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Spectral dichotomy methods of a matrix with respect to the general equation of the parabola

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Abstract. This paper presents methods of spectral dichotomy of a matrix which compute spectral projectors on the subspace associated with the eigenvalues external to the parabolas described by a general equation. These methods are modifications of the one proposed in [A. N. Malyshev and M. Sadkane, SIAM J. MATRIX ANAL. APPL. 18 (2), 265-278, 1997] which uses the spectral dichotomy Theoretical and method of a matrix with respect to the imaginary axis. algorithmic aspects of the methods are developed. Numerical results obtained by applying methods presented on matrices are reported.

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Key Words and Phrases: Spectral dichotomy method, spectral projector, eigensubspaces, eigenvalues.

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1. Introduction

Let $A \in \mathbb{R}^{n\times n}$ $(n > 1)$ be a matrix and $\Gamma(a, b, c)$ a parabola with an equation of the type

$$x = ay^2 + by + c \qquad a \neq 0. \tag{1}$$

The aim of this paper is to propose spectral dichotomy methods which partition the spectrum of matrix A into two parts : A first part inside the parabola and a second one outside. This will lead to the calculation of the projectors associated respectively with the eigenvalues inside and outside the parabola.

Equation (1) reduces to the following form

$$x = a\left[\left(y + \frac{b}{2a}\right)^2 - \frac{\mathbf{disc}}{4a^2}\right] \tag{2}$$

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where $\mathbf{disc} = b^2 - 4ac$; or again

$$\frac{1}{a}\left(x + \frac{\mathbf{disc}}{4a}\right) = \left(y + \frac{b}{2a}\right)^2 \tag{3}$$

Throughout this paper we assume that the coefficient a has a negative sign. Thus, the parabola with equation (3) takes the form

$$2p(d - x) = (y - pb)^2 \tag{4}$$

by setting

$$p = -\frac{1}{2a} \qquad \text{and} \qquad d = -\frac{\mathbf{disc}}{4a} = -\frac{b^2 - 4ac}{4a}$$

We assume that the matrix A has no eigenvalues on the parabola $\Gamma(a, b, c)$ for any variation of the parameters a, b and c with $a \neq 0$. From the work done in [13, 15], we propose in this paper spectral dichotomy methods which give the projector $\mathbb{P}$ on the subspace associated with the eigenvalues located outside of $\Gamma(a, b, c)$. The paper is organized as follows. Section 2 gives preliminaries used in the implementation of our proposed methods. It consists of three subsections. The first subsection summarizes the methods of spectral dichotomy of a matrix and a pencil of matrices with respect to a circle developed respectively by M. Dosso and al. in [2, 4] and M. Sadkane and al. in [15]. The second subsection makes a brief presentation of the spectral dichotomy method of a matrix with respect to the imaginary axis (see [15]). The last subsection presents the study made by A.N.Malyshev and M.Sadkane in [13]. Section 3 presents new methods of spectral dichotomy of a matrix with respect to the curve $\Gamma(a, b, c)$ for variations of parameters a, b and c with $a \neq 0$. Finally in section 4, numerical tests are used on various examples to illustrate the effectiveness of the methods presented.

Throughout this paper, the identity and zero matrices of order k are denoted by I_k and 0_k or just I and 0 whenever the order is clear from the context. The 2-norm of a matrix A is denoted by $\|A\|$.

2. Preliminaries on spectral dichotomy methods

2.1. Spectral dichotomy with respect to a circle

Let A be a matrix having no eigenvalues on the circle $\mathcal{C}(0, r)$ (where $r > 0$). The spectral projector on the subspace corresponding to the eigenvalues inside the unit circle is defined by

$$\mathbb{P} = \frac{1}{2i\pi}\int_{\mathcal{C}} (zI_n - A)^{-1} dz = \frac{1}{2\pi}\int_0^{2\pi} \left(I_n - \frac{e^{-i\theta}}{r}A\right)^{-1} d\theta \tag{5}$$

The computation of the spectral projector is accompanied by that of the Hermitian matrix defined by:

$$\mathbb{H} = H(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(I - \frac{e^{-i\theta}A}{r}\right)^{-*} H^{(0)}\left(I - \frac{e^{-i\theta}A}{r}\right)^{-1} \quad d\theta, \tag{6}$$

with $\mathbb{H}^{(0)} = (\mathbb{H}^{(0)})^* > 0$, an arbitrary Hermitian positive definite matrix used for scaling purpose.

Remark 1. *The spectral norm of* $\mathbb{H}$ *indicates the behavior of the spectral projector* $\mathbb{P}$*. The smaller* $\|\mathbb{H}\|$ *is, better is the quality of the dichotomy.*

The couple of matrices $(\mathbb{P}, \mathbb{H})$ is the only solution of the generalized Lyapunov's equation [8]

$$\begin{cases} r^2\mathbb{H} - A^*\mathbb{H}A = \mathbb{P}^*\mathbb{H}^{(0)}\mathbb{P} - (I-\mathbb{P})^*\mathbb{H}^{(0)}(I-\mathbb{P}) \\ \mathbb{P}A = A\mathbb{P} \\ \mathbb{P}^2 = \mathbb{P} \\ \mathbb{P}\mathbb{H} = (\mathbb{P}\mathbb{H})^* \end{cases} \tag{7}$$

That generalized equation was first proposed by Godunov in partial form in [7] and later by Bulgakov in complete form (7) in [1]. The most efficient numerical method for the circular dichotomy was first proposed in [10] and [12].
Moreover, for any vector x and for any integer k, we have the estimates [4, 6, 14]

$$\begin{aligned} \|A^k\mathbb{P}x\| &\leq \sqrt{\|\mathbb{H}\|\|\mathbb{H}^{-1}\|\|}\left(1 - \tfrac{1}{\|\mathbb{H}\|}\right)^{\frac{k}{2}}\|x\| \\ \|A^k\mathbb{P}x\| &\geq \frac{1}{\sqrt{\|\mathbb{H}\|\|\mathbb{H}^{-1}\|\|}}\left(1 + \tfrac{1}{\|\mathbb{H}\|}\right)^{\frac{k}{2}}\|(I-\mathbb{P})x\| \end{aligned} \tag{8}$$

which shows the importance of the quantity $\|\mathbb{H}\|$ on asymptotic decay to 0 (or growth to $+\infty$) of the powers of A.

Different authors have proposed methods for determining the projector $\mathbb{P}$ and the matrix $\mathbb{H}$. We summarize the most important steps of the method proposed in [2, 4]. Note that this method is a variant of an initial method proposed by S.K. Godunov and M. Sadkane in [9]. During their work, these authors have given some important results. The first proposition gives the link in the one hand between the sequences of matrices $Z_k^{(2^{j+1})}$ and $Z_k^{(2^j)}$, and in the other hand between $\mathbb{H}_{j+1}$ and $\mathbb{H}_j$.

Proposition 1. *For* $j = 0, 1, \cdots$ *and* $k = 0, 1, \cdots, 2^j$*, we have*

$$Z_k^{(2^{j+1})} = Z_k^{(2^j)} K_{j+1} \tag{9}$$

$$Z_{2j+k}^{(2^{j+1})} = Z_k^{(2^j)} L_{j+1} \tag{10}$$

$$H_{j+1} = (K_{j+1})^* H_j K_{j+1} + (L_{j+1})^* H_j L_{j+1}. \tag{11}$$

For the details of the proof, see in [2, 4].

In the second proposition, the sequences of matrices $(L_k)_{k\geq 0}$ and $(K_r)_{r\geq 0}$ are computed iteratively

Proposition 2. *For $j = 0, 1, \cdots$, we have*

$$\begin{pmatrix} B_j & A_j \\ A_j & B_j \end{pmatrix} \cdot \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix} \tag{12}$$

with $A_j = -AZ_1^{(2^j)}, \qquad B_j = Z_{2^j}^{(2^j)}$

For the details of the proof, see in [2, 4].

The third important result gives an estimate of the error $Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P}$ when j takes large values

Theorem 1. *There exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, we have*

$$\|Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P}\| \leq \kappa_2(X) \frac{\omega \gamma^{2^{j+1}}}{1 - \omega \gamma^{2^{j+1}}}.$$

with $\omega > 1$ and $0 < \gamma < 1$.

For the details of the proof, see in [2, 4].
This last result shows the fast convergence of $Z_{2^{j+1}}^{(2^{j+1})}$ to the projector $\mathbb{P}$.
These results led to the following algorithm

Algorithm 1 (DichoC1)**.**

- *Input variables: A and I_n such that the matrix pencil $zI_n - A$ has no eigenvalues on the circle $\mathcal{C}(O, r)$ with center O and the radius r.*

- *Output variables: The spectral projector $\mathbb{P}$ and the dichotomy criterion $\mathbb{H}$.*
  $\mathbb{P}$ being the projector on the right invariant space of $zI_n - A$ corresponding to the eigenvalues inside the circle $\mathcal{C}(O, r)$ and $\mathbb{H}$ the dichotomy criterion.

(i) Initialize

  (a) $A_0 = -\dfrac{A}{r}$.

  (b) resolve

$$\begin{pmatrix} A_0 & I_n \\ I_n & A_0 \end{pmatrix} \begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}.$$

  (c) Put $Z_1^{(2)} = K_1$, $Z_2^{(2)} = L_1$ *and compute* $H_1 = (Z_1^{(2)})^*(Z_1^{(2)}) + (Z_2^{(2)})^*(Z_2^{(2)})$.

(ii) Iterate : For $j = 1, 2, ...$

  (a) put

$$A_j = A_0 Z_1^{(2^j)}, \qquad B_j = Z_{2^j}^{(2^j)}.$$

(b) Resolve

$$\begin{pmatrix} B_J & A_j \\ A_j & B_j \end{pmatrix} \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}.$$

(c) Compute

$$\begin{aligned} Z_1^{(2^{j+1})} &= Z_1^{(2^j)} K_{j+1} \\ Z_{2^{j+1}}^{(2^{j+1})} &= Z_{2^j}^{(2^j)} L_{j+1} \\ H_{j+1} &= (K_{j+1})^* H_j K_{j+1} + (L_{j+1})^* H_j L_{j+1}. \end{aligned}$$

(iii)

$$\mathbb{P} = Z_{2^{j+1}}^{(2^{j+1})} \qquad \textit{and} \qquad \mathbb{H} = H_{j+1}.$$

Another spectral dichotomy method has been proposed by M. Sadkane and A. Touhami in [15]. We will just present the resulting algorithm of their work within the framework of the spectral dichotomy method of a pencil $\lambda B - A$

Algorithm 2 (DichoC2)**.**

- **Input:** $A, B \in \mathbb{C}^{n\times n}$ *such that the pencil* $\lambda B - A$ *is regular having no eigenvalues on the unit circle.*
  $\mathbb{H}^{(0)} = (\mathbb{H}^{(0)})^*$ *used for scaling. For instance* $\mathbb{H}^{(0)} = I_n$
- **Output:** $\mathbb{P}$ *the spectral projector onto the right deflating subspace of* $\lambda B - A$ *associated with the eigenvalues inside the unit circle.*
  $\mathbb{H}$ *the matrix integral whose norm* $\|\mathbb{H}\|$ *indicates the quality of the projector* $\mathbb{P}$*.*

1. **Initialization**
$H_0 = H^0$
First iteration

(i) **Compute** X, Y **solutions of the equations**
$X(B - A) = A, \qquad \textit{and} \qquad Y(B - A) = B$

(ii) **Compute** Δ_0, ∇_0 **solutions of the equations**
$(A + B)\Delta_0 = X, \qquad \textit{and} \qquad (A + B)\nabla_0 = Y$

(iii) **Compute** $H_1, Z_1^{(2)}, Z_2^{(2)}$ **:**
$H_1 = \Delta_0^* H_0 \Delta + \nabla_0^* H_0 \nabla_0$
$Z_1^2 = \Delta_0, \quad Z_2^{(2)} = \nabla_0$

2. **Next Iterations**
For $j = 2, 3 \cdots$ **until convergence Do :**
Update of A_{j-1}

(i) $A_{j-1} = -AZ_1^{(2^{j-1})}$

(ii) **Compute** Δ_{j-1}
$(2A_{j-1} - I_n)\Delta_{j-1} = A_{j-1}$

Computation of $H_j, Z_1^{(2^j)}, Z_{2^j}^{(2^j)}$ **:**

(iii) $H_j = \Delta_{j-1}^* H_{j-1} \Delta_{j-1} + (I - \Delta_{j-1})^* H_{j-1}(I_n - \Delta_{j-1})$

(iv) $Z_1^{(2^j)} = Z_1^{(2^{j-1})}\Delta_{j-1}, \qquad Z_{2^j}^{(2^j)} = Z_{2^{j-1}}^{(2^{j-1})}(I_n - \Delta_{j-1})$

EndFOR

3. $\mathbb{P} = Z_{2^j}^{(2^j)} B$ *and* $\mathbb{H} = H_j$

2.2. Spectral dichotomy with respect to the imaginary axis

We assume that $\lambda I_n - A$ does not have any eigenvalue on the imaginary axis. We summarize the computation of the spectral projector on the right eigenspace corresponding to the eigenvalues with positive real parts.

