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Closed Ideals and Annihilators of Distributive Dual Weakly Complemented Lattice

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Abstract. The goal of this paper is to study closed ideals and annihilators over the class of distributive dual weakly complemented lattices (DDWCLs). The algebraic structure of ideals, closed ideals, and dense ideals are shown. The connection between closed ideals and annihilators in this class is obtained.

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1. Introduction

A dual weakly complemented lattice was introduced by Wille and Kwuida in[9] and [18]. It is a bounded lattice equipped with a unary operation called a dual weak complementation. M.Mandelker introduced the concept of annihilator in lattices in[11]. Cornish defined an annihilator on a distributive lattice in [3] and [4] and discussed its properties. Later many authors discussed the concept of annihilator in different algebraic structures and classes e.g.[1],[2],[5],[7], [8], [10], [12],[13],[14],[15], [16] and [17].

This contribution connected the notion of annihilators of DDWCL with a certain type of ideals is called closed ideals. The given definition of closed ideal depends on the dual weak complementation operation on the lattice of all ideals $I(L)$ of L. Some important properties of closed and dense ideals are proved. A new entry to the concept of annihilator in the class of DDWCLs is displayed.

After preliminaries in section 2, the definition of closed and dense ideals are given and some properties are discussed in section 3. In addition, algebraic structures of them are obtained. In section 4, the definition of a closed annihilator is given, and we prove that

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the set of all closed annihilators forms a maximal Boolean algebra which is contained in the ortho lattice $S(I(L))$ of closed ideals of a DDWCL L. In the special case, if the unary operation is a pseudocomplementation, there is a one-to-one corresponding between the annihilators and the closed ideals of L.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed in the remaining parts.

Definition 1. *[9] A dual weakly complemented lattice is a bounded lattice L equipped with one unary operation $^{\blacktriangledown}$ called dual weak complementation, and satisfied the following conditions for all $a, b \in L$*
(1) $a \le a^{\blacktriangledown\blacktriangledown}$,
(2) If $a \le b$ *implies* $a^{\blacktriangledown} \ge b^{\blacktriangledown}$,
(3) $(a \vee b) \wedge (a \vee b^{\blacktriangledown}) = a$.

In a distributive lattice, the condition (3) becomes $a \wedge a^{\blacktriangledown} = 0$. The trivial dual weakly complemented lattice is the lattice in which every dual weak complementation of any non-zero element is zero.

A distributive pseudocomplemented lattice (p-algebra) is an algebra $< L; \wedge, \vee, ^*, 0, 1 >$, where $< L; \wedge, \vee, 0, 1 >$ is a bounded distributive lattice and the unary operation * is defined by:

$$x \le a^* \quad \text{iff} \quad x \wedge a = 0, \quad a \in L.$$

The operation * is called pseudocomplementation on L. The distributive pseudocomplemented lattice is a perfect example of the distributive dual weakly complemented lattice.

Some properties of the operation $^{\blacktriangledown}$ are listed in the following theorem.

Theorem 1. *[9] For any a, b of dual weakly complemented lattices L, we have the following:*
(1) $0^{\blacktriangledown} = 1$, $1^{\blacktriangledown} = 0$,
(2) $a^{\blacktriangledown\blacktriangledown\blacktriangledown} = a^{\blacktriangledown}$,
(3) $(a \vee a^{\blacktriangledown})^{\blacktriangledown} = 0$,
(4) $a^{\blacktriangledown} \ge b$ *iff* $b^{\blacktriangledown} \ge a$,
(5) $(a \vee b)^{\blacktriangledown} = a^{\blacktriangledown} \wedge b^{\blacktriangledown}$,
(6) $(a \vee b)^{\blacktriangledown\blacktriangledown} \ge a^{\blacktriangledown\blacktriangledown} \vee b^{\blacktriangledown\blacktriangledown}$,
(7) $a \vee b^{\blacktriangledown} \ge b$ *iff* $a \ge b$,
(8) If $a^{\blacktriangledown} \ge b$ *then* $a \wedge b = 0$,
(9) Tf $a \vee b = 1$ *then* $b^{\blacktriangledown} \le a$,
(10) If $a \vee a^{\blacktriangledown} = 1$ *then* $a = a^{\blacktriangledown\blacktriangledown}$,
(11) $a \wedge (a \wedge b)^{\blacktriangledown} \ge a \wedge b^{\blacktriangledown}$.

Definition 2. *[6] An orthocomplemented lattice (ortho lattice) is a bounded lattice* $< L; \wedge, \vee >$ *equipped with one unary operation* $^{\perp}$ *called orthocomplementation on* L *and satisfied the following conditions, for all* $a, b \in L$
(i) $a \leq b$ *implies* $a^{\perp} \geq b^{\perp}$,
(ii) $a^{\perp\perp} = a$,
(iii) $a \wedge a^{\perp} = 0$ *and* $a \vee a^{\perp} = 1$.

