conformable derivative properties, we get

$$Q(y) = a e^{y^\beta/\beta}$$

Apply $Q(0) = 1$ to get

$$Q(y) = e^{y^\beta/\beta}. \tag{9}$$

Consequently, if we want to get an atomic solution, $B(y)$ must equal to $e^{y^\beta/\beta}$.
Substitute in (3) to get

$$e^{y^\beta/\beta} P^{(2\alpha)}(x) + e^{y^\beta/\beta} P^{(\alpha)}(x) = e^{y^\beta/\beta} A(x).$$

Hence

$$P^{(2\alpha)}(x) + P^{(\alpha)}(x) = A(x). \tag{10}$$

The homogeneous equation $P^{(2\alpha)}(x) + P^{(\alpha)}(x) = 0$ is solved to give

$$P_h(x) = b_1 e^{-x^\alpha/\alpha} + b_2$$

According to [2], the particular solution of (10) is:

$$P_p(x) = -P_1(x)\int \frac{A(x)P_2(x)}{W[P_1,P_2](x)x^{2-2\alpha}}dx + P_2(x)\int \frac{A(x)P_1(x)}{W[P_1,P_2](x)x^{2-2\alpha}}dx,$$

where $P_1(x) = e^{-x^\alpha/\alpha}, P_2(x) = 1, W[P_1,P_2] = -\frac{x^{\alpha-1}}{\alpha}e^{-x^\alpha/\alpha}$.
Simplifying to get

$$P_p(x) = e^{-x^\alpha/\alpha}\int \frac{A(x)}{(\frac{x^{\alpha-1}}{\alpha}e^{-x^\alpha/\alpha})x^{2-2\alpha}}dx - \int \frac{A(x)e^{-x^\alpha/\alpha}}{(\frac{x^{\alpha-1}}{\alpha}e^{-x^\alpha/\alpha})x^{2-2\alpha}}dx,$$

$$P_p(x) = \alpha e^{-x^\alpha/\alpha}\int \frac{A(x)}{x^{1-\alpha}e^{-x^\alpha/\alpha}}dx - \alpha\int \frac{A(x)}{x^{1-\alpha}}dx.$$

So

$$P(x) = \alpha e^{-x^\alpha/\alpha}\int \frac{A(x)}{x^{1-\alpha}e^{-x^\alpha/\alpha}}dx - \alpha\int \frac{A(x)}{x^{1-\alpha}}dx + b_1 e^{-x^\alpha/\alpha} + b_2. \tag{11}$$

Hence, by (9) and (11)

$$u(x,y) = \alpha e^{y^\beta/\beta - x^\alpha/\alpha}\int \frac{A(x)}{x^{1-\alpha}e^{-x^\alpha/\alpha}}dx - \alpha e^{y^\beta/\beta}\int \frac{A(x)}{x^{1-\alpha}}dx + b_1 e^{y^\beta/\beta - x^\alpha/\alpha} + e^{y^\beta/\beta}b_2. \tag{12}$$

This is the atomic solution for case (ii).

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