Using the Cayley transformation $\varphi : \lambda \in \mathbb{C} \setminus \{1\} \longrightarrow z \in \mathbb{C} \setminus \{1\}$, defined by

$$\varphi(\lambda) = z = \frac{(\lambda + 1)}{(\lambda - 1)} \tag{13}$$

φ is a bijection from $\mathbb{C} \setminus \{1\}$ to $\mathbb{C} \setminus \{1\}$. The spectral dichotomy with respect to the imaginary axis can be transformed to the spectral dichotomy to the circle by the inverse of φ.
It is not difficult to show that the bijection φ transforms the interior (respectively exterior) of the circle to the left (respectively the right) half plane and the circle to the imaginary axis. We will briefly prove it below.
Let's assume that $z = x + iy$ then

$$z = \frac{\Re(\lambda) + i\Im(\lambda) + 1}{\Re(\lambda) + i\Im(\lambda) - 1}$$

$$= \frac{\Re(\lambda)^2 - 1 + \Im(\lambda)^2 - 2i\Im(\lambda)}{(\Re(\lambda) - 1)^2 + \Im(\lambda)^2}$$

So

$$x = \frac{\Re(\lambda)^2 - 1 + \Im(\lambda)^2}{(\Re(\lambda)-1)^2 + \Im(\lambda)^2} \qquad \text{and} \qquad y = \frac{-2\Im(\lambda)}{(\Re(\lambda)-1)^2 + \Im(\lambda)^2}$$

we have:

$$\begin{aligned} \lambda \in \mathcal{C}(O,1) &\Leftrightarrow |\lambda| = 1 \\ &\Leftrightarrow \Re(\lambda)^2 + \Im(\lambda)^2 = 1 \\ &\Leftrightarrow x = 0 \end{aligned}$$

which proves that φ maps bijectively the circle $\mathcal{C}(O,1)\backslash\{(1,0)\}$ onto the imaginary axis. Similarly we have,

$$\begin{aligned} |\lambda| < 1 &\Leftrightarrow \Re(\lambda)^2 + \Im(\lambda)^2 < 1 \\ &\Leftrightarrow x < 0 \end{aligned}$$

Consider the pencil $\lambda\mathcal{B} - \mathcal{A}$ where $\mathcal{B} = A - I_n$ and $\mathcal{A} = A + I_n$. The eigenvalues z of the matrix A and λ are linked by the relation $z = \dfrac{\lambda+1}{\lambda-1}$.
Therefore, the spectral dichotomy with respect to the imaginary axis can be transformed to the spectral dichotomy to the unit circle and their spectral projectors are the same. According to [3],[11],[12], the quality of the spectral dichotomy with respect to the imaginary axis is characterized by the numerical parameter

$$\alpha = \sup_{\Re(z)=0} \|(zI_n - A)^{-1}\| \tag{14}$$

Similarly, the quality of the dichotomy for the matrix pencil $\lambda\mathcal{B} - \mathcal{A}$ with respect to the unit circle is also given by

$$\tilde{\alpha} = \sup_{|\lambda|=1} \|(\lambda\mathcal{B} - \mathcal{A})^{-1}\| \tag{15}$$

The following proposition shows the relation between the two parameters.

Proposition 3. *We assume that $\|A\| = 1$ and let α and $\tilde{\alpha}$ be the two parameters defined by (14) and (15). Then*

$$\frac{1}{2}\alpha \le \tilde{\alpha} \le \alpha + \frac{1}{2} \tag{16}$$

Proof. Since

$$\begin{aligned} \lambda\mathcal{B} - \mathcal{A} &= \frac{z+1}{z-1}\mathcal{B} - (A + I_n) \\ &= \frac{1}{z-1}\left[(z+1)(A - I_n) - (z-1)(A + I_n)\right] \\ &= \frac{-2}{z-1}\left(zI_n - A\right) \end{aligned}$$

then $(\lambda\mathcal{B}-\mathcal{A})^{-1} = -\dfrac{z-1}{2}\,(zI_n - A)^{-1}$ and thus

$$\tilde{\alpha} = \sup_{\Re(z)=0} \frac{1}{2} \mid z-1 \mid \|(zI_n - A)^{-1}\| \geq \frac{1}{2}\alpha$$

We also have

$$\| (\lambda\mathcal{B}-\mathcal{A})^{-1} \| \leq \frac{1}{2}(1+|z|)\| (zI_n - A)^{-1} \|$$

We will discuss according to the values of $|z|$

- If $|z| \leq \dfrac{\alpha+1}{\alpha}$

$$\begin{aligned}\sup_{|\lambda|=1} \| (\lambda\mathcal{B}-\mathcal{A})^{-1} \| &\leq \frac{1}{2} \sup_{\Re(z)=0} (1+|z|)\| (zI_n - A)^{-1} \| \\ &\leq \alpha + \frac{1}{2}\end{aligned}$$

  then $\tilde{\alpha} \leq \alpha + \dfrac{1}{2}$

- If $|z| \geq \dfrac{\alpha+1}{\alpha}$ then with the assumption $\|A\| = 1$ it follows that $\left\|\dfrac{A}{z}\right\| \leq 1$. Which leads to

$$\begin{aligned}(zI_n - A)^{-1} &= \frac{1}{z}\left(I_n - \frac{A}{z}\right)^{-1} \\ &= \frac{1}{z}\left(I_n + \frac{A}{z}\sum_{m=0}^{+\infty}\frac{A^m}{z^m}\right) \\ &= \frac{1}{z}\left(I_n + A(zI_n - A)^{-1}\right)\end{aligned}$$

  Consequently, we obtain

$$\begin{aligned}\|(\lambda\mathcal{B}-\mathcal{A})^{-1}\| &\leq \frac{1}{2}(1+|z|)\|(zI_n - A)^{-1}\| \\ &\leq \frac{1}{2}(1+\frac{1}{|z|})\left(1+\|(zI_n - A)^{-1}\|\right) \\ &\leq \frac{2\alpha+1}{2(\alpha+1)}\left(1+\|(zI_n - A)^{-1}\|\right)\end{aligned}$$

  hence

$$\tilde{\alpha} \leq \frac{2\alpha+1}{2(\alpha+1)}\,(1+\alpha) = \alpha + \frac{1}{2}$$

In both cases

$$\tilde{\alpha} \leq \alpha + \frac{1}{2}$$

In conclusion

$$\frac{1}{2}\alpha \leq \tilde{\alpha} \leq \alpha + \frac{1}{2}$$

This proposition shows that the quality of the dichotomy of A with respect to the imaginary axis is equivalent to the dichotomy of the matrix pencil $\lambda\mathcal{B} - \mathcal{A}$ with respect to the unit circle. The following algorithm is used to calculate the values of the projector and the dichotomy criterion

Algorithm 3 (DichoI)**.**

- *Input variables: A and I_n such that the matrix sheaf $zI_n - A$ has no eigenvalues on the imaginary axis.*
- *Output variables: $\mathcal{P}$ and $\mathbb{H}$.*
  $\mathcal{P}$ is the projector onto the left deflating subspace of A corresponding to the eigenvalues with real positives parts and $\mathbb{H}$ the dichotomy's criterion.

1. *Set $\mathcal{A} = A + I$ and $\mathcal{B} = A - I$.*
2. *Using Algorithm 2 to $\lambda\mathcal{B} - \mathcal{A}$, compute the projectors $\mathcal{P}_i$ onto the right eigenspace of $\mathcal{A}$ associated with the eigenvalues inside the unit circle and the Hermitian matrice $\mathbb{H}$.*
3. *$\mathcal{P} = I_n - \mathcal{P}_i$.*

2.3. Spectral dichotomy with respect to a parabola

Consider the equation of the following parabola

$$2p\left(\frac{p}{2} - x\right) = y^2 \quad \text{with} \quad p > 0 \tag{17}$$

which was studied by Malyshev and Sadkane in [13]. We make a brief summary :

Consider the matrix A of order n $(n > 1)$ having no eigenvalues on the parabola $\Gamma = \Gamma(a, 0, c)$ of Equation (17). Let $\mathcal{A}$ be the matrix of order $2n$ defined by

$$\mathcal{A} = \begin{bmatrix} -\sqrt{\frac{p}{2}}I_n & A \\ & \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{bmatrix}$$

The respective eigenvalues λ and z of the matrices $\mathcal{A}$ and A satisfy the relation

$$z = \left(\lambda + \sqrt{\frac{p}{2}}\right)^2$$

In their study, Malyshev and Sadkane assumed in ([13]) that $\|A\| = 1$. Otherwise we set

$$A_1 = \frac{1}{\|A\|}A \text{ and } p_1 = \frac{p}{\|A\|}.$$

According to [13], the quality of the spectral dichotomy with respect to the imaginary axis is characterized by the numerical value

$$\alpha_{\mathcal{A}} = \sup_{\Re(\lambda)=0} \|(\lambda I_{2n} - \mathcal{A})^{-1}\|. \tag{18}$$

Similarly, the spectral dichotomy with respect to the parabola is also characterized by the numerical parameter

$$\alpha_A = \sup_{z\in\Gamma} \|(zI_n - A)^{-1}\| \tag{19}$$

The following proposition gives a relation between the parameters $\alpha_{\mathcal{A}}$ and α_A.

Proposition 4. *([13]) Let $\alpha_{\mathcal{A}}$ and α_A be the two parameters defined by* (18) *and* (19). *We have*

$$\alpha_A \le \alpha_{\mathcal{A}} \le \alpha_A + \sqrt{\alpha_A}\sqrt{1+\alpha_A} \tag{20}$$

Consider the spectral projectors

- $\mathbb{P} \in \mathbb{C}^{n\times n}$ on the right eigenspace of A associated with the eigenvalues outside the parabola Γ ;
- $\mathcal{P} \in \mathbb{C}^{2n\times 2n}$ on the right eigenspace of $\mathcal{A}$ associated with the eigenvalues in the right complex half-plane.

The following proposition characterizes the relation between $\mathbb{P}$ and $\mathcal{P}$

Proposition 5. *([13]) Consider a partition of the matrix $\mathcal{P}$ in the form*

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_2 \\ \mathcal{P}_3 & \mathcal{P}_4 \end{pmatrix} \qquad \textit{with} \qquad \mathcal{P}_i \in \mathbb{C}^{n\times n}, \qquad i = 1,4 \tag{21}$$

Then

$$\mathbb{P} = 2\mathcal{P}_1 = 2\mathcal{P}_4 = 4\mathcal{P}_2\mathcal{P}_3 \tag{22}$$

Moreover

$$\mathcal{P}_2 = \frac{1}{2}\left(\mathbb{P}A\right)^{\frac{1}{2}} \tag{23}$$

Algorithm 4 (DichoP).

- *Input variables: A and I_n such that the matrix sheaf $zI_n - A$ has no eigenvalues on the parabola of the equation $2p\left(\frac{p}{2} - x\right)^2 = y^2$ with $p > 0$*
- *Output variables: The spectral projector $\mathbb{P}$ and the dichotomy criterion $\mathbb{H}$.*
  $\mathbb{P}$ being the projector on the right invariant space of $zI_n - A$ corresponding to the eigenvalues outside the parabola and $\mathbb{H}$ the dichotomy criterion.

1. *Compute the matrix*

$$\mathcal{A} = \begin{pmatrix} -\sqrt{\frac{p}{2}}I_n & A \\ & \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{pmatrix}$$

2. *Using Algorithm 3 to $\lambda I_{2n} - \mathcal{A}$, compute the Projector $\mathcal{P}$ onto the right eigenspace of $\mathcal{A}$ associated with the eigenvalues on the right half-plane of the complex plane and the matrix $\mathbb{H}$;*
3. *If $\|\mathbb{H}\|$ is not large then determine the projector $\mathbb{P}$ by*

$$\mathbb{P} = 2\mathcal{P}_1$$

3. Presentation of new methods

In what follows, we will consider the general equation of the parabola (1) as announced in the introduction and we will determine the projector $\mathbb{P}$ for parameters a, b and $c \in \mathbb{R}$ with $a \neq 0$.

