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Hop Independent Sets in Graphs

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Abstract. Let G be an undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $S \subseteq V(G)$ is a hop independent set of G if any two distinct vertices in S are not at a distance two from each other, that is, $d_G(v,w) \neq 2$ for any distinct vertices $v, w \in S$. The maximum cardinality of a hop independent set of G, denoted by $\alpha_h(G)$, is called the hop independence number of G. In this paper, we show that the absolute difference of the independence number and the hop independence number of a graph can be made arbitrarily large. Furthermore, we determine the hop independence numbers of some graphs including those resulting from some binary operations of graphs.

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1. Introduction

In this paper we explore a parameter that is, in some sense, defined in a similar way that the well-known independence number of a graph is. Indeed, while an independent set of graph requires that no two distinct vertices in the set are at distance one from each other, the concept that we will be dealing with here imposes the condition that no two distinct vertices in the set are at distance two from each other. The motivation of introducing the concept is the ever increasing number of studies on hop domination and some of its variations. In fact, it can be shown that every maximum hop independent set of a graph is a hop dominating set. Consequently, the hop domination number of a graph is at most equal to the hop independence number of the graph.

The concept of hop domination was introduced and studied by Natarajan and Ayyaswamy in [4]. The concept and some of its variants are also studied in [1], [2],

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[3], [5], [6], [7], [8], and [9]. Alongside other previously defined parameters in a graph, the hop independence number of a graph may be used to give bounds on some hop-domination related parameters. Moreover, this newly defined concept may be utilized to introduce some concepts (say, a variant of hop domination) in the future.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic.* The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup\limits_{v \in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A.* The *open hop neighborhood* of a point u is the set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$. The *closed hop neighborhood* of u is $N_G^2[u] = N_G^2(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G^2(A) = \bigcup\limits_{v \in A} N_G^2(v)$ is called the *open hop neighborhood of A* and $N_G^2[A] = N_G^2(A) \cup A$ is called the *closed hop neighborhood of A.*

A set $S \subseteq V(G)$ is a *hop dominating* set if $N_G^2[S] = V(G)$. The minimum cardinality of a hop dominating set of a graph G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A set $S \subseteq V(G)$ is an *independent* set of G if no two pair of distinct vertices of S are adjacent. The maximum cardinality of an independent set of G, denoted by $\alpha(G)$, is called the *independence number* of G. Set S is a *hop independent* set of G if for any two distinct vertices v and w of S, $d_G(v,w) \neq 2$. The maximum cardinality of a hop independent set of G, denoted by $\alpha_h(G)$, is called the *hop independence number* of G. Any independent (hop independent) set with cardinality $\alpha(G)$ (resp. $\alpha_h(G)$) is referred to as a *maximum independent* set or α-set (resp. *maximum hop independent* set or α_h-set) of G.

A set S is *clique* of a graph G if the graph $\langle S \rangle$ induced by S is a complete graph. The maximum size or cardinality of a clique of G, denoted by $\omega(G)$, is called the *clique number* of G. Any clique in G with cardinality $\omega(G)$ is called an ω-set in G.

3. Results

Proposition 1. *Let G be any graph on n vertices. If S is a maximun hop independent set of G, then S is a hop dominating set. In particular, $\gamma_h(G) \leq \alpha_h(G)$.*

Proof. Let S be a maximun hop independent set of G and let $v \in V(G) \setminus S$. If $d_G(v,w) \neq 2$ for all $w \in S$, then $S \cup \{v\}$ is a hop independent set of G, contradicting the maximality of S. Thus, there exists $z \in S$ such that $d_G(v,z) = 2$, showing that S is a hop dominating set of G. Therefore $\gamma_h(G) \leq \alpha_h(G)$. □

Theorem 1. *Let G be any graph on n vertices. If S is a hop independent set of G, then every component of $\langle S \rangle$ is complete. Moreover,*

(i) *$\alpha_h(G) = n$ if and only if every component of G is complete; and*

(ii) *for $n \geq 3$, $\alpha_h(G) = n - 1$ if and only if all but a single component C of G are complete and $C \setminus v$ is a complete graph for some vertex $v \in V(C)$.*

Proof. Let S be a hop independent set of G. If some component C of $\langle S\rangle$ is not complete, then there exist distinct vertices $x, y \in C$ such that $d_G(x, y) = d_C(x, y) = 2$. This, however, contradicts our assumption of S. Hence, every component of $\langle S\rangle$ is complete.