A distributive ortho lattice is a Boolean algebra. The skeleton of dual weakly complemented lattice L is defined by $S(L) = \{a \in L : a = a^{\blacktriangledown\blacktriangledown}\}$. It forms an ortho lattice with the same meet operation of L and join "$\veebar$ "operation defined as: $a \veebar b = (a^{\blacktriangledown} \wedge b^{\blacktriangledown})^{\blacktriangledown}$, for any $a, b \in S(L)$. The dual weak complemented of an element $a \in S(L)$ is its orthocomplemented in $S(L)$. The set $D(L)$ of dense element of L defined as $D(L) = \{x \in L : x^{\blacktriangledown} = 0\}$. refer to [9].

Definition 3. *[6] A non-empty subset* I *of a lattice* L *is called an ideal of* L *if*
(i) $a, b \in I$ *implies* $a \vee b \in I$,
(ii) $a \in L, b \in I$ *and* $a \leq b$ *implies* $a \in I$.

The ideal $(a] = \{x \in L : x \leq a\}$ is called the principal ideal generated by $a \in L$. Let $I(L)$ be the set of all ideals of L under set inclusion forms a complete lattice with the smallest element $(0] = \{0\}$ and the largest element $(1] = L$. For any $I, K \in I(L)$, the infimum $I \wedge K = I \cap K$, and the supremum $I \vee K = \{x \in L : x \leq i \vee k$ for some $i \in I$ and some $k \in K\}$. The lattice of all ideals $I(L)$ of a distributive lattice L is distributive. For $a, b \in L$, meet and join operations of two principal ideals are given as: $(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$, see[6].

Definition 4. *[4] For a non-empty subset* A *of a lattice* L*. Define the set* $A^* = \{x \in L : x \wedge a = 0,$ *for all* $a \in A\}$ *is called the annihilator ideal of* A *in* L.

The set of annihilator ideals $A(L)$ of a distributive lattice L is a complete Boolean algebra with the smallest element $(0]$, the largest element L, set-theoretic intersection as the infimum, and the map $I \to I^*$ as complementation. The supremum of I and J in $A(L)$ is given by $I \veebar J = (I^* \cap J^*)^*$. If $I^* = (0]$ then I is called dense ideal. The set of all dense ideals of a distributive lattice L is denoted by $D^*(L)$ and it forms a distributive lattice too. An ideal of form $(a]^*$ is called an annulet of $a \in L$. Each annulet is an annihilator ideal, where $(a]^* = (a^*]$, a^* is the pseudocomplemented of a in L, see[4].

From now on, L stands for non-trivial distributive dual weakly complemented lattice.

3. Closed and dense Ideals of DDWCLs

In the present section, the dual weak complementation operation is defined on the lattice of all ideals $I(L)$ of L. Concepts of closed and dense ideals are introduced and it's basic properties are proved.

Definition 5. *Let I be an ideal. Define the set*

$$I^{\triangledown} = \{x \in L : x \leq a^{\blacktriangledown},\ \textit{for all}\ a \in I\}.$$

Note that, If I is an ideal and $x \in I^{\triangledown}$ then $x \wedge a = 0$, for all $a \in I$. Consequently $x \in I^*$ and so $I^{\triangledown} \subseteq I^*$.

Proposition 1. *A set $I^{\triangledown}$ is an ideal, moreover is the dual weak complemented of the ideal I in $I(L)$.*

Proof. Assume that $x, y \in I^{\triangledown}$, then $x, y \leq a^{\blacktriangledown}$ for all $a \in I$. Thus $x \vee y \leq a^{\blacktriangledown}$. Consequently, $x \vee y \in I^{\triangledown}$. Let $z \in L$ and $z \leq x$ for some $x \in I^{\triangledown}$. Then $z \leq a^{\blacktriangledown}$ for all $a \in I$. Therefore $I^{\triangledown}$ is an ideal of L.

Now we show that $I^{\triangledown}$ satisfies the conditions of dual weak complementation:
(1) If $a \in I$ and $b \in I^{\triangledown}$ then $b \leq a^{\blacktriangledown}$. From (4) in Theorem 1, this is equivalent that $a \leq b^{\blacktriangledown}$. Thus $a \in I^{\triangledown\triangledown}$.
(2) Let $I \leq K$ and $x \in K^{\triangledown}$. Then $x \leq b^{\blacktriangledown}$ for all $b \in K$. So $x \leq b^{\blacktriangledown}$ for all $b \in I$. It implies $x \in I^{\triangledown}$. So $K^{\triangledown} \leq I^{\triangledown}$,
(3) Clearly $I \wedge I^* = (0]$, then $I \wedge I^{\triangledown} = (0]$.