3.1. The case of a parabola of equation of type (1) with discriminant equal to 1.

Equation (1) of the parabola becomes

$$2p\left(\frac{p}{2} - x\right) = (y - pb)^2 \tag{24}$$

with $p > 0$.
For the parameter $b = 0$, we are in the case of the parabola studied by Malyshev and Sadkane in [13].
On the other hand, in this section, we are going to consider the coefficient $b \neq 0$. Which leads us to define the following parabola

$$\widetilde{\Gamma} = \{z = x + iy \setminus x + i(y - pb) \in \Gamma\}$$

of the equation

$$2p\left(\frac{p}{2}-x\right)=\tilde{y}^2 \tag{25}$$

where

$$\tilde{y}=y-pb.$$

Consider the matrix of order 2n defined by

$$\tilde{\mathcal{A}}=\begin{bmatrix} -\sqrt{\frac{p}{2}}I_n & A_b \\ & \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{bmatrix}$$

where $A_b=A-ipbI_n$

Remark 2. *The respective eigenvalues $\tilde{\lambda}$ and z of the matrices $\tilde{\mathcal{A}}$ and A are such that*

$$z=\left(\tilde{\lambda}+\sqrt{\frac{p}{2}}\right)^2+ipb$$

We also assume that $\|A_b\|=1$. Otherwise (i.e. $\|A_b\|\neq 1$), we can take

$$A_b^1=\frac{1}{\|A_b\|}A_b \text{ and } p_1=\frac{1}{\|A_b\|}p.$$

Consider the dichotomy quantities characterized by the following numerical parameters

$$\alpha_{\tilde{\mathcal{A}}}=\sup_{\Re(\tilde{\lambda})=0}\|(\tilde{\lambda}I_{2n}-\tilde{\mathcal{A}})^{-1}\| \quad \text{and} \quad \alpha_{A_b}=\sup_{z\in\widetilde{\Gamma}}\|(zI_n-A)^{-1}\| \tag{26}$$

We have the following proposition

Proposition 6. *Let $\alpha_{\tilde{\mathcal{A}}}$ and α_{A_b} be the two parameters defined in (26). Assume that*

$$\|A_b\|=1 \qquad \textit{and} \qquad |pb|<\frac{1}{\alpha_{A_b}}. \tag{27}$$

Then

$$\alpha_{A_b}\leq\alpha_{\tilde{\mathcal{A}}}\leq 2\left(\alpha_{A_b}+\sqrt{\alpha_{A_b}}\left(1+\sqrt{\alpha_{A_b}+1}\right)\right) \tag{28}$$

Proof. Consider the matrix

$$(\widetilde{\lambda}I_{2n}-\widetilde{\mathcal{A}})=\begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n & -A_b \\ & \\ -I_n & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n \end{bmatrix}.$$

We have

$$\begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n & A_b \\ I_n & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n \end{bmatrix} \times \begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n & -A_b \\ -I_n & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n \end{bmatrix} = \begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 I_n - A_b & 0 \\ 0 & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 I_n - A_b \end{bmatrix}$$

Therefore

$$(\widetilde{\lambda}I_{2n}-\widetilde{\mathcal{A}})^{-1} = \begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 I_n - A_b & 0 \\ 0 & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 I_n - A_b \end{bmatrix}^{-1} \times \begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n & A_b \\ I_n & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} ((\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 + ipb)I_n - A & 0 \\ 0 & ((\widetilde{\lambda}+\sqrt{\frac{p}{2}})^2 + ipb)I_n - A \end{bmatrix}^{-1} \times \begin{bmatrix} (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n & A - ipbI_n \\ I_n & (\widetilde{\lambda}+\sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} (zI_n - A)^{-1} & 0 \\ 0 & (zI_n - A)^{-1} \end{bmatrix} \times \begin{bmatrix} \sqrt{z-ipb}I_n & A - ipbI_n \\ I_n & \sqrt{z-ipb}I_n \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{z-ipb}(zI_n - A)^{-1} & (zI_n - A)^{-1}(A - ipbI_n) \\ (zI_n - A)^{-1} & \sqrt{z-ipb}(zI_n - A)^{-1} \end{bmatrix}$$

Knowing that the norm $(\widetilde{\lambda}I_{2n}-\widetilde{\mathcal{A}})^{-1}$ is greater than or equal to the norm of each of its block components taken individually, we can deduce that

$$\tilde{\alpha} = \sup_{\Re(\widetilde{\lambda})=0} \|(\widetilde{\lambda}I_{2n}-\widetilde{\mathcal{A}})^{-1}\| \geq \sup_{z\in\widetilde{\Gamma}} \|(zI_n - A)^{-1}\| = \alpha_{A_b}$$

and also

$$\begin{aligned}
\|(\widetilde{\lambda} I_{2n} - \widetilde{\mathcal{A}})^{-1}\| &\leq \left\| \begin{pmatrix} \sqrt{z - ipb} I_n & A - ipbI_n \\ \\ I_n & \sqrt{z - ipb} I_n \end{pmatrix} \right\| \|(zI_n - A)^{-1}\| \\
&\leq \left\| \begin{pmatrix} \|\sqrt{z - ipb} I_n\| & \|A - ipbI_n\| \\ \\ \|I_n\| & \|\sqrt{z - ipb} I_n\| \end{pmatrix} \right\| \|(zI_n - A)^{-1}\| \\
&\leq \left\| \begin{pmatrix} \sqrt{|z|} + \sqrt{|pb|} & 1 \\ \\ 1 & \sqrt{|z|} + \sqrt{|pb|} \end{pmatrix} \right\| \|(zI_n - A)^{-1}\| \\
&\leq \left((1 + \sqrt{|z|} + \sqrt{|pb|} \right) \|(zI_n - A)^{-1}\|
\end{aligned}$$

- If $|z| \leq \dfrac{\alpha_{A_b} + 1}{\alpha_{A_b}}$ then

$$\begin{aligned}
\|(\lambda I_{2n} - \widetilde{\mathcal{A}})^{-1}\| &\leq \alpha_{A_b}(\sqrt{|z|} + \sqrt{|pb|} + 1) \\
&\leq \alpha_{A_b} \left(\sqrt{\frac{\alpha_{A_b} + 1}{\alpha_{A_b}}} + \sqrt{\frac{1}{\alpha_{A_b}}} + 1 \right) \\
&\leq \alpha_{A_b} + \sqrt{\alpha_{A_b}} \left(1 + \sqrt{\alpha_{A_b} + 1}\right)
\end{aligned}$$

- If $|z| > \dfrac{\alpha_{A_b} + 1}{\alpha_{A_b}}$ then with the assumptions $\|A_b\| = 1$ and $|pb| < \dfrac{1}{\alpha_{A_b}}$ we note that

$$\begin{aligned}
\left\| \frac{A}{z} \right\| &< \frac{\alpha_{A_b}}{\alpha_{A_b} + 1} \left(\|A_b\| + |pb| \right) \\
&< \frac{\alpha_{A_b}}{\alpha_{A_b} + 1} \left(1 + \frac{1}{\alpha_{A_b}} \right) \\
&< 1
\end{aligned}$$

Which leads to

$$\begin{aligned}
(zI_n - A)^{-1} &= \frac{1}{z} \left(I_n - \frac{A}{z} \right)^{-1} \\
&= \frac{1}{z} \left(I_n + \frac{A}{z} \sum_{m=0}^{+\infty} \frac{A^m}{z^m} \right) \\
&= \frac{1}{z} \left(I_n + A(zI_n - A)^{-1} \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\|(\widetilde{\lambda}I_{2n}-\widetilde{\mathcal{A}})^{-1}\| &\leq \left\|\frac{1}{z}I_n+\frac{1}{z}A\left(zI_n-A\right)^{-1}\right\| \times \left\|1+\sqrt{|z|}+\sqrt{|pb|}\right\| \\
&\leq (\|A\|\|(zI_n-A)^{-1}\|+1)\frac{1+\sqrt{|z|}+\sqrt{|pb|}}{|z|} \\
&\leq ((1+|pb|)\alpha_{A_b}+1)\left(\frac{\alpha_{A_b}}{\alpha_{A_b}+1}+\frac{\sqrt{\alpha_{A_b}}}{\sqrt{\alpha_{A_b}+1}}+\frac{\alpha_{A_b}\sqrt{|pb|}}{\alpha_{A_b}+1}\right) \\
&\leq ((1+\frac{1}{\alpha_{A_b}})\alpha_{A_b}+1)\left(\frac{\alpha_{A_b}}{\alpha_{A_b}+1}+\frac{\sqrt{\alpha_{A_b}}}{\sqrt{\alpha_{A_b}+1}}+\frac{\sqrt{\alpha_{A_b}}}{\alpha_{A_b}+1}\right) \\
&\leq \frac{2+\alpha_{A_b}}{\alpha_{A_b}+1}\left(\alpha_{A_b}+\sqrt{\alpha_{A_b}}\sqrt{\alpha_{A_b}+1}+\sqrt{\alpha_{A_b}}\right) \\
&\leq 2\left(\alpha_{A_b}+\sqrt{\alpha_{A_b}}\left(1+\sqrt{\alpha_{A_b}+1}\right)\right).
\end{aligned}$$

Hence

$$\alpha_{A_b} \leq \tilde{\alpha} \leq 2\left(\alpha_{A_b}+\sqrt{\alpha_{A_b}}\left(1+\sqrt{\alpha_{A_b}+1}\right)\right).$$

This proves that the dichotomy parameters of a matrix with respect to a parabola and with respect to the imaginary axis are equivalent.

Consider spectral projectors

- $\widetilde{\mathbb{P}} \in \mathcal{C}^{n\times n}$ on the right subspace of A associated with its eigenvalues outside the parabola $\widetilde{\Gamma}$.
- $\widetilde{\mathcal{P}} \in \mathcal{C}^{2n\times 2n}$ on the right subspace of $\widetilde{\mathcal{A}}$ associated with its eigenvalues in the complex right half-plane.

We obtain the following proposition which characterizes the relation between $\widetilde{\mathbb{P}}$ and $\widetilde{\mathcal{P}}$

Proposition 7. *Consider a partition of the matrix $\widetilde{\mathcal{P}}$ in the form*

$$\widetilde{\mathcal{P}} = \begin{pmatrix} \widetilde{\mathcal{P}}_1 & \widetilde{\mathcal{P}}_2 \\ \widetilde{\mathcal{P}}_3 & \widetilde{\mathcal{P}}_4 \end{pmatrix} \qquad avec \qquad \widetilde{\mathcal{P}}_i \in \mathbb{C}^{n\times n}, \qquad i=1,4 \tag{29}$$

Then

$$\widetilde{\mathbb{P}} = 2\widetilde{\mathcal{P}}_1 = 2\widetilde{\mathcal{P}}_4 = 4\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_3 \tag{30}$$

Moreover

$$\widetilde{\mathbb{P}}A = 4 \times (\widetilde{\mathcal{P}}_2)^2 + 2ipb\widetilde{\mathcal{P}}_1 \tag{31}$$

Proof.
Let $\tilde{X}$ be a solution to the matrix equation

$$\left(\tilde{X} + \sqrt{\frac{p}{2}}I_n\right)^2 = A_b \tag{32}$$

Consider the matrix $\tilde{X}_1$ defined by

$$\tilde{X}_1 = -\tilde{X} - 2\sqrt{\frac{p}{2}}I_n.$$

We notice that

$$\begin{aligned}\left(\tilde{X}_1 + \sqrt{\frac{p}{2}}I_n\right)^2 &= \left(\tilde{X} + \sqrt{\frac{p}{2}}I_n\right)^2 \\ &= A_b\end{aligned}$$

Hence $\tilde{X}_1$ is also a solution of matrix equation (32).

$$\begin{bmatrix}\tilde{X} + \sqrt{\frac{p}{2}}I_n & -\tilde{X} - \sqrt{\frac{p}{2}}I_n \\ I_n & I_n\end{bmatrix} \times \begin{bmatrix}\tilde{X} & 0 \\ 0 & -\tilde{X} - 2\sqrt{\frac{p}{2}}I_n\end{bmatrix} \times \begin{bmatrix}\frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n \\ -\frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{X}(\tilde{X} + \sqrt{\frac{p}{2}}I_n) & (-\tilde{X} - \sqrt{\frac{p}{2}}I_n)(-\tilde{X} - 2\sqrt{\frac{p}{2}}I_n) \\ \tilde{X} & -\tilde{X} - 2\sqrt{\frac{p}{2}}I_n\end{bmatrix} \times \begin{bmatrix}\frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n \\ -\frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}\tilde{X} - \frac{1}{2}(\tilde{X} + 2\sqrt{\frac{p}{2}}I_n) & \frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)(2\tilde{X} + 2\sqrt{\frac{p}{2}}I_n) \\ \frac{1}{2}(\tilde{X} + \sqrt{\frac{p}{2}}I_n)^{-1}(2\tilde{X} + 2\sqrt{\frac{p}{2}}I_n) & \frac{1}{2}X - \frac{1}{2}(X + 2\sqrt{\frac{p}{2}}I_n)\end{bmatrix}$$

$$= \begin{bmatrix}-\sqrt{\frac{p}{2}}I_n & A_b \\ I_n & -\sqrt{\frac{p}{2}}I_n\end{bmatrix}$$

$= \widetilde{\mathcal{A}}.$

Let $\widetilde{X} = Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1}$ be the canonical jordan form of the matrix $\widetilde{X}$ with J_+ and J_- the Jordan blocks associated respectively with the eigenvalues located in the right half-plane and the left half-plane where J_+ if of order k.