(i) Now, if $\alpha_h(G) = n$, then $V(G)$ is a hop independent set of G. By the first part, this would imply that every component of G is complete.

Conversely, suppose that every component of G is complete. Then clearly, $V(G)$ is a hop independent set of G. Thus, $\alpha_h(G) = n$. This proves (i).

(ii) Suppose that $\alpha_h(G) = n - 1$. Then there exists $v \in V(G)$ such that $S = V(G) \setminus \{v\}$ is a hop independent set of G. Let $\Omega = \{C_1, C_2, \ldots, C_k\}$ be the set consisting of the components of $\langle S\rangle$. Again, by the first part, every component C_j of $\langle S\rangle$ is complete. Now, by (i) and the assumption, it follows that G has a component C that is not complete. Hence, $\langle\{v\}\rangle$ is not a component of G; otherwise, $C_1, C_2, \ldots, C_k, \langle\{v\}\rangle$ are the components of G which is not possible. This implies that there exists $z \in S$ such that $vz \in E(G)$. Let C_r be the component of $\langle S\rangle$ containing z. Since S is a hop independent set, $vq \notin E(G)$ for all $q \in \cup_{j\neq r} V(C_j)$. Let $D = V(C_r) \cup \{v\}$ and let $C = \langle D\rangle$. Then $(\Omega \setminus \{C_r\}) \cup \{C\}$ contains all the components of G. Consequently, C is not complete and $C \setminus v = C_r$ is complete.

Next, suppose that all but a single component C of G are complete and $C \setminus v$ is a complete graph for some vertex $v \in V(C)$. Then $\alpha_h(G \leq n-1$ by (i). Since $S' = V(G)\setminus\{v\}$ is a hop independent set of G, it follows that $\alpha_h(G) = n - 1$. □

The next result is immediate from Theorem 1.

Corollary 1. *Let G be a connected graph on n vertices. Then*

(i) *$\alpha_h(G) = n$ if and only if $G = K_n$; and*

(ii) *for $n \geq 3$, $\alpha_h(G) = n - 1$ if and only if $G \neq K_n$ and there exists $v \in V(G)$ such that $G \setminus v = K_{n-1}$.*

Proposition 2. *Let n be a positive integer.*

(i) *There exists a connected graph G such $\alpha_h(G) - \alpha(G) = n$.*

(ii) *There exists a connected graph G such $\alpha(G) - \alpha_h(G) = n$.*

Proof. For (i), consider $G = K_{n+1}$. Then $\alpha(G) = 1$, and by Corollary 1, $\alpha_h(G) = n+1$. Hence, $\alpha_h(G) - \alpha(G) = n$.

For (ii), consider $G = K_{1,n+2}$. Then $\alpha(G) = n+2$ and $\alpha_h(G) = 2$. Thus, $\alpha(G) - \alpha_h(G) = n$. □

Note that Proposition 2 implies that given a positive integer n, there exists a connected graph G such that $|\alpha(G) - \alpha_h(G)| = n$, i.e., the absolute difference of these two parameters can be made arbitrarily large.

Theorem 2. *Let a and b be positive integers such that $3 \leq a \leq b$. Then*

(i) *there exists a connected graph G such $\alpha_h(G) = a$ and $\alpha(G) = b$, and*

(ii) *there exists a connected graph G' such $\alpha(G') = a$ and $\alpha_h(G') = b$.*

Proof. Suppose first that $a = b$. Consider the graph G in Figure 1. Clearly, $S_1 = \{y_1, y_2, \ldots, y_a\}$ is both an α-set and an α_h-set of G. Hence, $\alpha(G) = \alpha_h(G) = a$.