If L is a distributive pseuodocomplemented lattice, then $I^{\triangledown} = I^*$, and the lattice $I(L)$ of its ideals become a distributive pseuodocomplemented lattice.

Lemma 1. *Let I, K be two ideals and $a \in L$. Then,:*
(1) $(a]^{\triangledown} = (a^{\blacktriangledown}]$,
(2) $I^{\triangledown} = \cap_{a \in I}(a]^{\triangledown}$,
(3) $(0]^{\triangledown} = (1]$ *and* $(1]^{\triangledown} = (0]$,
(4) $I^{\triangledown} = I^{\triangledown\triangledown\triangledown}$,
(5) $I^{\triangledown} \geq K$ *iff* $K^{\triangledown} \geq I$,
(6) $I^{\triangledown} \subseteq K^{\triangledown}$ *iff* $K^{\triangledown\triangledown} \subseteq I^{\triangledown\triangledown}$,
(7) $I \subseteq K^{\triangledown}$ *implies* $I \cap K = (0]$.

Proof. (1) Suppose $x \in (a^{\blacktriangledown}]$, then $x \leq y^{\blacktriangledown}$ for all $y \in (a]$.So $x \in (a]^{\triangledown}$. Conversely, let $y \in (a]^{\triangledown}$ and $a^{\blacktriangledown} \leq y$. Thus $y^{\blacktriangledown\blacktriangledown} \geq a^{\blacktriangledown}$ and by using (7) in Theorem 1 we get $y^{\blacktriangledown\blacktriangledown} \vee a^{\blacktriangledown\blacktriangledown} \geq a^{\blacktriangledown}$. Hence, $a^{\blacktriangledown} \wedge (y^{\blacktriangledown\blacktriangledown} \vee a^{\blacktriangledown\blacktriangledown}) = (a^{\blacktriangledown} \wedge y^{\blacktriangledown\blacktriangledown}) \vee (a^{\blacktriangledown} \wedge a^{\blacktriangledown\blacktriangledown}) \geq a^{\blacktriangledown}$, which is a contradiction. So, $y \leq a^{\blacktriangledown}$.Therefore, $(a]^{\triangledown} = (a^{\blacktriangledown}]$.

(2) Let $x \in \cap_{a\in I}(a]^{\triangledown}$ i.e., $x \in (a]^{\triangledown}$, for all $a \in I$. Then $x \leq a^{\blacktriangledown}$ and $x \in I^{\triangledown}$. Conversely, if $x \in I^{\triangledown}$ then $x \leq a^{\blacktriangledown}$ for all $a \in I$. It means $x \in \cap_{a\in I}(a]^{\triangledown}$. Therefore $I^{\triangledown} = \cap_{a\in I}(a]^{\triangledown}$.
(3) We get

$$(0]^{\triangledown} = \{x \in L : x \leq 0^{\blacktriangledown} = 1\} = (1],$$
$$(1]^{\triangledown} = \{x \in L : x \leq 1^{\blacktriangledown} = 0\} = (0].$$

(4) By using Proposition1, we get $I^{\triangledown\triangledown\triangledown} \leq I^{\triangledown}$. Conversely, if $b \in I^{\triangledown\triangledown\triangledown}$, then $b \leq a^{\blacktriangledown}$ for all $a \in I^{\triangledown\triangledown}$. But $I \subseteq I^{\triangledown\triangledown}$ so $x \leq c^{\blacktriangledown}$ for all $c \in I$.Thus $x \in I^{\triangledown}$.
(5) Suppose $I^{\triangledown} \geq K$ then $I \leq I^{\triangledown\triangledown} \leq K^{\triangledown}$ and vice versa.
(6) Let $I^{\triangledown} \leq K^{\triangledown}$. Then, from Proposition1, $I^{\triangledown\triangledown} \geq K^{\triangledown\triangledown}$. The opposite direction can be get by using (4).
(7) Assume $I \subseteq K^{\triangledown}$ then we get $I \wedge K^{\triangledown} = (0]$, since $K \wedge K^{\triangledown} = (0]$.