By replacing the decomposition of $\widetilde{X}$ in the matrix $\widetilde{\mathcal{A}}$, we get

$$\widetilde{\mathcal{A}} = \begin{bmatrix} Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} + \sqrt{\frac{p}{2}} I_n & -Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times$$

$$\begin{bmatrix} Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} & 0 \\ 0 & Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{1}{2} \left(Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2} \left(Q \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} Q^{-1} - \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix}$$

$$= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times$$

$$\begin{bmatrix} \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} \left(\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2} \left(\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

$$= \widetilde{\mathcal{Q}} \mathcal{J} \widetilde{\mathcal{Q}}^{-1}$$

with

$$\widetilde{\mathcal{Q}} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix}$$

and

$$\mathcal{J} = \begin{bmatrix} \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix}$$

Knowing that we have k eigenvalues of the matrix A_b in the right half-plane, we can therefore compute the associated projector

$$\begin{aligned}
\widetilde{\mathcal{P}} &= \widetilde{\mathcal{Q}} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \widetilde{\mathcal{Q}}^{-1} \\
&= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2}\left(\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2}\left(\begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \\
&= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \begin{bmatrix} J_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2} \begin{bmatrix} (J_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (J_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \\ -\frac{1}{2} \begin{bmatrix} (J_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (J_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \frac{1}{2}\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2}\begin{bmatrix} J_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} \\ \frac{1}{2}\begin{bmatrix} (J_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2}\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q\begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} & Q\begin{bmatrix} \frac{1}{2}(J_+ + \sqrt{\frac{p}{2}} I_k) & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ Q\begin{bmatrix} \frac{1}{2}(J_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} & Q\begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{\mathcal{P}}_1 & \widetilde{\mathcal{P}}_2 \\ \widetilde{\mathcal{P}}_3 & \widetilde{\mathcal{P}}_4 \end{bmatrix}$$

It follows that

$$\widetilde{\mathcal{P}}_1 = Q\begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = \frac{1}{2}\widetilde{\mathbb{P}}$$

$$\left.\begin{aligned} \widetilde{\mathcal{P}}_2 &= Q\frac{1}{2}\begin{bmatrix} J_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ \widetilde{\mathcal{P}}_3 &= Q\frac{1}{2}\begin{bmatrix} (J_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \end{aligned}\right\} \Longrightarrow \mathbb{P}_b = 4\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_3$$

$$\widetilde{\mathcal{P}}_4 = Q\begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = \frac{1}{2}\mathbb{P}_b$$

We also note that with

$$\begin{aligned} A &= A_b + ipbI_n \\ &= Q\begin{bmatrix} \left(J_+ + \sqrt{\frac{p}{2}} I_k\right)^2 + ipbI_k & 0 \\ 0 & \left(J_- + \sqrt{\frac{p}{2}} I_{n-k}\right)^2 + ipbI_{n-k} \end{bmatrix} Q^{-1} \end{aligned}$$

we have

$$
\begin{aligned}
\widetilde{\mathbb{P}}A &= Q\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \times Q \begin{bmatrix} \left(J_+ + \sqrt{\frac{p}{2}} I_k\right)^2 + ipbI_k & 0 \\ 0 & \left(J_- + \sqrt{\frac{p}{2}} I_{n-k}\right)^2 + ipbI_{n-k} \end{bmatrix} Q^{-1} \\
&= Q\begin{bmatrix} \left(J_+ + (\sqrt{\frac{p}{2}} I_k\right)^2 + ipbI_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\
&= 4\widetilde{\mathcal{P}}_2^2 + 2ipb\widetilde{\mathcal{P}}_1
\end{aligned}
$$

Thus we obtain the equality (30) and (31)

Remark 3. *if the parameter $b = 0$, whence equalities (31) are reduced to those of (23)*

Algorithm 5 (DichoPb)**.**

- *Input variables : A and I_n such that the matrix bundle $zI_n - A$ has no eigenvalues on the parabola with equation $2p\left(\frac{p}{2} - x\right)^2 = (y - pb)^2$ with $p > 0$*

- *Output variables: $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{H}}$*
  $\widetilde{\mathbb{P}}$ being the projector on the right subspace of $zI_n - A$ associated with the eigenvalues outside the parabola and the matrix $\widetilde{\mathbb{H}}$ whose norm gives the dichotomy criterion.

1. *Determine the matrix*

$$
\widetilde{\mathcal{A}} = \begin{pmatrix} -\sqrt{\frac{p}{2}} I_n & A - ipbI_n \\ I_n & -\sqrt{\frac{p}{2}} I_n \end{pmatrix}
$$

2. *Using Algorithm 3 to $\widetilde{\lambda} I_{2n} - \widetilde{\mathcal{A}}$, compute the Projector $\widetilde{\mathcal{P}}$ onto the right eigenspace of $\widetilde{\mathcal{A}}$ associted with the eigenvalues on the right half-plane of the complex plane and the matrix $\widetilde{\mathbb{H}}$;*

3. *If $\|\widetilde{\mathbb{H}}\|$ is not large, determine the projectors $\widetilde{\mathbb{P}}$ by*

$$
\widetilde{\mathbb{P}} = 2\widetilde{\mathcal{P}}_1
$$

3.2. The case of a parabola of equation of type (1) with discriminant different from 1

We consider the following change of variable in equation (4)

$$\tilde{x} = x + \frac{p}{2} - d$$

We get

$$2p\left(\frac{p}{2} - \widetilde{x}\right) = (y - pb)^2 \tag{33}$$

3.2.1. The spectral dichotomy method with the coefficient $b = 0$

Consider the set

$$\Gamma_d = \left\{ z_d = x + iy / x + (\frac{p}{2} - d) + iy \in \Gamma \right\}$$

described by the following equation

$$y^2 = 2p(\frac{p}{2} - \widetilde{x}).$$

Let the matrix

$$\mathcal{A}_d = \begin{bmatrix} -\sqrt{\frac{p}{2}} I_n & A_d \\ I_n & -\sqrt{\frac{p}{2}} I_n \end{bmatrix} \quad \text{where} \quad A_d = A + \left(\frac{p}{2} - d\right) I_n \tag{34}$$

Knowing that the eigenvalues z_d and z of the matrices A_d and A are linked by

$$z_d = z + \frac{p}{2} - d,$$

Remark 4. *The respective eigenvalues λ_d and z of the matrices $\mathcal{A}_d$ and A are such that*

$$z = \left(\lambda_d + \sqrt{\frac{p}{2}}\right)^2 - \frac{p}{2} + d$$

Furthermore, since $z = x + iy$, then we have

$$\begin{cases} x = \left(\Re(\lambda_d) + \sqrt{\frac{p}{2}}\right)^2 - \Im(\lambda_d)^2 - \frac{p}{2} + d \\ y = 2\left(\Re(\lambda_d) + \sqrt{\frac{p}{2}}\right)\Im(\lambda_d) \end{cases}$$

By setting that

$$p_d = 2\left(\Re(\lambda_d) + \sqrt{\frac{p}{2}}\right)^2$$

then

$$y^2 = 2p_d\left[\frac{p_d}{2} - x - \frac{p}{2} + d\right] = 2p_d\left(\frac{p_d}{2} - \widetilde{x}\right)$$

We also assume that $\|A_d\| = 1$. Otherwise we set (i.e. $\|A_d| \neq 1$), we can take

$$A_d^1 = \frac{1}{\|A_d\|}A_d \quad \text{and} \quad p_1 = \frac{1}{\|A_d\|}.$$

Consider the numerical parameters $\alpha_{\mathcal{A}_d}$ and α_{A_d} defined by

$$\alpha_{\mathcal{A}_d} = \sup_{\Re(\lambda_d)=0} \|(\lambda_d I_{2n} - \mathcal{A}_d)^{-1}\| \quad \text{et} \quad \alpha_{A_d} = \sup_{z\in\Gamma_d} \|(zI_n - A)^{-1}\| \tag{35}$$

The following proposition gives a relation between the parameters $\alpha_{\mathcal{A}_d}$ and α_{A_d}.

Proposition 8. *Let $\alpha_{\mathcal{A}_d}$ and α_{A_d} be the two parameters defined in* (35)*. Assume that*

$$\|A_d\| = 1 \qquad \textit{and} \qquad \left|\frac{p}{2} - d\right| < \frac{1}{\alpha_{A_d}}. \tag{36}$$

Then

$$\alpha_{A_d} \leq \alpha_{\mathcal{A}_d} \leq 2\left(\alpha_{A_d} + \sqrt{\alpha_{A_d}}\left(1 + \sqrt{\alpha_{A_d} + 1}\right)\right). \tag{37}$$

Proof.

Let the matrix

$$(\lambda_d I_{2n} - \mathcal{A}_d) = \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})I_n & -A_d \\ \\ -I_n & (\lambda_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

We have

$$\begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})I_n & A_d \\ \\ I_n & (\lambda_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix} \times \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})I_n & -A_d \\ \\ -I_n & (\lambda_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix} =$$

$$\begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - A_d & 0 \\ \\ 0 & (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - A_d \end{bmatrix}$$

with

$$(\lambda_d I_{2n} - \mathcal{A}_d)^{-1} = \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - A_d & 0 \\ 0 & (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - A_d \end{bmatrix}^{-1} \times \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}}) I_n & A_d \\ I_n & (\lambda_d + \sqrt{\frac{p}{2}}) I_n \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - (A + (\frac{p}{2} - d) I_n) & 0 \\ 0 & (\lambda_d + \sqrt{\frac{p}{2}})^2 I_n - (A + (\frac{p}{2} - d) I_n) \end{bmatrix}^{-1} \times \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}}) I_n & (A + (\frac{p}{2} - d) I_n) \\ I_n & (\lambda_d + \sqrt{\frac{p}{2}}) I_n \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d) I_n - A & 0 \\ 0 & ((\lambda_d + \sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d) I_n - A \end{bmatrix}^{-1} \times \begin{bmatrix} (\lambda_d + \sqrt{\frac{p}{2}}) I_n & A + (\frac{p}{2} - d) I_n \\ I_n & (\lambda_d + \sqrt{\frac{p}{2}}) I_n \end{bmatrix}$$

$$= \begin{bmatrix} (z I_n - A)^{-1} & 0 \\ 0 & (z I_n - A)^{-1} \end{bmatrix} \times \begin{bmatrix} \sqrt{z + \frac{p}{2} - d} I_n & A + (\frac{p}{2} - d) I_n \\ I_n & \sqrt{z + \frac{p}{2} - d} I_n \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{z + \frac{p}{2} - d} (z I_n - A)^{-1} & (z I_n - A)^{-1} (A + (\frac{p}{2} - d) I_n) \\ (z I_n - A)^{-1} & \sqrt{z + \frac{p}{2} - d} (z I_n - A)^{-1} \end{bmatrix}$$

Knowing that the norm of $(\lambda_d I_{2n} - \mathcal{A}_d)^{-1}$ is greater than or equal to the norm of each of its block components taken individually, we can deduce that

$$\alpha_{\mathcal{A}_d} = \sup_{\Re(\lambda_d)=0} \|(\lambda_d I_{2n} - \mathcal{A}_d)^{-1}\| \geq \sup_{z\in\Gamma_d} \|(zI_n - A)^{-1}\| = \alpha_{A_d}$$

and also

$$\begin{aligned}
\left\|(\lambda_d I_{2n} - \mathcal{A}_d)^{-1}\right\| &\leq \left\| \begin{pmatrix} \sqrt{z+\frac{p}{2}-d}I_n & A+(\frac{p}{2}-d)I_n \\ \\ I_n & \sqrt{z+\frac{p}{2}-d}I_n \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\| \\
&\leq \left\| \begin{pmatrix} \|\sqrt{z+\frac{p}{2}-d}I_n\| & \|A+(\frac{p}{2}-d)I_n\| \\ \\ \|I_n\| & \|\sqrt{z+\frac{p}{2}-d}I_n\| \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\| \\
&\leq \left\| \begin{pmatrix} \sqrt{|z|}+\sqrt{|\frac{p}{2}-d|} & 1 \\ \\ 1 & \sqrt{|z|}+\sqrt{|\frac{p}{2}-d|} \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\| \\
&\leq (\sqrt{|z|}+\sqrt{|\frac{p}{2}-d|}+1) \left\|(zI_n - A)^{-1}\right\|
\end{aligned}$$

- If $|z| \leq \dfrac{\alpha_{A_d}+1}{\alpha_{A_d}}$ then

$$\begin{aligned}
\|(\lambda I_{2n} - \mathcal{A}_d)^{-1}\| &\leq \alpha_{A_d}\left(\sqrt{\frac{\alpha_{A_d}+1}{\alpha_{A_d}}} + \sqrt{\left|\frac{p}{2}-d\right|}+1\right) \\
&\leq \alpha_{A_d}\left(1+\sqrt{\frac{1}{\alpha_{A_d}}}\right) + \sqrt{\alpha_{A_d}}\sqrt{1+\alpha_{A_d}} \\
&\leq \alpha_{A_d} + \sqrt{\alpha_{A_d}}\left(1+\sqrt{\alpha_{A_d}+1}\right)
\end{aligned}$$

- If $|z| > \dfrac{\alpha_{A_d}+1}{\alpha_{A_d}}$ with the conditions (36) we have $\left\|\dfrac{A}{z}\right\| < 1$.