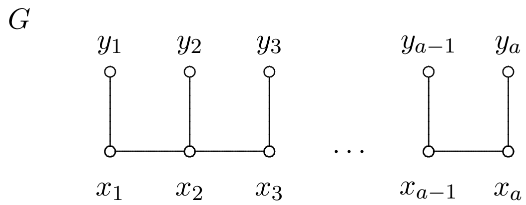

Figure 1

Next, suppose that $a < b$ and let $m = b - a + 1$. Consider the graph G in Figure 2. It can easily be verified that the set $S_1 = \{y_1, y_2, \ldots, y_{a-2}, x_a, z_1\}$ is an α_h-set and $S_2 = \{y_1, \ldots, y_{a-1}, z_1, \ldots, z_m\}$ is an α-set of G. Thus, $\alpha_h(G) = a$ and $\alpha(G) = b$.

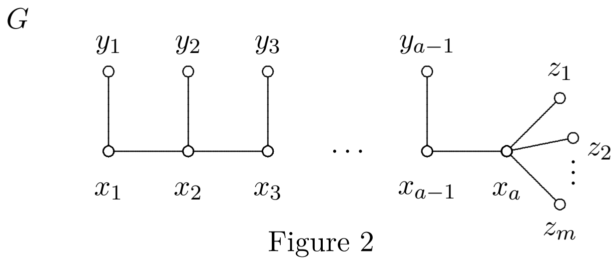

Figure 2

(ii) Suppose $a < b$ and let $m = b-a+1$. Consider the graph G' in Figure 3. It can easily be verified that the set $S = \{y_1, y_2, \ldots, y_{a-1}, x_a\}$ is an α-set and $S' = \{y_1, \ldots, y_{a-1}, z_1, \ldots, z_m\}$ is an α_h-set of G'. Thus, $\alpha(G') = a$ and $\alpha_h(G') = b$.

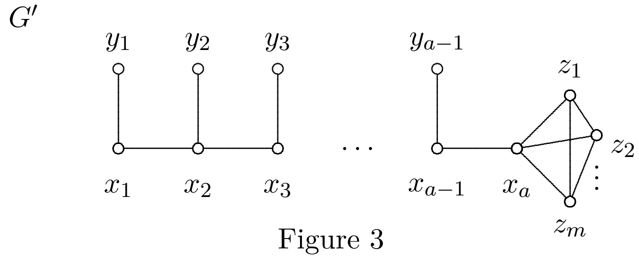

Figure 3

This proves the assertion. □

For any graph G, let $\delta_h(G) = \min\{|N_G^2(v)| : v \in V(G)\}$.

Theorem 3. *For any graph G on n vertices, $\alpha_h(G) \leq n - \delta_h(G)$.*

Proof. Let S be a maximum hop independent set of G and let $v \in S$. By definition, $\delta_h(G) \leq |N_G^2(v)|$. Since S is a hop independent set of G and $v \in S$, $N_G^2(v) \subseteq V(G) \setminus S$. Hence,

$$\delta_h(G) \leq |N_G^2(v)| \leq n - |S| = n - \alpha_h(G).$$

Therefore, $\alpha_h(G) \leq n - \delta_h(G)$. □

The *join* of two graphs G and H, denoted by $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 4. *Let G and H be graphs. Then S is a non-empty hop independent set of $G + H$ if and only if one of the following statements holds:*

(i) *$S \cap V(H) = \varnothing$ and $S \cap V(G)$ is a clique in G.*

(ii) *$S \cap V(G) = \varnothing$ and $S \cap V(H)$ is a clique in H*

(iii) *$S \cap V(G)$ and $S \cap V(G)$ are cliques in G and H, respectively.*

Proof. Suppose S is a hop independent set of $G + H$ and let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Suppose $S_H = \varnothing$. Then $S_G \neq \varnothing$. Let $a, b \in S_G$. Since S is a hop independent set of $G + H$, it follows that $d_{G+H}(a,b) = d_G(a,b) \neq 2$. This implies that $d_G(a,b) = 1$, showing that S_G is a clique in G. Hence, (i) holds. Similarly, (ii) holds. Next, suppose that $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Then, clearly, S_G and S_H are cliques in G and H, respectively.