Proposition 2. *Let $I, K \in I(L)$. Then:*
(1) $I \vee K^{\triangledown} \geq K$ *iff* $I \geq K$,
(2) $I \vee K = (1]$ *implies* $K^{\triangledown} \leq I$,
(3) $(I \vee K)^{\triangledown} = I^{\triangledown} \cap K^{\triangledown}$,
(4) $(I \cap K)^{\triangledown\triangledown} \leq I^{\triangledown\triangledown} \cap K^{\triangledown\triangledown}$,
(5) $(I \vee I^{\triangledown})^{\triangledown} = (0]$,
(6) $(I \vee K)^{\triangledown\triangledown} \geq I^{\triangledown\triangledown} \vee K^{\triangledown\triangledown}$,
(7) $I \cap (I \cap K)^{\triangledown} \geq I \cap K^{\triangledown}$.

Proof. (1) Suppose that $I \vee K^{\triangledown} \geq K$, we get $K = K \wedge (I \vee K^{\triangledown}) = (K \wedge I) \vee (K \wedge K^{\triangledown}) = K \wedge I$, which means $I \geq K$. Conversely, if $I \geq K$ then $I^{\triangledown} \leq K^{\triangledown}$. So $I \vee K^{\triangledown} \geq I \vee I^{\triangledown} \geq I \geq K$.
(2) Assume $I \vee K = (1]$. Meet each side by $K^{\triangledown}$ to get $K^{\triangledown} \wedge I = K^{\triangledown}$. Thus $K^{\triangledown} \leq I$.
(3) Let $x \in I^{\triangledown} \cap K^{\triangledown}$ and $z = a \vee b \in I \vee K$, where $a \in I$ and $b \in K$. Then $x \leq a^{\blacktriangledown}$ and $x \leq b^{\blacktriangledown}$. It implies $x \leq a^{\blacktriangledown} \wedge b^{\blacktriangledown} = (a \vee b)^{\blacktriangledown} = z^{\blacktriangledown}$. Hence, $x \in (I \vee K)^{\triangledown}$ and so $I^{\triangledown} \cap K^{\triangledown} \subseteq (I \vee K)^{\triangledown}$. The converse is trivial.
(4) From (3) we get $(I \cap K)^{\triangledown\triangledown} \leq I^{\triangledown\triangledown} \cap K^{\triangledown\triangledown}$.
(5) Using (3) we get $(I \vee I^{\triangledown})^{\triangledown} = I^{\triangledown} \cap I^{\triangledown\triangledown} = (0]$.
(6) Since $K, I \leq I \vee K$, then $I^{\triangledown\triangledown}, K^{\triangledown\triangledown} \leq (I \vee K)^{\triangledown\triangledown}$. Hence $I^{\triangledown\triangledown} \vee K^{\triangledown\triangledown} \leq (I \vee K)^{\triangledown\triangledown}$.
(7) Since $(I \cap K)^{\triangledown} \geq K^{\triangledown}$. Meeting both sides by I we get $I \cap (I \cap K)^{\triangledown} \geq I \cap K^{\triangledown}$.

The skeleton $S(I(L))$ and the set of dense elements of $I(L)$ are defined as:

$$S(I(L)) = \{I \in I(L) : I = I^{\triangledown\triangledown}\}, \text{ and}$$
$$D(I(L)) = \{I \in I(L) : I^{\triangledown} = (0]\}.$$

The elements of $S(I(L))$ and $D(I(L))$ are called closed and dense ideals, respectively.

Lemma 2. *Let I and K be two ideals. Then:*
(1) $(a]^{\triangledown\triangledown} = (a]$ *iff* $a \in S(L)$,
(2) $I \cap K = (I^{\triangledown} \vee K^{\triangledown})^{\triangledown}$, *for all* $I, K \in S(I(L))$,
(3) If $I \vee I^{\triangledown} = (1]$ *then* $I \in S(I(L))$,
(4) If I is an annihilator then $I \in S(I(L))$.

Proof.
(1) Immediately from (1) in Lemma 1.
(2) By using (3) in Proposition 2, $I \cap K = I^{\triangledown\triangledown} \cap K^{\triangledown\triangledown} = (I^{\triangledown} \vee K^{\triangledown})^{\triangledown}$.
(3) Assume $I \vee I^{\triangledown} = (1]$, meet both sides by $I^{\triangledown\triangledown}$ to get $I^{\triangledown\triangledown} = I^{\triangledown\triangledown} \wedge (I \vee I^{\triangledown}) = (I^{\triangledown\triangledown} \wedge I) \vee (I^{\triangledown\triangledown} \wedge I^{\triangledown}) = I^{\triangledown\triangledown} \wedge I = I$. Therefore $I = I^{\triangledown\triangledown}$.
(4) Let I be an annihilator and $I^{*\triangledown}$ be a proper subset of $I = I^{**}$. So there exists $x \in I^{**}$ such that $x \nleq a^{\blacktriangledown}$ for all $a \in I^{*}$. Thus $x \leq a^{*}$ and $x \nleq a^{*}$, at the same time, which is a contradiction. Therefore $I = I^{*\triangledown} \in S(I(L))$.