  Which leads to

$$\begin{aligned}
(zI_n - A)^{-1} &= \frac{1}{z}(I_n - \frac{A}{z})^{-1} \\
&= \frac{1}{z}\sum_{k=0}^{+\infty}\frac{A^k}{z^k}
\end{aligned}$$

$$= \frac{1}{z}\left(I_n + \frac{A}{z}\sum_{m=0}^{+\infty}\frac{A^m}{z^m}\right)$$
$$= \frac{1}{z}\left(I_n + \frac{A}{z}(I_n - \frac{A}{z})^{-1}\right)$$

Consequently

$$\|(\lambda_d I_{2n} - \mathcal{A}_d)^{-1}\| \leq \left\|\frac{1}{z}I_n + \frac{1}{z}A\,(zI_n - A)^{-1}\right\| \times \left(1 + \sqrt{|z|} + \sqrt{|\frac{p}{2} - d|}\right)$$

$$\leq \|A(zI_n - A)^{-1} + I_n\| \times \left(\frac{1}{|z|} + \frac{1}{\sqrt{|z|}} + \frac{\sqrt{|\frac{p}{2} - d|}}{|z|}\right)$$

$$\leq \left(\left(1 + \left|\frac{p}{2} - d\right|\right)\alpha_{A_d} + 1\right) \times \left(\frac{\alpha_{A_d}}{\alpha_{A_d} + 1} + \frac{\sqrt{\alpha_{A_d}}}{\sqrt{\alpha_{A_d} + 1}} + \frac{\sqrt{\alpha_{A_d}}}{\alpha_{A_d} + 1}\right)$$

$$\leq \frac{(2 + \alpha_{A_d})}{\alpha_{A_d} + 1}\left(\alpha_{A_d} + \sqrt{\alpha_{A_d}}\sqrt{\alpha_{A_d} + 1} + \sqrt{\alpha_{A_d}}\right)$$

$$\leq 2\left(\alpha_{A_d} + \sqrt{\alpha_{A_d}}\left(1 + \sqrt{\alpha_{A_d} + 1}\right)\right).$$

Consider spectral projectors

- $\mathbb{P}_d \in \mathbb{C}^{n\times n}$ on the right eigensubspace associated with the eigenvalues of A outside the parabola Γ_d
- $\mathcal{P}_d \in \mathbb{C}^{2n\times 2n}$ on the right eigensubspace associated to the eigenvalues of $\mathcal{A}_d$ in the right complex half-plane.

The following proposition characterizes the relation between $\mathbb{P}_d$ and $\mathcal{P}_d$

Proposition 9. *Consider a partition of the matrix $\mathcal{P}_d$ in the form*

$$\mathcal{P}_d = \begin{pmatrix} \mathcal{P}_1^{(d)} & \mathcal{P}_2^{(d)} \\ \mathcal{P}_3^{(d)} & \mathcal{P}_4^{(d)} \end{pmatrix} \qquad \textit{with} \qquad \mathcal{P}_i^{(d)} \in \mathbb{C}^{n\times n}, \qquad i = 1,4 \tag{38}$$

Then

$$\mathbb{P}_d = 2\mathcal{P}_1^{(d)} = 2\mathcal{P}_4^{(d)} = 4\mathcal{P}_2^{(d)}\mathcal{P}_3^{(d)} \tag{39}$$

Moreover

$$\mathbb{P}_d A = 4(\mathcal{P}_2^{(d)})^2 - (p-2d)\mathcal{P}_1^{(d)} \tag{40}$$

Proof.
Let X_d be a solution of the matrix equation

$$\left(X_d + \sqrt{\frac{p}{2}}I_n\right)^2 = A_d. \tag{41}$$

Following the same calculation as in the proof of Proposition 7, we get

$$\mathcal{A}_d = \begin{bmatrix} X_d + \sqrt{\frac{p}{2}}I_n & -X_d - \sqrt{\frac{p}{2}}I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} X_d & 0 \\ 0 & -X_d - 2\sqrt{\frac{p}{2}}I_n \end{bmatrix} \times \begin{bmatrix} \frac{1}{2}(X_d + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n \\ -\frac{1}{2}(X_d + \sqrt{\frac{p}{2}}I_n)^{-1} & \frac{1}{2}I_n \end{bmatrix}$$

Let $X_d = Q_d \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_d^{-1}$ be the canonical Jordan form of the matrix X_d with M_+ and M_- the Jordan blocks associated respectively with the eigenvalues of X_d located in the right half-plane and the left half-plane.

By replacing the decomposition of X_d in the matrix $\mathcal{A}_d$, we get

$$\mathcal{A}_d = \widetilde{\mathcal{Q}}_d \mathcal{M} (\widetilde{\mathcal{Q}}_d)^{-1}$$

with

$$\widetilde{\mathcal{Q}}_d = \begin{bmatrix} Q_d & 0 \\ 0 & Q_d \end{bmatrix} \begin{bmatrix} \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} + \sqrt{\frac{p}{2}}I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}}I_n \\ I_n & I_n \end{bmatrix}$$

and

$$\mathcal{M} = \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - 2\sqrt{\frac{p}{2}}I_n \end{bmatrix}$$

Therefore we can compute the associated projector

$$\mathcal{P}_d = (\widetilde{\mathcal{Q}}_d) \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\mathcal{Q}}_d)^{-1}$$

$$= \begin{bmatrix} Q_d & 0 \\ 0 & Q_d \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{2}\left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2}\left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix} \begin{bmatrix} Q_d^{-1} & 0 \\ 0 & Q_d^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_d & 0 \\ 0 & Q_d \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \\ -\frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_d^{-1} & 0 \\ 0 & Q_d^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_d & 0 \\ 0 & Q_d \end{bmatrix} \begin{bmatrix} \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} \\ \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_d^{-1} & 0 \\ 0 & Q_d^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_d \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} & Q \begin{bmatrix} \frac{1}{2}(M_+ + \sqrt{\frac{p}{2}} I_k) & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} \\ Q_d \begin{bmatrix} \frac{1}{2}(M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} & Q_d \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{P}_1^{(d)} & \mathcal{P}_2^{(d)} \\ \mathcal{P}_3^{(d)} & \mathcal{P}_4^{(d)} \end{bmatrix}$$

It follows that

$$\mathcal{P}_1^{(d)} = Q_d \begin{bmatrix} \frac{1}{2}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} = \frac{1}{2}\mathbb{P}_d$$

$$\left.\begin{array}{l} \mathcal{P}_2^{(d)} = Q_d \frac{1}{2} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} \\ \mathcal{P}_3^{(d)} = Q_d \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}}I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} \end{array}\right\} \Longrightarrow \mathbb{P}_d = 4\mathcal{P}_2^{(d)}\mathcal{P}_3^{(d)}$$

$$\mathcal{P}_4^{(d)} = Q_d \begin{bmatrix} \frac{1}{2}I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} = \frac{1}{2}\mathbb{P}_d$$

With $X_d = Q \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_d^{-1}$ we have

$$A = A_d - \left(\frac{p}{2} - d\right) I_n$$

$$= Q_d \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - \left(\frac{p}{2} - d\right) I_k & 0 \\ 0 & \left(M_- + \sqrt{\frac{p}{2}}I_{n-k}\right)^2 - \left(\frac{p}{2} - d\right) I_{n-k} \end{bmatrix} Q^{-1}$$

and

$$\mathbb{P}_d A = Q \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1} \times Q \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - (\frac{p}{2} - d)I_k & 0 \\ 0 & \left(M_- + \sqrt{\frac{p}{2}}I_{n-k}\right)^2 - (\frac{p}{2} - d)I_{n-k} \end{bmatrix} Q_d^{-1}$$

$$= Q_d \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}}I_k\right)^2 - (\frac{p}{2} - d)I_k & 0 \\ 0 & 0 \end{bmatrix} Q_d^{-1}$$

$$= 4(\mathcal{P}_2^{(d)})^2 - (p - 2d)\mathcal{P}_1^{(d)}$$

Hence (39) and (40).

Remark 5. *If the parameter $d = \frac{p}{2}$, whence equalities (40) are reduced to those of equalities (23).*

Algorithm 6 (DichoPd)**.**

- *Input variables: the matrices A , I_n and the real numbers d and p such that the matrix pencil $zI_n - A$ has no eigenvalues on the parabola with equation $2p\,(d-x) = y^2$ with $p > 0$ et $d > 0$*
- *Output variables: $\mathbb{P}_d$ and $\mathbb{H}_d$.*
  $\mathbb{P}_d$ being the projector on the right subspace of $zI_n - A$ associated with the eigenvalues outside the parabola and $\mathbb{H}_d$ the matrix whose norm defines the dichotomy criterion.

1. *Determine the matrix*

$$\mathcal{A}_d = \begin{pmatrix} -\sqrt{\frac{p}{2}}I_n & A + (\frac{p}{2} - d)I_n \\ \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{pmatrix}$$

2. *Using Algorithm 3 to $\lambda_d I_{2n} - \mathcal{A}_d$, compute the Projector $\mathcal{P}_d$ onto the right eigenspace of $\mathcal{A}_d$ associted with the eigenvalues on the right half-plane of the complex plane and the matrix $\mathbb{H}_d$.*
3. *If $\|\mathbb{H}_d\|$ is not large, determine the projector $\mathbb{P}_d$ by*

$$\mathbb{P}_d = 2\mathcal{P}_1^{(d)}.$$

3.2.2. The spectral dichotomy method with the coefficient $b \neq 0$

Consider the set

$$\widetilde{\Gamma}_d = \left\{ z = x + iy / x + \left(\frac{p}{2} - d\right) + i(y - pb) \in \Gamma \right\}$$

described by the following equation (33).

We consider the following order matrix $2n$

$$\widetilde{\mathcal{A}}_d = \begin{bmatrix} -\sqrt{\frac{p}{2}}I_n & A_{db} \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{bmatrix} \quad \text{with} \quad A_{db} = A + \left(\frac{p}{2} - d - ipb\right) I_n.$$

The respective eigenvalues $\widetilde{\lambda}_d$ and z_{db} of the matrices $\widetilde{\mathcal{A}}_d$ and A_{db} verify the relationship

$$z_{db} = \left(\sqrt{\frac{p}{2}} + \widetilde{\lambda}_d\right)^2.$$

This leads to the following remark

Remark 6. *The respective eigenvalues $\widetilde{\lambda}_d$ and z of the matrices $\widetilde{\mathcal{A}}_d$ and A satisfy the relation*

$$z = \left(\sqrt{\frac{p}{2}} + \widetilde{\lambda}_d\right)^2 - \left(\frac{p}{2} - d - ipb\right)$$

Furthermore, we get

$$\begin{cases} x &= \left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}}\right)^2 - \Im(\widetilde{\lambda}_d)^2 + \frac{p}{2} - d \\ \\ y &= 2\left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}}\right)\Im(\widetilde{\lambda}_d) - pb \end{cases}$$

Thus

$$\begin{aligned} y^2 &= 4\left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}}\right)^2 \Im(\widetilde{\lambda}_d)^2 - 4\left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}}\right)\Im(\widetilde{\lambda}_{db})pb + p^2b^2 \\ &= \left[\left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}}\right)^2 - x + \sqrt{\frac{p}{2}} - d - ipb\right] \end{aligned}$$

By setting

$$\widetilde{p}_d = 2\left(\Re(\widetilde{\lambda}_d) + \sqrt{\frac{p}{2}} - pb\right)^2$$

we have

$$y^2 = 2\widetilde{p}_{db}\left(\frac{\widetilde{p}_{db}}{2} - x - \frac{p}{2} + db\right) = 2\widetilde{p}_{db}\left(\frac{\widetilde{p}_{db}}{2} - \widetilde{x}\right)$$

We also assume that $\|A_{db}\| = 1$. Otherwise (if $\|A_{db}\| \neq 1$), we can take

$$A^1_{db} = \frac{1}{\|A_{db}\|}A_{db} \qquad \text{and} \qquad \widetilde{p}^1_{db} = \frac{1}{\|A_{db}\|}.$$

Consider the numerical parameters $\alpha_{\widetilde{\mathcal{A}}_d}$ and $\alpha_{A_{db}}$ defined by

$$\alpha_{\widetilde{\mathcal{A}}_d} = \sup_{\Re(\widetilde{\lambda}_d)=0} \|(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1}\| \text{ and } \alpha_{A_{db}} = \sup_{z\in\widetilde{\Gamma}_d} \|(zI_n - A)^{-1}\| \tag{42}$$

The following proposition gives a relation between the parameters $\alpha_{\widetilde{\mathcal{A}}_d}$ and $\alpha_{A_{db}}$.