The converse is clear. □

The next result is a consequence of Theorem 4.

Corollary 2. *Let G and H be graphs. Then $\alpha_h(G + H) = \omega(G) + \omega(H)$. In particular, we have* (*i*) $\alpha_h(K_n + H) = n + \omega(H)$ *for all* $n \geq 1$*;*

(*ii*) $\alpha_h(W_n) = \alpha_h(K_1 + C_n) = 3$ *for all* $n \geq 4$*;*

(*iii*) $\alpha_h(F_n) = \alpha_h(K_1 + P_n) = 3$ *for all* $n \geq 1$*; and*

(*iv*) $\alpha_h(K_{1,n}) = \alpha_h(K_1 + \overline{K}_n) = 2$ *for all* $n \geq 1$*.*

The *corona* of graphs G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$.

Theorem 5. *Let G be a non-trivial connected graph and let H be any graph. Then S is a hop independent set in $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$ and satisfies the following conditions:*

(*i*) *A is a hop independent set in G.*

(*ii*) *S_v is empty or a clique in H^v for each $v \in V(G) \setminus N_G(A)$.*

(*iii*) *$S_v = \varnothing$ for each $v \in N_G(A)$.*

Proof. Suppose S is a hop independent set in $G \circ H$ and let $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Since S is a hop independent set in $G \circ H$, A is a hop independent set in G. This shows that (*i*) holds. Next, let $v \in V(G)$. Suppose first that $v \in V(G) \setminus N_G(A)$. Suppose further that $S_v \neq \varnothing$. Clearly, if $|S_v| = 1$, then it is a clique. So suppose $|S_v| \geq 2$ and let $x, y \in S_v$. Since S is a hop independent set, $d_{G \circ H}(x, y) = d_{H^v}(x, y) \neq 2$, i.e., $d_{H^v}(x, y) = 1$. This shows that (*ii*) holds. Suppose now that $v \in N_G(A)$, say $vw \in E(G \circ H)$ for some $w \in A$. Since S is a hop independent set and $d_{G \circ H}(w, z) = 2$ for all $z \in V(H^v)$, it follows that $S_v = \varnothing$, showing that (*iii*) holds.

For the converse, suppose that S has the given form and satisfies (*i*), (*ii*), and (*iii*). Let $a, b \in S$, where $a \neq b$, and let $v, w \in V(G)$ such that $a \in V(v + H^v)$ and $b \in V(w + H^w)$. Consider the following cases:

Case 1. $v \neq w$.

Suppose $a = v$ and $b = w$. Then $a, b \in A$. Since A is a hop independent set of G, $d_{G \circ H}(a, b) = d_G(a, b) \neq 2$. Suppose now that $a = v$ and $b \neq w$ (or $a \neq v$ and $b = w$). Then $a \in A$ and $b \in S_w$. By (*iii*), $w \notin N_G(A)$. Hence, $wv \notin E(G)$ and $d_{G \circ H}(a, b) \neq 2$. If if $a \neq v$ and $b \neq w$, then $a \in S_v$ and $b \in S_w$. Clearly, $d_{G \circ H}(a, b) \neq 2$.

Case 2. $v = w$.

If one of a and b is v, say $a = v$, then $b \in S_v$ and $d_{G \circ H}(a, b) = 1 \neq 2$. If $a \neq v$ and $b \neq w$, then $a, b \in S_v$. By (*ii*), S_v is a clique in H^v and so $d_{G \circ H}(a, b) = 1 \neq 2$.