Theorem 2. *The skeleton $S(I(L))$ of the lattice $I(L)$ forms an ortho lattice.*

Proof. Two binary operations $\cap$ and $\sqcup$ will be defined for $I, K \in S(I(L))$ by:

$$I \cap K = (I^{\triangledown} \vee K^{\triangledown})^{\triangledown} \quad \text{and} \quad I \sqcup K = (I^{\triangledown} \cap K^{\triangledown})^{\triangledown}.$$

It is clear that the meet operation $\cap$ is the usual intersection of $I(L)$. Now we prove that the supremum $I \sqcup K$ of any two ideals I and K of $S(I(L))$ equals $(I^{\triangledown} \cap K^{\triangledown})^{\triangledown}$:
Since $I^{\triangledown} \cap K^{\triangledown} \leq I^{\triangledown}, K^{\triangledown}$, we get $(I^{\triangledown} \cap K^{\triangledown})^{\triangledown} \geq I^{\triangledown\triangledown} = I, K^{\triangledown\triangledown} = K$. If $J \in S(I(L))$ and $J \geq I, K$, then $J^{\triangledown} \leq I^{\triangledown}, K^{\triangledown}$. It implies $J^{\triangledown} \leq I^{\triangledown} \cap K^{\triangledown}$ i.e., $J^{\triangledown\triangledown} = J \geq (I^{\triangledown} \cap K^{\triangledown})^{\triangledown}$.
If $I \in S(I(L))$,then $I^{\triangledown}$ is the orthcomplemented of I. especially, $(0]^{\triangledown} = (1]$ and $(1]^{\triangledown} = (0]$.
Therefore $< S(I(L)); \cap, \sqcup, ^{\triangledown}, (0], (1] >$ forms an ortho lattice.

Theorem 3. *The skeleton $S(L)$ of the lattice L is embedded in the skeleton $S(I(L))$ of its lattice of ideals $I(L)$.*

Proof. Consider the map ψ from $S(L)$ into $S(I(L))$ which mapping the element $a \in S(L)$ to $(a]^{\triangledown\triangledown} \in S(I(L))$ is a well defined and satisfies the following:
$\psi(a \wedge b) = (a \wedge b]^{\triangledown\triangledown} = (a \wedge b] = (a] \cap (b] = (a]^{\triangledown\triangledown} \wedge (b]^{\triangledown\triangledown} = \psi(a) \cap \psi(b)$,
$\psi(a \vee b) = \psi((a^{\blacktriangledown} \wedge b^{\blacktriangledown})^{\blacktriangledown}) = ((a^{\blacktriangledown} \wedge b^{\blacktriangledown})^{\blacktriangledown}]^{\triangledown\triangledown} = ((a^{\blacktriangledown} \wedge b^{\blacktriangledown})^{\blacktriangledown}] = (a^{\blacktriangledown} \wedge b^{\blacktriangledown}]^{\triangledown} = ((a^{\blacktriangledown}] \wedge (b^{\blacktriangledown}])^{\triangledown}$
$= ((a]^{\triangledown} \wedge (b]^{\triangledown})^{\triangledown} = (a] \sqcup (b] = (a]^{\triangledown\triangledown} \sqcup (b]^{\triangledown\triangledown} = \psi(a) \sqcup \psi(b)$,
$\psi(a^{\blacktriangledown}) = (a^{\blacktriangledown}]^{\triangledown\triangledown} = (a]^{\triangledown\triangledown\triangledown} = (\psi(a))^{\triangledown}$,
$\psi(0) = (0]^{\triangledown\triangledown} = (0]$, and $\psi(1) = (1]^{\triangledown\triangledown} = L$.
It is easy to show that ψ is an injective map and this completed the proof.

Theorem 4. *The following statements are equivalent:*
(1) I is closed ideal,
(2) If $a \in L$ *and* $a \leq b^{\blacktriangledown}$, *for all* $b \in I^{\triangledown}$ *then* $a \in I$,
(3) $I = K^{\triangledown}$ *for some ideal* $K \in I(L)$.

Proof. Let I be a closed ideal and $a \in L$ such that $a \leq b^{\blacktriangledown}$ for all $b \in I^{\triangledown}$, then $a \in I^{\triangledown\triangledown} = I$.
Let (2) be satisfied and let $x \in I^{\triangledown\triangledown}$. Then $x \leq b^{\blacktriangledown}$ for all $b \in I^{\triangledown}$. Therefore $x \in I$.
Assume $I = I^{\triangledown\triangledown}$, set $K = I^{\triangledown}$ then $K^{\triangledown} = I \in S(I(L))$.
Let $I = K^{\triangledown}$ for some $K \in I(L)$. Thus $I^{\triangledown\triangledown} = K^{\triangledown\triangledown\triangledown} = K^{\triangledown} = I$.