Proposition 10. *Let $\alpha_{\widetilde{\mathcal{A}}_d}$ and $\alpha_{A_{db}}$ be the two parameters defined in* (46)*. Assume that*

$$\|A_{db}\| = 1 \qquad \textit{and} \qquad \left|\frac{p}{2} - d - ipb\right| < \frac{1}{\alpha_{db}} \tag{43}$$

Then

$$\alpha_{A_{db}} \leq \alpha_{\widetilde{\mathcal{A}}_d} \leq 2\left(\alpha_{A_{db}} + \sqrt{\alpha_{A_{db}}}\left(\sqrt{1+\alpha_{A_{db}}} + 1\right)\right). \tag{44}$$

Proof. Let the matrix

$$\left(\widetilde{\lambda} I_{2n} - \widetilde{\mathcal{A}}_d\right) = \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & -A_{db} \\ \\ -I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

We have

$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & A_{db} \\ \\ I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n! \end{bmatrix} \times \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & -A_{db} \\ \\ -I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix} =$$
$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - A_{db} & 0 \\ \\ 0 & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - A_{db} \end{bmatrix}$$

With

$$(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1} = \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - A_{db} & 0 \\ \\ 0 & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - A_{db} \end{bmatrix}^{-1} \times$$
$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & A_{db} \\ \\ I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - (A + (\frac{p}{2} - d - ipb)I_n) & 0 \\ \\ 0 & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 I_n - (A + (\frac{p}{2} - d - ipb)I_n) \end{bmatrix}^{-1} \times$$
$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & (A + (\frac{p}{2} - d - ipb)I_n) \\ \\ I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} ((\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d - ipb)I_n - A & 0 \\ \\ 0 & ((\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})^2 - \frac{p}{2} + d - ipb)I_n - A \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n & A + (\frac{p}{2} - d - ipb)I_n) \\ \\ I_n & (\widetilde{\lambda}_d + \sqrt{\frac{p}{2}})I_n \end{bmatrix}$$

$$= \begin{bmatrix} (zI_n - A)^{-1} & 0 \\ \\ 0 & (zI_n - A)^{-1} \end{bmatrix} \times \begin{bmatrix} \sqrt{z + \frac{p}{2} - d - ipb} I_n & A + (\frac{p}{2} - d - ipb)I_n \\ \\ I_n & \sqrt{z + \frac{p}{2} - d - ipb} I_n \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{z + \frac{p}{2} - d - ipb}(zI_n - A)^{-1} & (zI_n - A)^{-1}(A + (\frac{p}{2} - d - ipb)I_n) \\ \\ (zI_n - A)^{-1} & \sqrt{z + \frac{p}{2} - d - ipb}(zI_n - A)^{-1} \end{bmatrix}$$

Knowing that the norm of $(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1}$ is greater than or equal to the norm of each of its block components taken individually, we can deduce that

$$\alpha_{\widetilde{\mathcal{A}}_d} = \sup_{\Re(\widetilde{\lambda}_d)=0} \|(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1}\| \geq \sup_{z \in \widetilde{\Gamma}_d} \|(zI_n - A)^{-1}\| = \alpha_{A_{db}}$$

and also

$$\left\|(\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d)^{-1}\right\| \leq \left\| \begin{pmatrix} \sqrt{z + \frac{p}{2} - d - ipb} I_n & A + (\frac{p}{2} - d - ipb)I_n \\ \\ I_n & \sqrt{z + \frac{p}{2} - d - ipb} I_n \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\|$$

$$\leq \left\| \begin{pmatrix} \|\sqrt{z + \frac{p}{2} - d - ipb} I_n\| & \|A + (\frac{p}{2} - d - ipb)I_n\| \\ \\ \|I_n\| & \|\sqrt{z + \frac{p}{2} - d - ipb} I_n\| \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\|$$

$$\leq \left\| \begin{pmatrix} \sqrt{|z|} + \sqrt{|\frac{p}{2} - d - ipb|} & 1 \\ \\ 1 & \sqrt{|z|} + \sqrt{|\frac{p}{2} - d - ipb|} \end{pmatrix} \right\| \left\|(zI_n - A)^{-1}\right\|$$

$$\leq \left(\sqrt{|z|} + \sqrt{|\frac{p}{2} - d - ipb|} + 1 \right) \left\|(zI_n - A)^{-1}\right\|$$

- If $|z| \leq \dfrac{\alpha_{A_{db}}+1}{\alpha_{A_{db}}}$ then

$$\begin{aligned}
\|(\lambda I_{2n}-\widetilde{\mathcal{A}}_d)^{-1}\| &\leq \alpha_{A_{db}}(\sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ipb|}+1)\\
&\leq \alpha_{A_{db}}\left(\sqrt{\frac{\alpha_{A_{db}}+1}{\alpha_{A_{db}}}}+\frac{1}{\sqrt{\alpha}}+1\right)\\
&\leq \alpha_{A_{db}}+\sqrt{\alpha_{db}}(\sqrt{\alpha_{db}+1}+1)
\end{aligned}$$

- If $|z| > \dfrac{\alpha_{A_{db}}+1}{\alpha_{A_{db}}}$ with the conditions (43) we have $\left\|\dfrac{A}{z}\right\| < 1.$ we note that

$$\begin{aligned}
\left\|\frac{A}{z}\right\| &< \frac{\alpha_{db}}{\alpha_{db}+1}\left(\|A_{db}\|+|\frac{p}{2}-d-ipb|\right)\\
&< \frac{\alpha_{db}}{\alpha_{db}+1}\left(1+\frac{1}{\alpha_{db}}\right)\\
&< 1
\end{aligned}$$

Which leads to

$$\begin{aligned}
(zI_n-A)^{-1} &= \frac{1}{z}(I_n-\frac{A}{z})^{-1}\\
&= \frac{1}{z}\left(I_n+\sum_{k=1}^{+\infty}\frac{A^k}{z^k}\right)\\
&= \frac{1}{z}\left(I_n+\frac{A}{z}\sum_{m=0}^{+\infty}\frac{A^m}{z^m}\right)\\
&= \frac{1}{z}\left(I_n+\frac{A}{z}(I_n-\frac{A}{z})^{-1}\right)
\end{aligned}$$

Consequently

$$\|(\widetilde{\lambda}_d I_{2n}-\widetilde{\mathcal{A}}_d)^{-1}\| \leq \left\|\frac{1}{z}I_n+\frac{1}{z}A\,(zI_n-A)^{-1}\right\| \times \left\|1+\sqrt{|z|}+\sqrt{\left|\frac{p}{2}-d-ipb\right|}\right\|$$

$$\leq (\|A(zI_n-A)^{-1}\|+1)\frac{1+\sqrt{|z|}+\sqrt{|\frac{p}{2}-d-ipb|}}{|z|}$$

$$\leq \left(\left(1+|\frac{p}{2}-p-ipb|\right)\alpha_{A_{db}}+1\right)\left(\frac{1}{\sqrt{|z|}}+\frac{1}{|z|}+\frac{\sqrt{|\frac{p}{2}-d-ipb|}}{|z|}\right)$$

$$\leq \left(\left(1+\frac{1}{\alpha_{A_{db}}}\right)+1\right)\left(\frac{\sqrt{\alpha_{A_{db}}}}{\sqrt{\alpha_{A_{db}}+1}}+\frac{\alpha_{A_{db}}}{\alpha_{A_{db}}+1}+\frac{\sqrt{\alpha_{A_{db}}}}{\alpha_{A_{db}+1}}\right)$$

$$\leq \left(\frac{\alpha_{A_{db}}+2}{\alpha_{A_{db}}+1}\right)\left(\sqrt{\alpha_{A_{db}}}\sqrt{\alpha_{A_{db}}+1}+\sqrt{\alpha_{A_{db}}}+\alpha_{db}\right)$$
$$\leq 2\left(\alpha_{A_{db}}+\sqrt{\alpha_{A_{db}}}\left(\sqrt{\alpha_{A_{db}+1}}+1\right)\right)$$

Finally

$$\alpha_{A_{db}} \leq \alpha_{\mathcal{A}_{db}} \leq 2\left(\alpha_{A_{db}}+\sqrt{\alpha_{A_{db}}}\left(\sqrt{\alpha_{A_{db}+1}}+1\right)\right).$$

Consider the projector

- $\widetilde{\mathbb{P}_d} \in \mathbb{C}^{n\times n}$ on right eigenspace of A_{db} associated with eigenvalues outside the parabola $\widetilde{\Gamma_d}$
- $\widetilde{\mathcal{P}}_d \in \mathbb{C}^{2n\times 2n}$ on right eigenspace of $\widetilde{\mathcal{A}}_d$ associated with eigenvalues in the right complex half-plane.

The following proposition characterizes the relation between $\widetilde{\mathbb{P}_d}$ and $\widetilde{\mathcal{P}}_d$

Proposition 11. *Consider a partition of the matrix $\widetilde{\mathcal{P}}_d$ in the form*

$$\widetilde{\mathcal{P}}_d = \begin{pmatrix} \widetilde{\mathcal{P}}_1^{(d)} & \widetilde{\mathcal{P}}_2^{(d)} \\ \widetilde{\mathcal{P}}_3^{(d)} & \widetilde{\mathcal{P}}_4^{(d)} \end{pmatrix} \qquad \text{with} \qquad \widetilde{\mathcal{P}}_i^{(d)} \in \mathbb{C}^{n\times n}, \qquad i=1,4 \tag{45}$$

Then

$$\widetilde{\mathbb{P}_d} = 2\widetilde{\mathcal{P}}_1^{(d)} = 2\widetilde{\mathcal{P}}_4^{(d)} = 4\widetilde{\mathcal{P}}_2^{(d)}\widetilde{\mathcal{P}}_3^{(d)} \tag{46}$$

Moreover

$$\widetilde{\mathbb{P}_d}A = 4(\widetilde{\mathcal{P}}_2^{(d)})^2 - (p-2d-2ipb)\widetilde{\mathcal{P}}_1^{(d)} \tag{47}$$

Proof.
Let $\widetilde{X_d}$ be a solution of the matrix equation

$$\left(\widetilde{X_d}+\sqrt{\frac{p}{2}}I_n\right)^2 = A_{db}. \tag{48}$$

Following the same calculation as in the proof of Proposition 7, we get

$$\widetilde{\mathcal{A}_d} = \begin{bmatrix} \widetilde{X_d} + \sqrt{\frac{p}{2}} I_n & -\widetilde{X_d} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \widetilde{X_d} & 0 \\ 0 & -\widetilde{X_d} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix} \times \begin{bmatrix} \frac{1}{2}(\widetilde{X_d} + \sqrt{\frac{p}{2}} I_n)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2}(\widetilde{X_d} + \sqrt{\frac{p}{2}} I_n)^{-1} & \frac{1}{2} I_n \end{bmatrix}$$

Let $\widetilde{X_d} = Q_{db} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_{db}^{-1}$ be the canonical Jordan form of the matrix $\widetilde{X_d}$ with M_+ and M_- the Jordan blocks associated respectively with the eigenvalues of $\widetilde{X_d}$ located in the right half-plane and the left half-plane.