Therefore, S is a hop independent set of $G \circ H$. □

Lemma 1. *Let G be a non-trivial connected graph and let A be a hop independent set of G. Then $|A| \leq |N_G(A)|$.*

Proof. Note that $A = (A \setminus N_G(A)) \cup (A \cap N_G(A))$. Since G is a non-trivial connected graph, $N_G(a) \neq \varnothing$ for each $a \in A \setminus N_G(A)$. Now let $a, b \in A \setminus N_G(A)$ with $a \neq b$. Suppose $N_G(a) \cap N_G(b) \neq \varnothing$, say $x \in N_G(a) \cap N_G(b)$. Since A is a hop independent set of G, $d_G(a, b) \neq 2$. Hence, $ab \in E(G)$, implying that $a \in A \cap N_G(A)$. This contradicts the assumption that $a \in A \setminus N_G(A)$. Therefore, $N_G(a) \cap N_G(b) = \varnothing$ for any two distinct vertices a and b in $A \setminus N_G(A)$. For each $a \in A \setminus N_G(A)$, choose $v_a \in (V(G) \setminus A) \cap N_G(a)$ (such vertex v_a exists because G is non-trivial and connected) and let $D = \{v_a : a \in A \setminus N_G(A)\}$. Then $D \subseteq N_G(A)$ and $|D| = |A \setminus N_G(A)|$. Thus,

$$|A| = |A \cap N_G(A)| + |A \setminus N_G(A)| = |A \cap N_G(A)| + |D| \leq |N_G(A)|.$$

This proves the assertion. □

Corollary 3. *Let G be a non-trivial connected graph and let H be any graph. Then $\alpha_h(G \circ H) = |V(G)|\omega(H)$.*

Proof. Let S_v be an ω-set of H^v for each $v \in V(G)$. Then $S = \cup_{v \in V(G)} S_v$ is a hop independent set of $G \circ H$ by Theorem 5. This implies that $\alpha_h(G \circ H) \geq |S| = |V(G)|\omega(H)$.

Next, let S^* be a α_h-set of $G \circ H$. Then $S^* = A \cup (\cup_{v \in V(G)} R_v)$ and satisfies (i), (ii), and (iii) of Theorem 5. Hence, by Theorem 5 and Lemma 1, we have

$$\begin{aligned}
\alpha_h(G \circ H) &= |S^*| = |A| + \sum_{v \in V(G)} |R_v| \\
&= |A| + \sum_{u \in N_G(A)} |R_u| + \sum_{v \notin N_G(A)} |R_v| \\
&= |A| + \sum_{v \notin N_G(A)} |R_v| \\
&\leq |A| + (|V(G)| - |N_G(A)|)\omega(H) \\
&= |A| - |N_G(A)|\omega(H) + |V(G)|\omega(H) \\
&\leq |A| - |N_G(A)| + |V(G)|\omega(H) \\
&\leq |V(G)|\omega(H).
\end{aligned}$$

This proves the desired equality. □

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 6. *Let G and H be non-trivial connected graphs. Then $C = \bigcup_{x \in S} [\{x\} \times T_x]$,* *where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a hop independent set of $G[H]$ if and only if the following conditions hold.*

(i) S is a hop independent set of G.

(ii) T_x is a clique in H for each $x \in S$.

Proof. Suppose $C = \bigcup_{x \in S} [\{x\} \times T_x]$ is a hop independent set of $G[H]$. Let $v, w \in S$ with $v \neq w$ and let $a \in T_v$ and $b \in T_w$. Since $(v, a), (w, b) \in C$ and C is a hop independent set of $G[H]$, it follows that $d_{G[H]}((v, a), (w, b)) = d_G(v, w) \neq 2$. This implies that S is a hop independent set of G, showing that (i) holds. Next, let $x \in S$. If $|T_x| = 1$, then T_x is a clique in H. Suppose $|T_x| \geq 2$ and let $p, q \in T_x$, where $p \neq q$. Then (x, p) and (x, q) are distinct elements of C. Since C is a hop independent set of $G[H]$, $d_{G[H]}((x, p), (x, q)) \neq 2$. Now, since G is non-trivial and connected, it follows that $d_H(p, q) = 1$. Thus, T_x is a clique in H, showing that (ii) holds.

For the converse, suppose that $C = \bigcup_{x \in S} [\{x\} \times T_x]$ and satisfies (i) and (ii). Let $(y, a), (z, b) \in C$ with $(y, a) \neq (z, b)$. Consider the following cases:

Case 1. $y = z$.

Then $a, b \in T_y$. From condition (ii), T_x is a clique in H and so $d_H(a, b) = 1$. Hence, $d_{G[H]}((y, a), (y, b)) = 1 \neq 2$.