Lemma 3. *Let $D(I(L))$ be the set of all dense ideals of L. Then:*
(1) $(1] \in D(I(L))$,
(2) If $I, K \in I(L)$ such that $I \subseteq K$ and $I \in D(I(L))$ then $K \in D(I(L))$,
(3)If $I, K \in D(I(L))$, then $I \vee K \in D(I(L))$,
(4)If $I \in D(I(L))$, then $I \vee I^{\triangledown} \in D(I(L))$,
(5)If $I \cap D(L) \neq \phi$ then $I \in D(I(L))$.
(6) $D^{*}(L) \subseteq D(I(L))$.

Proof. (1) Since $(1]^{\triangledown} = (0]$, then $(1] \in D(I(L))$.
(2) Let $I, K \in I(L)$, $I \subseteq K$ and $I^{\triangledown} = (0]$. This implies $(0] = I^{\triangledown} \supseteq K^{\triangledown}$ i.e., $K^{\triangledown} = (0]$ and $K \in D(I(L))$.
(3) Assume $I, K \in D(I(L))$, from (3) in Proposition 2, we get $(I \vee K)^{\triangledown} = I^{\triangledown} \cap K^{\triangledown} = (0]$. Thus $I \vee K \in D(I(L))$.
(4) If $I \in D(I(L))$ then $I \subseteq I \vee I^{\triangledown}$. It implies $(0] = I^{\triangledown} \supseteq (I \vee I^{\triangledown})^{\triangledown}$. Therefore $I \vee I^{\triangledown} \in D(I(L))$.
(5) Assume $I \cap D(L) \neq \phi$, then there exists a non-zero element $a \in I$ such that $a^{\blacktriangledown} = 0$. Hence $I^{\triangledown} = (0]$ and $I \in D(I(L))$.
(6) Assume $I \in D^{*}(L)$. Since $I^{\triangledown} \subseteq I^{*} = (0]$, then $I \in D(I(L))$.

Theorem 5. *$D(I(L))$ forms a join semilattice with one.*

4. Closed Annihilators of DDWCLs

This section introduces the concept of a closed annihilator. The structure of the set of all closed annihilators of DDWCL L and the connection with the set of all closed ideals $S(I(L))$are investegated.

Definition 6. *An ideal I is called a closed annihilator iff $I = I^{\triangledown *}$.*

The set of all annihilators is denoted by $I^{\triangledown *}(L)$. The only dense ideal belongs to $I^{\triangledown *}(L)$ is $(1]$.

Lemma 4.
(1) If I is an ideal then $I^{*\triangledown} \subseteq I^{\triangledown\triangledown} \subseteq I^{\triangledown *}$,

(2) If I is a closed annihilator then $I^* = I^\triangledown$,
(3) If I and K are closed annihilators then $I \cap K$ and $I \sqcup K$ are too,
(4) $I \in A(L)$ iff $I \in I^{\triangledown *}(L)$.

Proof.
(1) We have that $I^\triangledown \subseteq I^*$. then $I^{\triangledown\triangledown} \supseteq I^{*\triangledown}$. Suppose $x \in I^{\triangledown\triangledown}$ i.e., $x \leq a^\blacktriangledown$ for all $a \in I^\triangledown$.It implies $x \wedge a = 0$. Therefore $I^{*\triangledown} \subseteq I^{\triangledown\triangledown} \subseteq I^{\triangledown *}$.
(2) If I is a closed annihilator then $I = I^{\triangledown *} = I^{*\triangledown}$. The dual weak complementation of I is given by $I^\triangledown = I^{\triangledown * \triangledown} = I^{*\triangledown\triangledown} \supseteq I^*$. Therefore $I^* = I^\triangledown$.
(3) For meet operation we have $I \cap K \leq I, K$. Thus $(I \cap K)^{\triangledown *} \leq I^{\triangledown *}, K^{\triangledown *}$. Accordingly, $(I \cap K)^{\triangledown *} \leq I^{\triangledown *} \cap K^{\triangledown *} = I \cap K$, and $I \cap K \subseteq (I \cap K)^{\triangledown\triangledown} \subseteq (I \cap K)^{\triangledown *}$. Then $I \cap K \in I^{\triangledown *}(L)$.For join operation we have $[I \sqcup K]^{\triangledown *} = (I^\triangledown \cap K^\triangledown)^{\triangledown\triangledown *} = (I^\triangledown \cap K^\triangledown)^\triangledown = I \sqcup K$.
(4) If $I \in A(L)$ then $I = I^{*\triangledown} = I^{**}$ and $I^* \in A(L)$. Hence $I^{\triangledown *} = I^{*\triangledown\triangledown *} = I^{**} = I$. Conversely, if $I \in I^{\triangledown *}(L)$ then $I = I^{\triangledown *} = I^{**}$ and $I^* = I^\triangledown \in I^{\triangledown *}$. So $I^{*\triangledown} = I^{\triangledown ** \triangledown} = I^{\triangledown\triangledown} = I$.