By replacing the decomposition of $\widetilde{X_d}$ in the matrix $\widetilde{\mathcal{A}}_d$, we get

$$\widetilde{\mathcal{A}}_d = \widetilde{\mathcal{Q}}_{db} \mathcal{M} (\widetilde{\mathcal{Q}}_{db})^{-1}$$

with

$$\widetilde{\mathcal{Q}_{db}} = \begin{bmatrix} Q_{db} & 0 \\ 0 & Q_{db} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix}$$

et

$$\mathcal{M} = \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - 2\sqrt{\frac{p}{2}} I_n \end{bmatrix}$$

We can therefore calculate the associated projector

$$\begin{aligned}
\widetilde{\mathcal{P}}_d &= (\widetilde{\mathcal{Q}}_{db}) \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\mathcal{Q}}_{db})^{-1} \\
&= \begin{bmatrix} Q_{db} & 0 \\ 0 & Q_{db} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n & -\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \\ I_n & I_n \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2} \left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \\ -\frac{1}{2} \left(\begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} - \sqrt{\frac{p}{2}} I_n \right)^{-1} & \frac{1}{2} I_n \end{bmatrix} \begin{bmatrix} Q_{db}^{-1} & 0 \\ 0 & Q_{db}^{-1} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} Q_{db} & 0 \\ 0 & Q_{db} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \\ -\frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & (M_- + \sqrt{\frac{p}{2}} I_{n-k})^{-1} \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_{db}^{-1} & 0 \\ 0 & Q_{db}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{db} & 0 \\ 0 & Q_{db} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} \\ \frac{1}{2} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \frac{1}{2} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} Q_{db}^{-1} & 0 \\ 0 & Q_{db}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{db} \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} & Q_{db} \begin{bmatrix} \frac{1}{2}(M_+ + \sqrt{\frac{p}{2}} I_k) & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} \\ Q_{db} \begin{bmatrix} \frac{1}{2}(M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} & Q_{db} \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{\mathcal{P}}_1^{(d)} & \widetilde{\mathcal{P}}_2^{(d)} \\ \widetilde{\mathcal{P}}_3^{(d)} & \widetilde{\mathcal{P}}_4^{(d)} \end{bmatrix}$$

It follows that

$$\widetilde{\mathcal{P}}_1^{(d)} = Q_d \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} = \frac{1}{2} \widetilde{\mathbb{P}_d}$$

$$\widetilde{\mathcal{P}}_2^{(d)} = \frac{1}{2} Q_{db} \begin{bmatrix} M_+ + \sqrt{\frac{p}{2}} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1}$$

$$\widetilde{\mathcal{P}}_3^{(d)} = \frac{1}{2} Q_{db} \begin{bmatrix} (M_+ + \sqrt{\frac{p}{2}} I_k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1}$$

$$\widetilde{\mathcal{P}}_4^{(d)} = Q_{db} \begin{bmatrix} \frac{1}{2} I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} = \frac{1}{2} \widetilde{\mathbb{P}_d}$$

With $\widetilde{X_d} = Q_{db} \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} Q_{db}^{-1}$ we have

$$\begin{aligned} A &= A_{db} - \left(\frac{p}{2} - d - ipb\right) I_n \\ &= Q_d \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}} I_k\right)^2 - \left(\frac{p}{2} - d - ipb\right) I_k & 0 \\ 0 & \left(M_- + \sqrt{\frac{p}{2}} I_{n-k}\right)^2 - \left(\frac{p}{2} - d - ipb\right) I_{n-k} \end{bmatrix} Q^{-1} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathbb{P}_d} A &= Q_{db} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}} I_k\right)^2 - (\frac{p}{2} - d - ipb) I_k & 0 \\ 0 & \left(M_- + \sqrt{\frac{p}{2}} I_{n-k}\right)^2 - (\frac{p}{2} - d - ipb) I_{n-k} \end{bmatrix} Q_{db}^{-1} \\ &= Q_{db} \begin{bmatrix} \left(M_+ + \sqrt{\frac{p}{2}} I_k\right)^2 - (\frac{p}{2} - d - ipb) I_k & 0 \\ 0 & 0 \end{bmatrix} Q_{db}^{-1} \\ &= 4(\widetilde{\mathcal{P}}_2^{(d)})^2 - (p - 2d - 2ipb) \widetilde{\mathcal{P}}_1^{(db)} \end{aligned}$$

Hence (46) and (47).

Remark 7. *We note that :*

- *if the parameter* $d = \frac{p}{2}$*, whence equalities (47) are reduced to those of equalities (31).*
- *if the parameter* $b = 0$*, whence equalities (47) are reduced to those of equalities (40).*
- *if the parameters* $b = 0, d = \frac{p}{2}$*, whence equalities (47) are reduced to those of equalities (23).*

Algorithm 7 (DichoPdb)**.**

- *Input variables: the matrices A, I_n and the real numbers b, d and p such that the matrix pencil $zI_n - A$ has no eigenvalues on the parabola with equation $2p(d-x) = (y-ipb)^2$ with $p>0$ et $d>0$*
- *Output variables: $\widetilde{\mathbb{P}_d}$, $\widetilde{\mathbb{H}_d}$ and $b \neq 0$.*
  $\widetilde{\mathbb{P}_d}$ being the projector on the right subspace of $zI_n - A$ associated with the eigenvalues outside the parabola and $\widetilde{\mathbb{H}_d}$ the matrix whose norm defines the dichotomy criterion.

1. *Determine the matrix*

$$\widetilde{\mathcal{A}}_d = \begin{pmatrix} -\sqrt{\frac{p}{2}}I_n & A + (\frac{p}{2} - d - ipb)I_n \\ \\ I_n & -\sqrt{\frac{p}{2}}I_n \end{pmatrix}$$

2. *Using Algorithm 3 to $\widetilde{\lambda}_d I_{2n} - \widetilde{\mathcal{A}}_d$, compute the Projector $\widetilde{\mathcal{P}}_d$ onto the right eigenspace of $\widetilde{\mathcal{A}}_d$ associted with the eigenvalues on the right half-plane of the complex plane and the matrix $\widetilde{\mathbb{H}_d}$.*
3. *If $\|\widetilde{\mathbb{H}_d}\|$ is not large, determine the projector $\widetilde{\mathbb{P}_d}$ by*

$$\widetilde{\mathbb{P}_d} = 2\widetilde{\mathcal{P}}_1^{(d)}.$$

4. Numerical experiments

In this section, we illustrate numerical examples using a matrix function from [2, 4, 5] on which we apply the algorithms 4, 5, 6 and 7 for positive parameters p, b and d given.

$$W(t) = \begin{bmatrix} -(A(s(t)))^{-T}\cos(w(t)) & -(A(s(t)))^{-T}\sin(w(t)) \\ \\ A(s(t))\sin(w(t)) & (A(s(t)))^{-T}\cos(w(t)) \end{bmatrix} \tag{49}$$

avec

$$A(s) = \begin{pmatrix} 1-s^2 & -1 \\ s^2 & 1-s^2 \end{pmatrix}, \quad w(t) = \pi\left(\frac{1}{2} - \frac{1}{3}\sin(3t)\right) \qquad \text{et} \quad s(t) = 4\sin(3t)$$

- Applying Algorithm 4 to matrix function (49) gives the following results:
  - At $t = 3.5$,
    Those different graphs on Figure 1 show how a parabola Γ can realise a dichotomy on the eigenvalues of a given matrix. We have three possibilities: when all the eigenvalues are in the interior,then the computed projector is the null matrix. When all the eigenvalues are at the exterior of the parabola, the

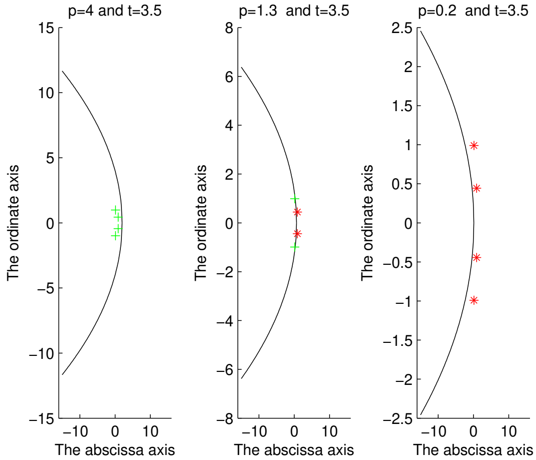

Figure 1: Partition of the spectrum of the matrix $W(t)$ for $t = 3.5$ by parabolas of equation $2p(\frac{p}{2} - x) = y^2$.

Table 1: Traces, norms and quality of spectral projectors $\mathbb{P}$ by applying the DichoP algorithm for three values of p

p	$tr(\mathbb{P})$	$\|\mathbb{P}\|$	$\|\mathbb{P}^2 - \mathbb{P}\|$	$\|\mathbb{P}W(t) - W(t)\mathbb{P}\|$	$\|\mathbb{H}\|$
4	0	0	0	0	4.0502
1.3	2	1.6305	$2.3747\ 10^{-15}$	$5.6077\ 10^{-15}$	63.1478
0.2	4	1	$2.0540\ 10^{-15}$	$5.3639\ 10^{-15}$	6.2228

computed projector is the identity matrix. A part of the eigenvalues can be in the interior of the parabola and another part of the eigenvalues can be at the exterior. In this case, the projector is different of the null matrix and the identity matrix. In the above table of values, the trace $tr(\mathbb{P})$ of $\mathbb{P}$ denotes the number of eigenvalues outside of the parabola. Moreover, the values of $\|\mathbb{P}\|$ confirm what was said above. The computing of $\|\mathbb{P}^2 - \mathbb{P}\|$ and $\|\mathbb{P}W(t) - W(t)\mathbb{P}\|$ prove that $\mathbb{P}$ is a projector and the values obtained for $\|\mathbb{H}\|$ show the good quality of the dichotomy. This shows the effectiveness of the method.

· The partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with the parameters $p \in \{0.5, 1, 2, 4\}$ gives us Figure 2.

Those graphs describe the spectral portrait of $W(t)$,$\forall t \in [0; 2\pi]$

This spectrum dichotomy realised by the parabola Γ is illustrated with colors (the green color for the inside eigenvalues and the red color for the outside eigenvalues).

- Applying Algorithm 5 to matrix function 49 gives the following results:

  · At $t = 6$,

  Those different graphs on Figure 3 show how a parabola $\widetilde{\Gamma}$ can realise a dichotomy on the eigenvalues of a given matrix. Similarly to the case seen for

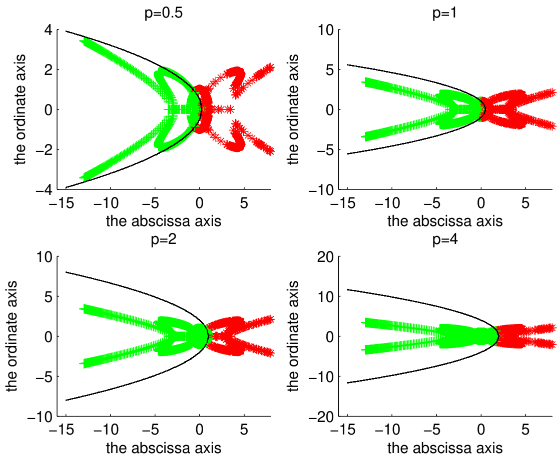

Figure 2: Partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ for $p \in \{0.5, 1, 2, 4\}$.

Table 2: Traces, norms and quality of spectral projectors $\widetilde{\mathbb{P}}$ by applying the DichoPb algorithm for differents values of p and b

p	b	$tr(\widetilde{\mathbb{P}})$	$\|\|\widetilde{\mathbb{P}}\|\|$	$\|\|\widetilde{\mathbb{P}}^2 - \widetilde{\mathbb{P}}\|\|$	$\|\|\widetilde{\mathbb{P}}W(t) - W(t)\widetilde{\mathbb{P}}\|\|$	$\|\|\widetilde{\mathbb{H}}\|\|$
2	1	3	1.0268	$1.4726\ 10^{-15}$	$2.2659\ 10^{-15}$	3.9802
2	3	4	1	$2.6170\ 10^{-15}$	$3.9255\ 10^{-15}$	0.9799
2	0.1	0	$4.9838\ 10^{-16}$	$4.9838\ 10^{-16}$	$8.0143\ 10^{-16}$	29.4351
0.1	0.5	4	1	$2.8478\ 10^{-15}$	$5.1651\ 10^{-15}$	1.1260
1	0.5	3	1.6455	$2.6392\ 10^{-15}$	$2.6392\ 10^{-15}$	7.7339
4	0.5	0	$4.8120\ 10^{-16}$	$4.8120\ 10^{-16}$	$6.0288\ 10^{-16}$	3.7896

the parabola Γ, the values obtained for $||\widetilde{\mathbb{P}}^2 - \widetilde{\mathbb{P}}||$ and $||\widetilde{\mathbb{P}}W(t) - W(t)\widetilde{\mathbb{P}}||$ proved that $\widetilde{\mathbb{P}}$ is a projector. Moreover, the values of $||\widetilde{\mathbb{H}}||$ show the good quality of the dichotomy.

· The partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with the parameters $(p, b) \in \{(0.5, 2),\ (1, -2), (2, 0), (4, 3)\}$ gives us Figure 4. This shows the effectiveness of the method.

Those graphs describe the spectral portrait of $W(t)$,$\forall t \in [0; 2\pi]$. This spectrum dichotomy realised by the parabola $\widetilde{\Gamma}$ is illustrated with colors (the green color for the inside eigenvalues and the red color for the outside eigenvalues).