Case 2. $y \neq z$.

Since $y, z \in S$ and S is a hop independent set of G, $d_G(y, z) \neq 2$. It follows that $d_{G[H]}((y, a), (z, b)) = d_G(y, z) \neq 2$.

Accordingly, C is a hop independent set of $G[H]$. □

Corollary 4. *Let G and H be non-trivial connected graphs. Then*

$$\alpha_h(G[H]) = \alpha_h(G)\omega(H).$$

Proof. Let S be a α_h-set of G and let D be a clique in H with $|D| = \omega(H)$. For each $x \in S$, set $T_x = D$. Then $C = \bigcup_{x \in S} [\{x\} \times T_x] = S \times D$ is a hop independent set of $G[H]$ by Theorem 6. Hence,

$$\alpha_h(G[H]) \geq |C| = \alpha_h(G)\omega(H).$$

Next, let $C_0 = \bigcup_{x \in S_0} [\{x\} \times R_x]$ be a α_h-set of $G[H]$. By Theorem 6, S_0 is a hop independent set of G and T_x is a clique in H. Hence,

$$\alpha_h(G[H]) = |C_0| = \sum_{x \in S_0} |T_x| \leq |S_0|\omega(H) \leq \alpha_h(G)\omega(H).$$

This establishes the desired equality. □

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $a = b$ and $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Theorem 7. *Let G and H be non-trivial connected graphs. Then $C = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a hop independent set of $G\Box H$ if and only if the following conditions hold.*

(i) T_x is a hop independent set of H for each $x \in S$;

(ii) For each $x \in S \cap N_G(S)$ and for each $y \in S \cap N_G(x)$, it holds that $d_H(p,q) \neq 1$ for all $p \in T_x$ and $q \in T_y$; and

(iii) For each $v \in S \cap N_G^2(S)$ and for each $w \in S \cap N_G^2(v)$, it holds that $d_H(a,b) \geq 1$ for all $a \in T_v$ and $b \in T_w$.

Proof. Suppose $C = \bigcup_{x\in S}[\{x\} \times T_x]$ is a hop independent set of $G\Box H$. Let $x \in S$ and let $a, b \in T_x$ with $a \neq b$. Since (x,a) and (x,b) are distinct elements of C and C is a hop independent set of $G\Box H$, $d_H(a,b) = d_{G\Box H}((x,a),(x,b)) \neq 2$, showing that (i) holds, i.e., T_x is a hop independent set of H. Next, let $x \in S\cap N_G(S)$ and let $y \in S\cap N_G(x)$. Take any $p \in T_x$ and $q \in T_y$. Suppose $d_H(p,q) = 1$. Clearly, (x,p) and (y,q) are distinct elements of C and since $p \neq q$, $d_{G\Box H}((x,p),(y,q)) \neq 1$. Since $[(x,p),(x,q),(y,q)]$ is an (x,p)-(y,q) geodetic in $G\Box H$, $d_{G\Box H}((x,p),(y,q)) = 2$, contrary to the fact that C is a hop independent set of $G\Box H$. Thus, $d_H(p,q) \neq 1$, showing that (ii) holds. Finally, let $v \in S \cap N_G^2(S)$ and let $w \in S \cap N_G^2(v)$. Choose any $a \in T_v$ and $b \in T_w$. Then $(v,a),(w,b) \in C$. Again, since C is a hop independent set of $G\Box H$, $d_{G\Box H}((v,a),(w,b)) \neq 2$. Since $d_G(v,w) = 2$, $a \neq b$. Thus, $d_H(a,b) \geq 1$, showing that (iii) holds.

For the converse, suppose that C satisfies conditions (i), (ii), and (iii). Let $(x,p),(y,q) \in C$ such that $(x,p) \neq (y,q)$. Consider the following cases:

Case 1. $x = y$. Then $p \neq q$ and $p,q \in T_x$. From condition (i), it follows that $d_{G\Box H}((x,p),(y,q)) = d_H(p,q) \neq 2$.

Case 2. $x \neq y$.