The subset of ortho lattice L which forms a Boolean algebra under the same operations of L is called a Boolean algebra induced from L.

Theorem 6. *The set $I^{\triangledown *}(L)$, of all closed annihilators forms a maximal Boolean algebra indued from $S(I(L))$.*

Proof. To prove $I^{\triangledown *}(L)$ forms a Boolean algebra it is enough to prove the distributivity of it. Initially, we prove the inequality (1): for $I, J, K, H \in I^{\triangledown *}(L)$

$$(I \sqcup J) \cap H \leq I \sqcup (J \cap H) \;...(1)$$

Since,

$$H \cap I^\triangledown \cap (J \cap H)^\triangledown \leq I^\triangledown ...(2)$$

and, $J \cap H \cap (I^\triangledown \cap (J \cap H)^\triangledown) = I^\triangledown \cap (J \cap H) \cap (J \cap H)^\triangledown = (0]$. It implies that, $H \cap I^\triangledown \cap (J \cap H)^\triangledown \leq J^\triangledown$ Thus,

$$H \cap I^\triangledown \cap (J \cap H)^\triangledown \leq J^\triangledown ...(3)$$

From (2) and (3), $H \cap I^\triangledown \cap (J \cap H)^\triangledown \leq I^\triangledown \cap J^\triangledown$ and from the properties of "$^\triangledown$", we get $(H \cap I^\triangledown \cap (J \cap H)^\triangledown) \cap (I^\triangledown \cap J^\triangledown)^\triangledown = (0]$, implies that, $((I^\triangledown \cap J^\triangledown)^\triangledown \cap H) \cap (I^\triangledown \cap (J \cap H)^\triangledown) = (0]$. So, $H \cap (I^\triangledown \cap J^\triangledown)^\triangledown \leq (I^\triangledown \cap (J \cap H)^\triangledown)^\triangledown$. Therefore, $(I^\triangledown \cap J^\triangledown)^\triangledown \cap H \leq (I^\triangledown \cap (J \cap H)^\triangledown)^\triangledown$. By putting $H = I \sqcup K$ in inq.(1), we get $(I \sqcup J) \cap (I \sqcup K) \leq I \sqcup [(J \cap I \sqcup K)] \leq I \sqcup [I \sqcup (J \cap K)] = I \sqcup (J \cap K)$. Also, $J \cap K \leq (I \sqcup J) \cap (I \sqcup K)$ and $I \leq (I \sqcup J) \cap (I \sqcup K)$. Then, $I \sqcup (J \cap K) \leq (I \sqcup J) \cap (I \sqcup K)$. Consequently, $I \sqcup (J \cap K) = (I \sqcup J) \cap (I \sqcup K)$. Therefore, $< I^{\triangledown *}(L); \sqcup, \cap, ^\triangledown, (0], (1] >$ forms a Boolean algebra.

Next, we prove the maximality of $I^{\triangledown *}(L)$. Let B be a Boolean algebra induced from $S(I(L))$ such that $I^{\triangledown *}(L) \subset B$. Then there exist ideal $H \in B$ and $H \notin I^{\triangledown *}(L)$, then $H \subset H^{\triangledown *}$. Since $H^\triangledown \cap H^{\triangledown *} = (0]$ then. $H^{\triangledown *} \subseteq H^{\triangledown\triangledown} = H$. So that $H = H^{\triangledown *}$, which is a contradiction.

Theorem 7. *(1) $A(L)$ isomorphic to $I^{\triangledown *}(L)$,*
(2) If $I^{\triangledown} = K^{\triangledown}$, for any closed annihilator I and any ideal K such that $I \subset K$, then $S(I(L))$ is a Boolean algebra,
(3) If there exists closed annihilator I and ideal K such that $I \subset K$ and $K^{\triangledown} \subset I^{\triangledown}$ then $S(I(L))$ is not Boolean algebra.