- Applying Algorithm 6 to matrix function 49 gives the following results:

  · At $t = 2\pi$,

  Those different graphs on Figure 5 show how a parabola Γ_d can realise a dichotomy on the eigenvalues of a given matrix.
  Similarly to the case seen for the parabola Γ, the values obtained for $||\mathbb{P}_d^2 - \mathbb{P}_d||$

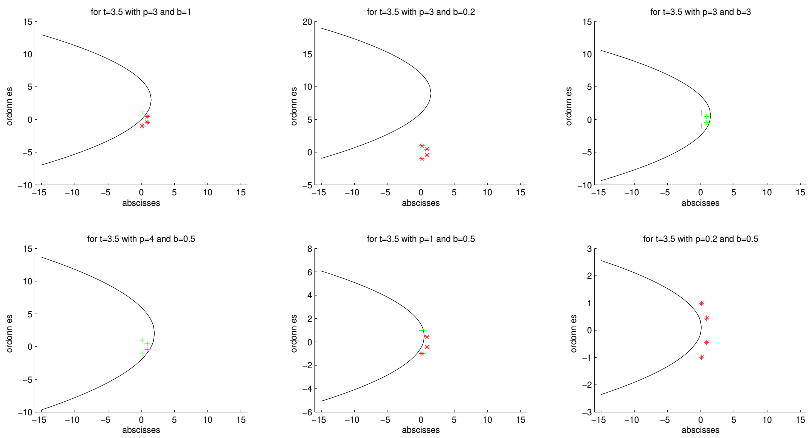

Figure 3: Partition of the spectrum of the matrix $W(t)$ for $t = 6$ by parabolas of equation $2p(\frac{p}{2}-x) = (y-ipb)^2$.

Table 3: Traces, norms and quality of spectral projectors $\mathbb{P}_d$ by applying the DichoPd algorithm for differents values of p and d

p	d	$tr(\mathbb{P}_d)$	$\|\mathbb{P}_d\|$	$\|\mathbb{P}_d^2 - \mathbb{P}_d\|$	$\|\mathbb{P}_d W(t) - W(t)\mathbb{P}_d\|$	$\|\mathbb{H}_d\|$
0.1	5	0	$1.9703\ 10^{-17}$	$1.9703\ 10^{-17}$	$4.1735\ 10^{-17}$	37.0029
0.1	2	4	1	$1.3784\ 10^{-15}$	$1.9182\ 10^{-15}$	11.0360
0.1	3.5	2	1	$2.8954\ 10^{-15}$	$5.0227\ 10^{-15}$	96.0484
0.15	1.5	4	1	$2.1579\ 10^{-15}$	$3.6774\ 10^{-15}$	15.1757
0.25	1.5	2	1	$4.4977\ 10^{-16}$	$2.0476\ 10^{-15}$	9.6088
2	1.5	0	$6.2936\ 10^{-17}$	$6.2936\ 10^{-17}$	$1.0107\ 10^{-16}$	1.1828

and $\|\mathbb{P}_d W - W\mathbb{P}_d\|$ proved that $\mathbb{P}_d$ is a projector. Moreover, the values of $\|\mathbb{H}_d\|$ show the good quality of the dichotomy.

· The partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with the parameters $(p, d) \in \{(0.5, 0.5), (2, 1), (0.2, 2), (0.2, 7)\}$ gives us Figure 6. This shows the effectiveness of the method.

Those graphs describe the spectral portrait of $W(t)$,$\forall t \in [0; 2\pi]$ This spectrum dichotomy realised by the parabola Γ_d is illustrated with colors (the green color for the inside eigenvalues and the red color for the outside eigenvalues).

- Applying Algorithm 7 to matrix function 49 gives the following results:

· At $t = 3.5$,

Those different graphs on Figure 7 show how a parabola $\widetilde{\Gamma}_d$ can realise a dichotomy on the eigenvalues of a given matrix.

Similarly to the case seen for the parabola Γ, the values obtained for $\|\widetilde{\mathbb{P}_d}^2 - \widetilde{\mathbb{P}_d}\|$ and $\|\widetilde{\mathbb{P}_d}W(t) - W(t)\widetilde{\mathbb{P}_d}\|$ proved that $\widetilde{\mathbb{P}_d}$ is a projector. Moreover, the values of $\|\widetilde{\mathbb{H}_d}\|$ show the good quality of the dichotomy.

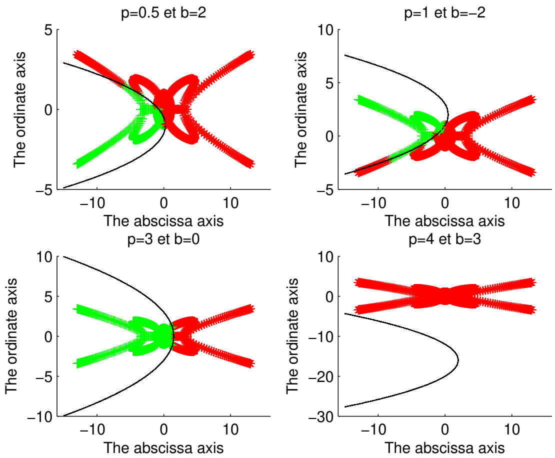

Figure 4: Partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with $(p, b) \in \{(0.5, 2), (1, -2), (2, 0), (4, 3)\}$.

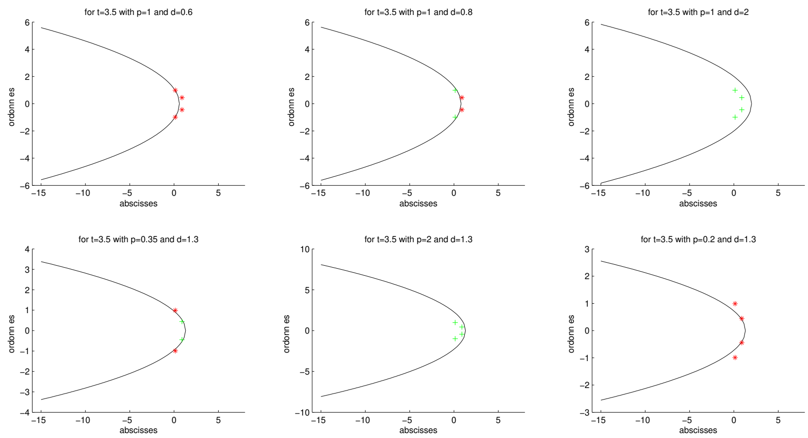

Figure 5: Partition of the spectrum of the matrix $W(t)$ for $t = 2\pi$ by parabolas of equation $2p(d - x) = y^2$.

· The partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with the parameters $(p, d, b) \in \{(0.5, 0.5, 2), (2, 1, -2), (0.2, 2, 0), (0.2, 7, 3)\}$ gives us Figure 8. This shows the effectiveness of the method.

Those graphs describe the spectral portrait of $W(t)$,$\forall t \in [0; 2\pi]$ This spectrum dichotomy realised by the parabola $\widetilde{\Gamma}_d$ is illustrated with colors (the green color for the inside eigenvalues and the red color for the outside eigenvalues).

5. Conclusion

In this work, we proposed methods of spectral dichotomies of a matrix with respect to the general equation $x = ay^2 + by + c$ wth $a \neq 0$ of a parabola $\Gamma_{a,b,c}$. These methods are modifications of the Algorithm proposed by A. N. Malyshev and M. Sadkane in [13]. In

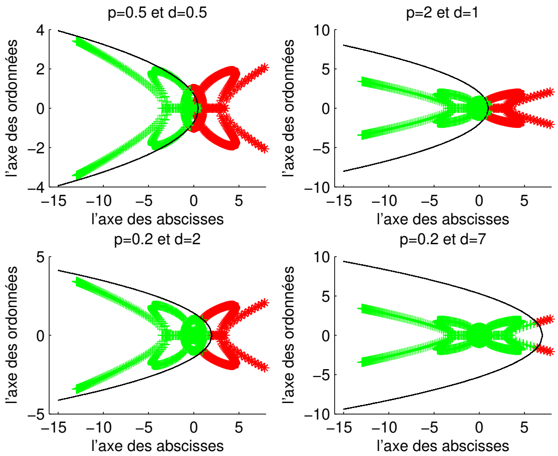

Figure 6: Partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with $(p, d) \in \{(0.5, 0.5), (2, 1), (0.2, 2), (0.2, 7)\}$.

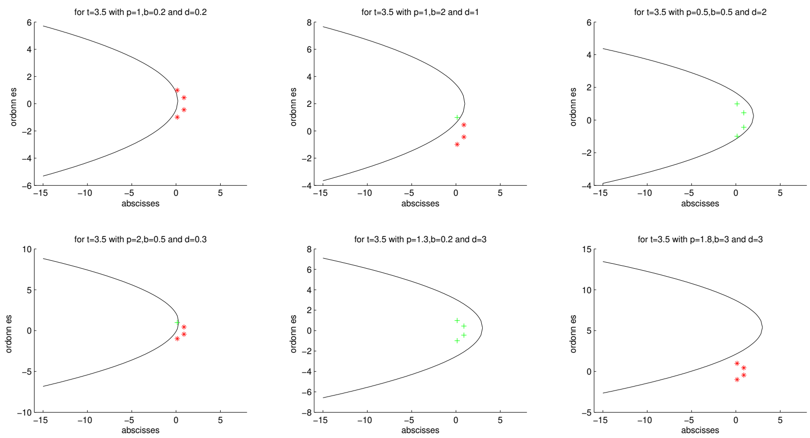

Figure 7: Partition of the spectrum of the matrix $W(t)$ for $t = 2.8$ by parabolas of equation $2p(d-x) = (y-pb)^2$.

this study, if the discriminant is equal to 1 in the canonical form of the general equation with non-zero parameter b, the matrix A is replaced by the matrix $A_b = A - ipbI_n$ in the algorithm. Moreover, if the discriminant is different from 1, we replace the matrix A by $A_d = A + \left(\frac{p}{2} - d\right) I_n$ (respectively $A_{db} = A + \left(\frac{p}{2} - d - ipb\right) I_n$) in the DichoP algorithm when the parameter b is zero (respectively b is not zero).

A theoretical analysis of the proposed new methods shows how to calculate the projector spectral associated with the eigenvalues outside at a given parabola $\Gamma_(a, b, c)$. An analysis of the new method shows how to extract the projector.

Thus, In the numerical experiments, the application of the four algorithms DichoP, DichoPb, DichoPd and DichoPdb to a matrix function shows the efficient calculation of the projector which allows a separation of its spectrum into two parts with respect to the parabola (i. e. inside and outside the parabola).

Table 4: Traces, norms and quality of spectral projectors $\widetilde{\mathbb{P}_d}$ by applying the DichoPdb algorithm for different values of p b and d

p	b	d	$tr(\widetilde{\mathbb{P}_d})$	$\|\widetilde{\mathbb{P}_d}\|$	$\|\widetilde{\mathbb{P}_d}^2 - \widetilde{\mathbb{P}_d}\|$	$\|\widetilde{\mathbb{P}_d}W(t) - W(t)\widetilde{\mathbb{P}_d}\|$	$\|\widetilde{\mathbb{H}_d}\|$
1	0.2	0.2	2	1.3292	$3.3183\ 10^{-15}$	$7.2032\ 10^{-15}$	7.3681
1	4	1	4	1	$3.8213\ 10^{-15}$	$5.3470\ 10^{-15}$	1.3453
0.5	0.5	2	0	$3.2139\ 10^{-17}$	$3.2139\ 10^{-17}$	$4.7630\ 10^{-16}$	9.9291
2	0.5	0.3	1	1.3499	$1.7537\ 10^{-15}$	$5.6241\ 10^{-15}$	4.3035
1.3	0.2	3	0	$3.6050\ 10^{-17}$	$3.6050\ 10^{-17}$	$6.0976\ 10^{-17}$	1.3319
3	5	3	4	1	$2.6922\ 10^{-15}$	$5.4784\ 10^{-15}$	1.1007

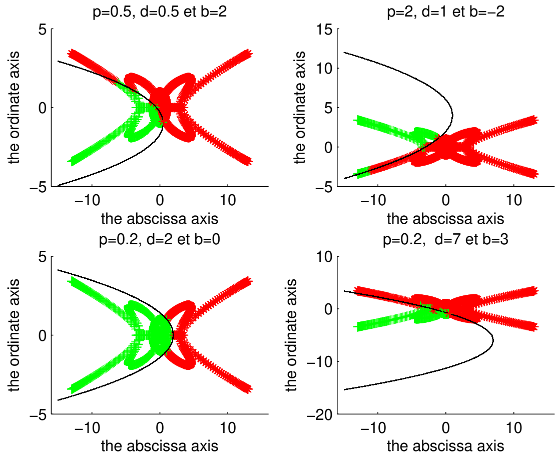

Figure 8: Partition of the eigenvalues of $W(t)$, $\forall t \in [0, \pi]$ with $(p, d, b) \in \{(0.5, 0.5, 2), (2, 1, -2), (0.2, 2, 0), (0.2, 7, 3)\}$.

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