Clearly, if $d_G(x,y) \geq 3$, then $d_{G\Box H}((x,p),(y,q)) \neq 2$. Next, suppose that $d_G(x,y) = 1$. Then by condition (ii), $d_H(p,q) \neq 1$. If $d_H(p,q) = 0$, then $d_{G\Box H}((x,p),(y,q)) = 1 \neq 2$. If $d_H(p,q) \geq 2$, then $d_{G\Box H}((x,p),(y,q)) = 1 + d_H(p,q) \geq 3$. Suppose now that $d_G(x,y) = 2$. Then by (iii), $d_H(p,q) \geq 1$. Hence, $d_{G\Box H}((x,p),(y,q)) = d_G(x,y) + d_H(p,q) \geq 3$.

Therefore, C is a hop independent set of $G\Box H$. $\square$

A set S is a 3-*hop set* of a connected graph G if $d_G(v,w) = 3$ for every pair of distinct vertices $v, w \in S$. The maximum cardinality of a 3-hop set of G is denoted by $\alpha_h^3(G)$.

Corollary 5. *Let G and H be non-trivial connected graphs. Then*

$$\alpha_h(G\square H) \geq \max\{\alpha_h^3(G)\alpha_h(H), \alpha_h^3(H)\alpha_h(G)\}.$$

Proof. Let S be a 3-hop set of G with $|S| = \alpha_h^3(G)$ and let D be an α_h-set of H. Set $T_x = D$ for each $x \in S$. Then $C = \bigcup_{x\in S}[\{x\} \times T_x] = S \times D$ is a hop independent set of $G\square H$ by Theorem 7. Hence, $\alpha_h(G\square H) \geq |C| = |S||D| = \alpha_h^3(G)\alpha_h(H)$. Since $G\square H$ and $H\square G$ are isomorphic, the assertion holds. □

The bound given in Corollary 5 is attainable. To see this, consider $P_4\square K_4$. Note that $\alpha_h(K_4) = 4$ and $\alpha_h^3(P_4) = 2$. One can easily verify that $\alpha_h(P_4\square K_4) = 8 = \alpha_h^3(P_4)\alpha_h(K_4)$. The bound, however, may not be attained. Consider, for example, $P_4\square P_4$. It can also be verified that $\alpha_h(P_4\square K_4) = 6 > 4 = \alpha_h^3(P_4)\alpha_h(P_4)$.

4. Conclusion

The concept of hop independent set in a graph, though maybe considered informally previously, has been introduced formally and investigated initially in this study. It is shown that the hop independence number of a graph is an upper bound of the hop domination number of the graph and that the absolute difference of the independence number and hop independence number can be made arbitrarily large. Just like the independence number, the hop independence number of a graph may be used to give bounds on some graph-theoretic parameters. In this paper the concept has been investigated for the join, corona, lexicographic and Cartesian products of graphs. Finding better bounds on the hop independence number of the Cartesian product of some graphs is recommended. Also, this newly defined parameter can be studied further for other types of graphs.

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References

[1] S. Arriola and Jr. S. Canoy. $(1,2)^*$-domination in graphs. *Advances and Applications in Discrete Mathematics.*, 18(2):179–190, 2017.

[2] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences*, 125(4):449–455, 2015.

[3] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics.*, 33(4):913–927, 2017.

[4] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[5] Jr. S. Canoy and G. Malacas. Determining the intruder's location in a given network: Locating-dominating sets in a graph. *NRCP Research Journal.*, 13(1):1–8, 2013.

[6] Jr. S. Canoy and G. Malacas. Differentiating-dominating sets in graphs under binary operations. *Tamkang J. Math.*, 46(1):51–60, 2015.

[7] Jr. S. Canoy, R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *Eur. J. Pure Appl. Math.*, 12(4):1455–1463, 2019.

[8] Jr. S. Canoy and G. Salasalan. Locating-hop domination in graphs. *Kyungpook Mathematical Journal*, 62:193–204, 2022.

[9] G. Salasalan and Jr. S. Canoy. Global hop domination numbers of graphs. *Eur. J. Pure Appl. Math.*, 14(1):112–125, 2021.