Proof. (1) We defined a map $\alpha : A(L) \to I^{\triangledown *}(L)$ as: $\alpha(I) = I^{\triangledown *}$.
Assume $I, J \in A(L)$ then:
$\alpha(I \cap J) = (I \cap J)^{\triangledown *} = I \cap J = I^{\triangledown *} \cap J^{\triangledown *}$,
$\alpha(I \veebar J) = (I \veebar J)^{\triangledown *} = (I^* \cap J^*)^{*\triangledown *} = (I^{\triangledown} \cap J^{\triangledown})^{\triangledown} = I \sqcup J$,
$\alpha(I^*) = I^{*\triangledown *} = I^* = I^{\triangledown}$,
Clearly that α is bijective map.

When for any closed annihilator I and any ideal K such that $I \subset K$ we get $K^{\triangledown} = I^{\triangledown} = I^*$. Then the set of all closed ideals $S(I(L))$ coincides with the set $I^{\triangledown *}(L)$ of all closed annihilators. Consequently, $S(I(L))$ becomes a Boolean algebra. While if there exists closed annihilator I and ideal K such that $I \subset K$ and $K^{\triangledown} \subset I^{\triangledown}$ then $K^{\triangledown}$ and its dual weak complementation are added to $S(I(L))$. So it does not Boolean algebra.

The following corollary is immediately proved from Theorem 7

Corollary 1. *Let L be a distributive pseudocomplemented lattice. Then $I^{\triangledown *}(L)$, $A(L)$ and $S(I(L))$ are isomorphic.*

Conclusion

This work introduces the annihilator concept for the class of DDWCLs, the characterization and important properties of closed and dense ideals are proved. The one-to-one correspondence between closed annihilators and usual annihilators of a distributive lattice is shown. Especially, over the distributive pseudocomplemented lattice there is correspondence between closed annihilators, closed ideals, and usual annihilators. This new generalization of the annihilator concept for the class of DDWCLs can be extended to the classes of distributive weakly complemented and dicomplemented lattices, refer to [9].

References

[1] Gezahagne Mulat Addis. Annihilators in universal algebras: A new approach. *Journal of Mathematics*, 2020, 2020.

[2] I. Chajad and R. Halas. Annihilators in universal algebras. *Contributions to General Algebra*, 14:29–43, 2004.

[3] W.H. Cornish. Normal lattices. *J. Austral. Math. Soc.*, 14:200–215, 1972.

[4] W.H. Cornish. Annulets and α-ideals in distributive lattices. *J. Austral. Math. Soc.*, 15:70–77, 1973.

[5] E.Mehdi-Nezhad and A. M. Rahimi. The annihilator graphs of commutator posets and lattices with respect to an elements. *Journal of Algebra and its Applications*, 16, 2017.

[6] G. Gretzer. *Lattice Theory: Foundation.* Springer Basel, 2011.

[7] G. Muhiuddin H. Bordbar and Abdulaziz M. Alanazi. Primeness of relative annihilators in bck-algebra. *Symmetry*, 12(286), 2020.

[8] M. Jastrzebska. Rings with boolean lattices of one-sided annihilators. *Symmetry*, 13, 2021.

[9] L. Kwuida. *Dicomplemented Lattices. A Contextual Generalization of Boolean Algebras.* PhD thesis, TU Dresden, 2004.

[10] S.S.Khopade M.A.Gandhi and Y.S.Pawar. Epimorphisms and ideals of distributive nearlattices. *Annals of Pure and Applied Mathematics*, 18(2):175–179, 2018.

[11] M. Mandelker. Relative annihilators in lattices. *Duke Math. J.*, 37:377–386, 1970.

[12] G. C. Rao and M. Sambasiva Rao. Annihilator ideals in almost distributive lattices. *International Mathematical Forum*, 15(4):733–746, 2009.

[13] G. Nanaji Rao and T. G. Beyene. Relative annihilators and filters in almost semilattice. *Southeast Asian Bulletin of Mathematics*, 43:553–576, 2019.

[14] G. Nanaji Rao and R. Venkata Aravinda Raju. Annihilator ideals in 0-distributive almost lattices. *Bull. Int. Math. Virtual Inst.*, 11(1):1–13, 2021.

[15] M. Sambasiva Rao. On annihilator ideals of c-algebras. *Asian-Eur. J. Math.*, 6(1), 2013.

[16] A. Taherifar and T. Dube. On the lattice of annihilator ideals and its applications. *Communications in Algebra*, 49, 2021.

[17] Lia Vas. Annihilator ideals of graph algebras. *arXivLabs*, 2021.

[18] R. Wille. Boolean concept logic. In B. Ganter and G. W. Mineau, editors, *ICCS 2000 Conceptual Structures: Logical, Linguistic and Computational Issues.*, pages 14–18, Germany, 2000. Darmstadt.